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An Extremal Result for Unknown Interval Linear Systems

Daniel N. Mohsenizadeh * **L.H. Keel** ** **S.P. Bhattacharyya** ***

* *Department of Mechanical Engineering, Texas A&M University, College Station, TX 77843 USA (email: danielmz@tamu.edu).*
** *Department of Electrical and Computer Engineering, Tennessee State University, Nashville, TN 37209 USA (e-mail: keel@gauss.tsuniv.edu).*
*** *Department of Electrical and Computer Engineering, Texas A&M University, College Station, TX 77843 USA (email: bhatt@ee.tamu.edu).*

Abstract: This paper explores some important characteristics of a system of linear equations containing parameters. Such a system of equations arises in many branches of engineering including electrical circuits, hydraulic networks and truss structures. A parametrized solution of a set of linear equations can be obtained by applying Cramer's rule. In many practically important cases the parameters appear with rank one dependency, resulting in parametrized solutions to be of a rational multilinear form, which will be monotonic in each parameter. This monotonic characteristic has practical importance in the analysis and design of linear systems with parameters having interval uncertainties. In particular, extremal values of system variables occur at the vertices of the parameter boxes.

Keywords: Linear systems, parametrized solution, interval analysis.

1. INTRODUCTION

The problem of analyzing and controlling interval systems is important for practical applications and has been open for the last few decades. Several results concerning robustness analysis of systems with real parametric uncertainty can be found in the early works of Horowitz [1963], Siljak [1969], Ackermann [1980, 1993] and Bhattacharyya et al. [1995]. Kharitonov [1978] theorem, later generalized by Chapellat and Bhattacharyya [1989], provided a means to evaluate the stability of an interval plant by testing a finite number of polynomials for stability. An extension of the Kharitonov's theorem, known as the edge theorem, discovered by Bartlett et al. [1988], states that the stability of a polytope of polynomials is equivalent to the stability of its one-dimensional exposed edge polynomials. Jetto and Orsini [2009] showed that the Schur stability analysis of an interval polynomial family can be performed through a uniquely defined extreme polynomial. The sign-definite decomposition method, introduced by Elizondo-Gonzalez [2000] and followed up by Knap et al. [2011], can be used to decide the robust positivity (or negativity) of a polynomial over a box of uncertain parameters by evaluating the sign of the decomposed polynomials at the vertices of the box. An application of sign-definite decomposition method to the synthesis of stabilizing controllers of a fixed structure is studied by Mohsenizadeh et al. [2011]. A method is also proposed by Anai and Hara [2000] to design fixed-structure robust controllers based on a special Quantifier Elimination (QE) technique and a sign-definite condition. Also, recent results on the robust control of linear systems are provided by Bhattacharyya et al. [2009].

This paper concentrates on the class linear systems containing real parameters with interval uncertainties and presents an extremal result. It will be shown that if in an unknown linear system the uncertain parameters appear with rank one dependency, then the extremal values of some set of system variables over a box in the parameter space occur at the vertices of that box. This enables us to evaluate the performance of an unknown interval system over a box of uncertain parameters by checking the respective performance index at the vertices.

An application of the Cramer's rule to a set of linear equations containing parameters yields an expression for a parametrized solution of the set. A parametrized solution can be seen as a system variable whose value is to be evaluated over a box in the parameter space. The determinants appearing in Cramer's formula can be expanded as polynomial functions of the parameters, resulting in a rational polynomial form for the parametrized solutions. We will show that the coefficients of these polynomials can be determined directly from a small number of measurements and does not require the knowledge of the linear equations describing the system. If the uncertain parameters appear with rank one dependency, which is the case in many applications, then the rational polynomial form for the parametrized solutions reduces to a rational multilinear function, being monotonic in each parameter. This monotonic characteristic leads us to extract our extremal results.

This paper is organized as follows. In Section 2 we provide some mathematical preliminaries on the parametrized solution of a set of linear equations. Section 3 presents our extremal result for unknown linear systems with parameters appearing with rank one dependency. Some illustrative examples of current, power level and flow rate control problems are given in Section 4. Finally, we summarize with our concluding remarks in Section 5.

2. LINEAR EQUATIONS WITH PARAMETERS

Consider the system of linear equations

$$Ax = b, \tag{1}$$

where A is an $n \times n$ matrix, and x and b are $n \times 1$ vectors, all with real or complex entries. Let $|.|$ denotes the determinant. Assuming that $|A| \neq 0$, there exists a unique solution x and, by Cramer's rule, the i^{th} element x_i of x is given by

$$x_i = \frac{|B|}{|A|}, \qquad i = 1, 2, \ldots, n \tag{2}$$

where B is the matrix obtained by replacing the i^{th} column of A by b.

In order to show the parameter dependency of A and b explicitly, let us rewrite (1) as

$$A(p)x = b(q), \tag{3}$$

where $p = [p_1, p_2, \ldots, p_l]^T$ and $q = [q_1, q_2, \ldots, q_m]^T$ are vectors of system parameters. With this notation, (2) becomes

$$x_i(p, q) = \frac{|B(p, q)|}{|A(p)|}, \qquad i = 1, 2, \ldots, n. \tag{4}$$

Suppose that the parameter vector p appears *affinely* in $A(p)$. Thus, we can write

$$A(p) = A_0 + p_1 A_1 + p_2 A_2 + \cdots + p_l A_l. \tag{5}$$

To proceed, consider the special case of a scalar parameter $p = p_1$ and

$$A(p) = A_0 + p_1 A_1. \tag{6}$$

Lemma 1. With $A(p)$ as in (6), $|A(p)|$ is a polynomial of degree at most r_1 in p_1 where

$$r_1 = \operatorname{rank}[A_1]. \tag{7}$$

Proof. The proof follows easily from the properties of determinants. □

According to the form (6) and the rank condition in (7), we say that p_1 appears in $A(p)$ with rank r_1 dependency. The statement of Lemma 1 can be generalized to the case with more than one parameter.

Lemma 2. With $A(p)$ as in (5), let

$$r_i = \operatorname{rank}[A_i], \quad i = 1, 2, \ldots, l. \tag{8}$$

Then, $|A(p)|$ is a multivariate polynomial in p of degree r_i or less in p_i, $i = 1, 2, \ldots, l$ and

$$|A(p)| = \sum_{i_l=0}^{r_l} \cdots \sum_{i_2=0}^{r_2} \sum_{i_1=0}^{r_1} \alpha_{i_1 i_2 \cdots i_l} p_1^{i_1} p_2^{i_2} \cdots p_l^{i_l}. \tag{9}$$

Proof. This follows immediately from Lemma 1. □

Remark 1. In the formula (9), the number of coefficients $\alpha_{i_1 i_2 \cdots i_l}$ are $\prod_{i=1}^{l}(r_i + 1)$.

According to the form (5) and the rank conditions in (8), we say that p_i appears in $A(p)$ with rank r_i dependency.

Let us assume that $b(q)$ can be decomposed as

$$b(q) = b_1 q_1 + b_2 q_2 + \cdots + b_m q_m, \tag{10}$$

where $b_1, b_2, \ldots, b_m$ are $n \times 1$ vectors with real or complex entries. The decomposition given in (10) is a characteristic of linear systems obeying the Superposition Principle. Based on the above lemmas, we have the following characterization of parametrized solutions.

Theorem 1. With $A(p)$ as in (5),

$$x_i(p, q) = \frac{|B(p, q)|}{|A(p)|} := \frac{\beta(p, q)}{\alpha(p)}, \qquad i = 1, 2, \ldots, n, \tag{11}$$

where $\beta(p, q)$ and $\alpha(p)$ are multivariate polynomials in (p, q) and p, respectively.

Proof. The proof follows from (4) and Lemma 2. □

Remark 2. The matrix $B(p, q)$ in (11) can be written as

$$\begin{aligned} B(p, q) = B_0 + p_1 B_1 + \cdots + p_l B_l \\ + q_1 B_{l+1} + \cdots + q_m B_{l+m}. \end{aligned} \tag{12}$$

But, since $q_1, q_2, \ldots, q_m$ appear only in one column of $B(p, q)$, then

$$\operatorname{rank}[B_i] = 1, \quad i = l + 1, l + 2, \ldots, l + m. \tag{13}$$

Thus, $\beta(p, q)$ in (11) becomes

$$\begin{aligned} \beta(p, q) = \sum_{i_{l+m}=0}^{1} \cdots \sum_{i_{l+1}=0}^{1} \sum_{i_l=0}^{r_l} \cdots \sum_{i_1=0}^{r_1} \\ \left(\beta_{i_1 \cdots i_l i_{l+1} \cdots i_{l+m}}\, p_1^{i_1} \cdots p_l^{i_l} q_1^{i_{l+1}} \cdots q_m^{i_{l+m}} \right). \end{aligned} \tag{14}$$

The number of coefficients $\beta_{i_1 \cdots i_l i_{l+1} \cdots i_{l+m}}$ in (14) are $2^m \left(\prod_{i=1}^{l} (r_i + 1) \right)$.

Remark 3. In physical systems, the parameters p *usually* appear in $A(p)$ with rank one dependency. For instance, branch resistors, impedances and dependent sources in an electrical circuit, mechanical properties of links in a truss structure, pipe resistances in a linear hydraulic network, and blocks in a signal flow block diagram, all appear with rank one dependency in the characteristic matrix of the system.

3. MAIN RESULTS

Suppose that a linear physical system can be described by the following set of linear equations

$$A(p)x = b(q), \tag{15}$$

where $A(p)$ is referred to as the system characteristic matrix, p and q are vectors of system parameters and inputs, respectively, and x is the vector of unknown system variables, such as currents in an electrical circuit or flow rates in a hydraulic network. We make the following crucial assumption regarding the set of equations (15).

Assumption 1. There exists no p such that $A(p)$ is a singular matrix.

This assumption is usually true for physical systems, because if there exists a vector p_0 so that $A(p_0)$ becomes a singular matrix, then the corresponding vector of system variables, x in (15), will not have a unique value which is not the case for physical systems.

We define the following sets:

$$\mathcal{P} := \{p, q\} = \{p_1, p_2, \ldots, p_l, q_1, q_2, \ldots, q_m\}, \tag{16}$$
$$\mathcal{X} := \{x_1, x_2, \ldots, x_n\}. \tag{17}$$

Let us consider the i^{th} element of $\mathcal{X}$, x_i, whose value over a box in the parameter space $\mathcal{D}$, where $\mathcal{D} \subset \mathcal{P}$, is to be evaluated. In the following subsections we summarize our results for 3 cases:

(1) $\mathcal{D} = \{p_1\}$,
(2) $\mathcal{D} = \{p_1, p_2\}$,
(3) $\mathcal{D} = \mathcal{P}$.

3.1 Case 1: $\mathcal{D} = \{p_1\}$

In this case there is only one parameter, p_1. We state the following theorem.

Theorem 2. Supposing that $\text{rank}[A_1] = 1$ *in* (6)*, the function $x_i(p_1)$ in* (11) *can be determined by setting p_1 to 3 different values and measuring the corresponding x_i values.*

Proof. Since $\text{rank}[A_1] = 1$, and based on Lemma 1, then $x_i(p_1)$ can be expressed as:

$$x_i(p_1) = \frac{\tilde{\beta}_0 + \tilde{\beta}_1 p_1}{\tilde{\alpha_0} + \tilde{\alpha_1} p_1}. \tag{18}$$

We note that for $\tilde{\alpha_0} = \tilde{\alpha_1} = 0$, $x_i \to \infty, \forall p_1$, which is not physically possible. Hence, we rule out this case. If $\tilde{\alpha_1} \neq 0$, then the numerator and denominator of (18) can be divided by $\tilde{\alpha}_1$:

$$x_i(p_1) = \frac{\beta_0 + \beta_1 p_1}{\alpha_0 + p_1}. \tag{19}$$

The function $x_i(p_1)$ in (19) can be determined by setting p_1 to 3 different values, measuring the corresponding x_i values and solving the following set of *measurement* equations:

$$\underbrace{\begin{bmatrix} 1 & p_1^1 & -x_i^1 \\ 1 & p_1^2 & -x_i^2 \\ 1 & p_1^3 & -x_i^3 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \beta_0 \\ \beta_1 \\ \alpha_0 \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} x_i^1 p_1^1 \\ x_i^2 p_1^2 \\ x_i^3 p_1^3 \end{bmatrix}}_{m}. \tag{20}$$

The set of equations (20) has a unique solution for β_0, β_1 and α_0 if and only if $|M| \neq 0$. If $|M| = 0$, then as the first two columns of M are linearly independent, x_i will be:

$$x_i(p_1) = \beta_0 + \beta_1 p_1, \tag{21}$$

where β_0 and β_1 can be obtained from any 2 experiments conducted earlier. Equation (21) corresponds to the case where $\tilde{\alpha_1} = 0$ in (18) and the numerator and denominator of (18) are divided by $\tilde{\alpha_0}$. □

The linear fractional form in (19) has some important practical aspects which is explained below.

Remark 4. Taking the derivative of (19) with respect to p_1 yields:

$$\frac{dx_i}{dp_1} = \frac{\beta_1 \alpha_0 - \beta_0}{(\alpha_0 + p_1)^2}. \tag{22}$$

Therefore, we can state the followings:

(1) The function in (19) is *monotonic* in p_1. For example, if $\beta_1\alpha_0 - \beta_0 > 0$ (see Fig. 1), then x_i will monotonically increase as p_1 increases. The upper and lower bounds of x_i for this case are:

$$\frac{\beta_0}{\alpha_0} \leq x_i \leq \beta_1. \tag{23}$$

The range in (23) is called the achievable range.

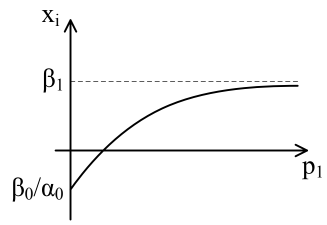

Fig. 1. $x_i(p_1)$ for the case where $\beta_1\alpha_0 - \beta_0 > 0$

(2) This monotonic characteristic is beneficial in solving design problems. For instance, suppose that the system variable x_i is to lie within the range $x_i^- \leq x_i \leq x_i^+$ by adjusting p_1. If $[x_i^-, x_i^+]$ is inside the achievable range, then there exists a unique interval of values for p_1, $p_1^- \leq p_1 \leq p_1^+$, such that the constraint on x_i is satisfied.

The parameter p_1 can be viewed as an uncertain parameter varying in an interval $\mathcal{I} = [p_1^-, p_1^+]$. We now state our first extremal result.

Theorem 3. Assuming that $\text{rank}[A_1] = 1$ *in* (6)*, and p_1 is varying in an interval, $\mathcal{I} = [p_1^-, p_1^+]$, then the extremal values of x_i can be obtained from:*

$$\min_{p_1 \in \mathcal{I}} x_i(p_1) = \min\{x_i(p_1^-), x_i(p_1^+)\},$$
$$\max_{p_1 \in \mathcal{I}} x_i(p_1) = \max\{x_i(p_1^-), x_i(p_1^+)\}.$$

Proof. The proof follows from Theorem 2 and Remark 4. □

3.2 Case 2: $\mathcal{D} = \{p_1, p_2\}$

Here there are two parameters, p_1 and p_2, and therefore the characteristic matrix $A(p)$ can written as

$$A(p) = A_0 + p_1 A_1 + p_2 A_2. \tag{24}$$

We state the following theorem.

Theorem 4. Supposing that $\text{rank}[A_1] = \text{rank}[A_2] = 1$ *in* (24)*, the function $x_i(p_1, p_2)$ in* (11) *can be determined by assigning 7 different sets of values to (p_1, p_2) and measuring the corresponding x_i values.*

Proof. According to Lemma 2, since $\text{rank}[A_1] = \text{rank}[A_2] = 1$, by following the same strategy described in the proof on Theorem 2, $x_i(p_1, p_2)$ will be:

$$x_i(p_1, p_2) = \frac{\beta_0 + \beta_1 p_1 + \beta_2 p_2 + \beta_3 p_1 p_2}{\alpha_0 + \alpha_1 p_1 + \alpha_2 p_2 + p_1 p_2}. \tag{25}$$

A corresponding function for $x_i(p_1, p_2)$ can be obtained if $|M| = 0$ in this case (see proof of Theorem 2). □

Remark 5. Taking the derivative of x_i in (25) with respect to p_1 and fixing $p_2 = p_2^*$ yields

$$\left[\frac{dx_i}{dp_1}\right]_{p_2 = p_2^*} = \frac{a + bp_2^* + cp_2^{*2}}{(\alpha_0 + \alpha_2 p_2^* + (\alpha_1 + p_2^*)p_1)^2}, \tag{26}$$

where

$$a = \alpha_0\beta_1 - \alpha_1\beta_0, \tag{27}$$
$$b = \alpha_0\beta_3 + \alpha_2\beta_1 - \alpha_1\beta_2 - \beta_0, \tag{28}$$
$$c = \alpha_2\beta_3 - \beta_2, \tag{29}$$

which is of the form in (22) and is monotonic in p_1. A similar relationship for $[(dx_i/dp_2)]_{p_1=p_1^*}$ can be derived. Therefore, the function $x_i(p_1, p_2)$ in (25) is monotonic in each parameter p_1 and p_2.

Theorem 3 can be generalized for this case as below.

Theorem 5. If $\text{rank}[A_1] = \text{rank}[A_2] = 1$ *in* (24)*, and* p_1 *and* p_2 *are varying in a rectangle,* $\mathcal{R}$ *(see Fig. 2),*

$$\mathcal{R} = \{(p_1, p_2) \mid p_1^- \le p_1 \le p_1^+,\ p_2^- \le p_2 \le p_2^+\}, \tag{30}$$

with vertices:

$$A = (p_1^-, p_2^-),\ B = (p_1^-, p_2^+),$$
$$C = (p_1^+, p_2^+),\ D = (p_1^+, p_2^-),$$

then the extremal values of x_i *happen at the vertices of* $\mathcal{R}$*:*

$$\min_{p_1,p_2\in\mathcal{R}} x_i(p_1, p_2) = \min\{x_i(A), x_i(B), x_i(C), x_i(D)\},$$
$$\max_{p_1,p_2\in\mathcal{R}} x_i(p_1, p_2) = \max\{x_i(A), x_i(B), x_i(C), x_i(D)\}.$$

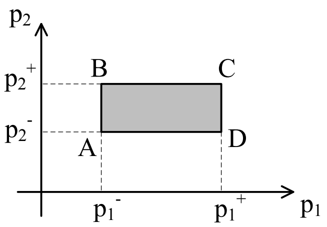

Fig. 2. Rectangle of (p_1, p_2)

Proof. The proof follows immediately from Remark 5. □

3.3 Case 3: $\mathcal{D} = \mathcal{P}$

The results developed in the previous subsections can be generalized to the case where all system parameters (p, q) are considered. In this case $A(p)$ can be decomposed as the form given in (5),

$$A(p) = A_0 + p_1A_1 + p_2A_2 + \cdots + p_lA_l, \tag{31}$$

and $B(p, q)$ will be

$$B(p, q) = B_0 + p_1B_1 + \cdots + p_lB_l + q_1B_{l+1} + \cdots + q_mB_{l+m}, \tag{32}$$

as described in (12). We now state the following general theorems. The proofs follow from the results provided in the previous subsections and are thus omitted here.

Theorem 6. If $\text{rank}[A_i] = 1,\ i = 1, 2, \ldots, l$ *in* (31)*, the function* $x_i(p, q)$ *in* (11) *can be determined by assigning* $2^l(2^m + 1) - 1$ *linearly independent sets of values to* (p, q)*, measuring the corresponding values of* x_i *and solving a system of measurement equations.*

Theorem 7. If $\text{rank}[A_i] = 1,\ i = 1, 2, \ldots, l$ *in* (31)*, and* (p, q) *are varying in a box,* $\mathcal{B}$*,*

$$\mathcal{B} = \{(p, q) \mid p_i^- \le p_i \le p_i^+,\ i = 1, 2, \ldots, l,\quad q_j^- \le q_j \le q_j^+,\ j = 1, 2, \ldots, m\}, \tag{33}$$

with $v := 2^{l+m}$ *vertices, labeled* $V_1, V_2, \ldots, V_v$*, then the extremal values of* x_i *occur at the vertices of* $\mathcal{B}$*:*

$$\min_{p,q\in\mathcal{B}} x_i(p, q) = \min\{x_i(V_1), x_i(V_2), \ldots, x_i(V_v)\},$$
$$\max_{p,q\in\mathcal{B}} x_i(p, q) = \max\{x_i(V_1), x_i(V_2), \ldots, x_i(V_v)\}.$$

Before ending this section, we mention that the evaluation of extremal values of x_i can be accomplished by either of the following ways:

(1) Directly assign values corresponding to the vertices of $\mathcal{B}$, to the vector of parameters and measure x_i, or
(2) First, find the functional dependency for x_i, as states in Theorem 6 by conducting a small number of measurements, and then evaluate that function at the vertices of $\mathcal{B}$.

4. ILLUSTRATIVE EXAMPLES

In this section three illustrative examples are presented to explain the results developed in Section 3.

Example 1. Consider the linear DC circuit shown in Fig. 3. This system can be described mathematically by the following set of linear equations

$$A(p)x = b(q), \tag{34}$$

where $p = [R_1, R_2, \ldots, R_{13}, K_1, K_2]^T$, $q = [V, J_1, J_2]^T$, and x is the vector of unknown currents. In this example $R_i,\ i = 1, 2, \ldots, 13,\ i \ne 5$ are resistors, R_5 is a gyrator resistance, V, J_1, J_2 are independent sources and V_1, V_2 are dependent sources with amplifier gains K_1 and K_2, respectively. We assume that the system is unknown, implying that p and q are unknown.

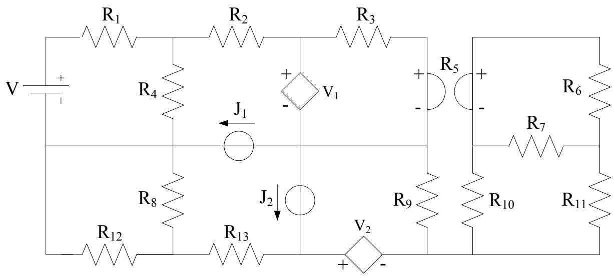

Fig. 3. An unknown DC circuit

Suppose that the objective is to find the extremal values of I_2, if R_1 is varying in the interval $\mathcal{I} = [R_1^-, R_1^+] = [10, 30]$ (Ω). Since the circuit is unknown, $A(p)$ and $b(q)$ in (34) are unknown; but, in fact, one can write

$$A(R_1) = A_0 + R_1A_1, \tag{35}$$

with $\text{rank}[A_1] = 1$. This infers that R_1 appears in $A(p)$ with rank one dependency, and accordingly the results of Section 3.1 can be applied. Based on Theorem 3, the extremal values of I_2 occur at $R_1^- = 10$ (Ω) and $R_1^+ = 30$ (Ω). Assigning these values to R_1 gives:

$$I_{2,\min} = 4.7\ (A),$$
$$I_{2,\max} = 6.3\ (A). \tag{36}$$

An alternative approach to evaluate the extremal values of I_2 is to firstly find the function $I_2(R_1)$. Based on Theorem 2, one can find the function $I_2(R_1)$ by assigning 3 different values to R_1, measuring the corresponding current I_2, and solving the measurement equations (20) for β_0, β_1 and α_0. Table 1 shows the numerical values of the measurements for this example. Solving (20) for β_0, β_1 and α_0 and substituting these constants into (19) yields

$$I_2(R_1) = \frac{21.9 + 8R_1}{11.7 + R_1}, \tag{37}$$

which is plotted in Fig. 4. It can be verified from Fig. 4 that the extremal values of I_2 are as obtained in (36).

Exp. No.	R_1 (Ω)	I_2 (A)
1	7	4.2
2	18	5.6
3	32	6.4

Table 1. Numerical values of the measurements for Example 1

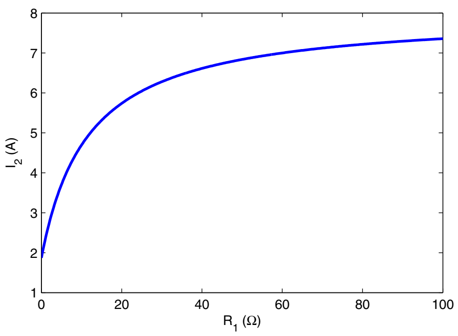

Fig. 4. $I_2(R_1)$ for Example 1

Example 2. In this example we consider the same circuit as in the Example 1. Suppose that the uncertain parameters R_1 and R_6 are varying in the rectangle,

$$\mathcal{R} = \{(R_1, R_6) \mid 5 \le R_1 \le 15,\ 2 \le R_6 \le 5\ (\Omega)\}, \tag{38}$$

with vertices:

$$A = (5, 2),\ B = (5, 5),$$
$$C = (15, 5),\ D = (15, 2),$$

and one is interested to evaluate the extremal values of the power level P_3, in the resistor $R_3 = 10$ (Ω), over the rectangle $\mathcal{R}$ in (38). The power level P_3 can be expressed in the terms of the uncertain parameters as

$$P_3(R_1, R_6) = R_3\ I_3^2(R_1, R_6), \tag{39}$$

but since, according to Remark 5, $I_3(R_1, R_6)$ is monotonic in R_1 and R_6, Theorem 5 is valid to evaluate the extremal values of P_3 at the vertices. Setting (R_1, R_6) to the values corresponding to vertices A, B, C, D, one gets:

$$P_{3,\min} = 49.4\ (W) \text{ at vertex B,}$$
$$P_{3,\max} = 150\ (W) \text{ at vertex D.} \tag{40}$$

Also, one can plot the function $P_3(R_1, R_6)$ (see Fig. 5) following Theorem 4 and by conducting 7 experiments. The rectangle $\mathcal{R}$, defined in (38), is also shown in Fig. 5. It can be seen that the extremal values of P_3 are the same as those obtained in (40).

Example 3. Consider the unknown hydraulic network shown in Fig. 6. Assuming that the flows are in the laminar state, the system can be described, by applying Kirchhoff's laws, as a set of linear equations

$$A(p)x = b(q), \tag{41}$$

where p denotes the vector of pipe resistances, q is the vector of inputs such as pump pressures, and x is the vector

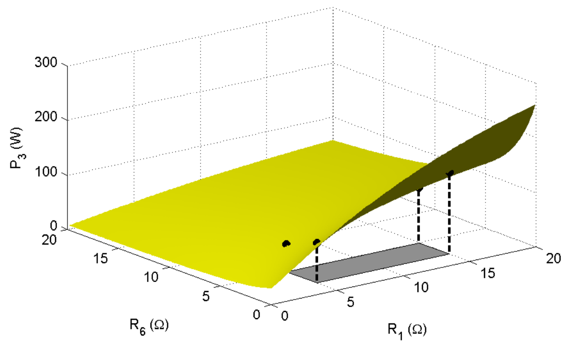

Fig. 5. $P_3(R_1, R_6)$ for Example 2

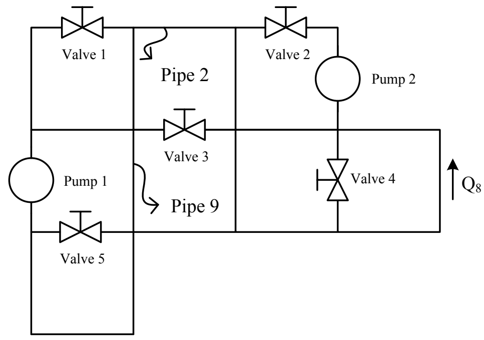

Fig. 6. An unknown hydraulic network

of unknown flow rates. A pipe resistance is related to the properties of the fluid flowing through it and its geometric dimensions by

$$R = \frac{8\mu L}{\pi r^4}, \tag{42}$$

where μ is the dynamic viscosity of the fluid, and L and r represent the length and radius of the pipe. It can be observed that each pipe resistance appears with a rank one dependency in $A(p)$. Suppose that the radii of pipes numbered 2 and 9 are varying in ranges described by

$$\mathcal{R} = \{(r_2, r_9) \mid 0.08 \le r_2 \le 0.14,\ 0.07 \le r_9 \le 0.10\ (m)\}, \tag{43}$$

where the vertices are labelled as:

$$A = (0.08, 0.07),\ B = (0.08, 0.10),$$
$$C = (0.14, 0.10),\ D = (0.14, 0.07).$$

It is of interest to evaluate the extremal values of the flow rate Q_8 over the rectangle $\mathcal{R}$ in (43). Similar to the previous example, since the assumptions in Theorem 5 hold, the extremal values of Q_8 occur at the vertices of the rectangle $\mathcal{R}$:

$$Q_{8,\min} = 0.045\ (m^3/s) \text{ at vertex A,}$$
$$Q_{8,\max} = 0.053\ (m^3/s) \text{ at vertex C.} \tag{44}$$

The function $Q_8(r_2, r_9)$ can be found by taking 7 measurements as explained in Theorem 4, and is depicted in Fig. 7. The rectangle $\mathcal{R}$, defined in (43), is also shown.

5. CONCLUSIONS

In this paper we described some important characteristics of parametrized solutions of a system of linear equations. If the uncertain parameters appear with rank one

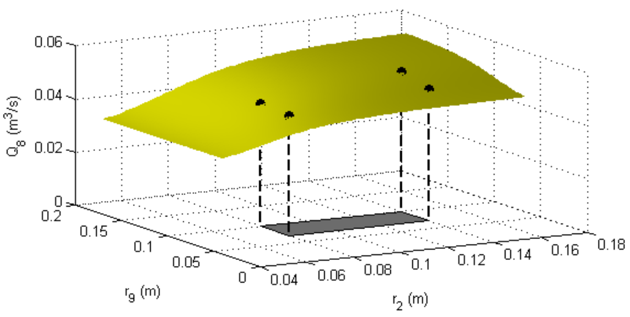

Fig. 7. $Q_8(r_2, r_9)$ for Example 3

dependency in the characteristic matrix of the system, which is usually the case in practical applications, then the parametrized solutions will be monotonic in these parameters. This fact is used to show that the extremal values of the parametrized solutions over a box in the parameter space occur at the vertices of the box. This result is explained through illustrative examples.

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