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# Delayed feedback stabilization of unstable equilibria

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**Abstract:** Analytical results of stabilizing of unstable equilibria of two- and three-dimensional linear systems are presented. The stabilization method is based on time-delayed feedback. Effective necessary and/or sufficient conditions for stabilizing the considered systems in terms of its parameters are obtained. The potential of delayed state and output feedback controls is shown. The theorems proved show that such a delayed feedback approach allows one to extend the possibilities available with static time-invariant state and output feedbacks for stabilizability of unstable linear time-invariant controllable systems. It is used two types of time-delayed feedback: conventional and by Pyragas' ones. A comparison of the stabilizability domains via both types of feedback is given.

Keywords: delayed feedback control, stabilization, equilibrium, asymptotic stability, linear controllable system.

## 1. INTRODUCTION

The control of dynamical systems is a classical object in engineering sciences. One of the main problems in control theory is stabilization of systems. The different questions concerning to stabilization problem have been studied in the past four decades (see the comprehensive bibliography in [Leonov et al., 2012] and in the surveys [Syrmos et al., 1997, Polyak et al., 2005]). The increasing interest to the stabilization problems is motivated by the needs of the practice of control and also the open problems formulated by many famous scholars (see [Zubov, 2001, Wonham, 1979, Bernstain, 1992, Brockett, 1999, Rosenthal et al., 1999]). One of the problems, which stimulated many publications was the Brockett problem on the stabilizability of time-invariant linear system by means of constructing a static time-varying output linear feedback. This problem was solved in many important cases in the works [Leonov, 2001, Moreau et al., 2004], where the algorithms of low-frequency and high-frequency stabilization were constructed. It was shown that in the case of two- and three-dimensional systems the introduction of time-varying output feedback enlarges its possibilities.

A question arises: *exist there other manners for stabilization of linear systems, which would enlarge the possibilities of stationary stabilization? Is it possible to stabilize a time-invariant linear system by introduction of time-delayed feedback with a constant feedback gain? What is the potential of stationary time-delayed feedback control for stabilization of linear systems?*

Another motivation for the stabilization of unstable linear systems by time-delayed feedback control occurs in the problems of stabilization of unstable periodic orbits and unstable equilibria in chaotic systems. The latter has been a field of extensive research during the last two decades (see surveys [Tian et al., 2005, Pyragas, 2006, 2012]). This interest is motivated by different applications in chemistry, biology, medicine, economics, engineering and physical sciences. A variety of control schemes have been developed to control periodic orbits and equilibria ([Ott et al., 1990, Baba et al., 2002, Hövel et al., 2005, Schöll et al., 2008, Ahlborn et al., 2005, Yanchuk et al., 2006, Hooton et al., 2011, 2013], see also surveys [Tian et al., 2005, Pyragas, 2006, 2012]).

In the work ([Pyragas, 1992]) K. Pyragas introduced a simple and efficient control scheme called a time-delayed feedback control (TDFC), which stabilizes unstable periodic orbits embedded in a chaotic attractor. This control method generates a continuous feedback such that a control force is proportional to the difference of the output signal (which is a function of the current system state) and the output signal delayed by some time in the past. Later, the use of (TDFC) was extended to the problem on stabilization of equilibrium points (see [Hövel et al., 2005, Yanchuk et al., 2006, Pyragas, 1995, Ahlborn et al., 2004, Dahms et al., 2007, Gjurchinovski et al., 2008, Huijberts et al., 2009]). Now the TDFC method is one of the most popular methods in chaos control research.

However, since a closed-system is a delayed differential equation, it is quite difficult to analyze its stability and, therefore, it is also difficult to give analytic and effective stabilization criteria [Pyragas, 2006, 2012, Just et al., 1997]. Even the linear stability analysis of such systems is quite complicated on account of the infinite number of Floquet exponents (in the case of controlled orbits) or infinite number of characteristic roots determined by transcendental equation (in the case of controlled equilibria).

Here it arises first of all a problem of stabilizability of unstable equilibria of linear dynamical systems by means of time-delayed feedback. This class of control systems is important for the stability analysis of nonlinear control systems in the neighborhood of an equilibrium point.

Further it will be given an answer to the question whether the two- and three-dimensional linear systems can be stabilized by means of a time-delayed output/state feedback. Note that for two-dimensional systems the stabilization problem was considered in the work [Huijberts et al., 2009] by using analytical results with numerical simulations and computer experiments. The method is based on the use of eigenvalue optimization approach. Below it will be presented the results concerning to the problem of stabilization of unstable equilibria of two- and three-dynamical controllable systems. An approach to the problem considered is far different from the one, used in [Huijberts et al., 2009]. It based on the method of $D$-decomposition of the space of system parameters using Rushe's theorem on zeros of analytical function and the investigation of behavior of the boundary of the stabilizability region when the controller parameters both feedback gain and delay-time, are changed. The advantage of this approach in contrast with that in [Huijberts et al., 2009] lies in the fact that the investigation of the problem considered is purely analytical, the proofs of the theorems are more simple, and much less mathematical tools are used. The results presented show that a delayed feedback approach allows one to extend the possibilities concerning the static time-invariant output feedback (without delay-time) for stabilizability of second- and third-order linear control. The results can be applied to the stabilization of unstable equilibria, embedded in a strange attractor of chaotic systems, and be used for a wide range of systems in physics, chemistry, engineering, technology, and the sciences, where the problem of stabilization of unstable fixed points occur.

## 2. PROBLEM STATEMENT

Consider a linear control system

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases} \tag{1}$$

where $x = x(t) \in \mathbb{R}^n$ is a state, $u = u(t)) \in \mathbb{R}^n$ is an input (control), $y = y(t) \in \mathbb{R}^n$ is an output, the matrices $A, B$ and $C$ are real constant matrices of dimensions $n \times n$, $n \times m$, $l \times n$, respectively. System (1) can be considered as linearized system of a nonlinear control system around equilibrium point $(x, u) = (0, 0)$.

Consider for system (1) the classical feedback stabilization problem:

*Find an appropriate stabilizing law under the assumption that the uncontrolled system is unstable.*

It is well-known that in the case $C = I$ (I is an identity $n \times n$-matrix) the solving of the above problem is given by Zubov's and Wonham's theorem on a pole assignment ([Zubov, 1966, Wonham, 1967]) (see also [Leonov et al., 2010]). In this case the stabilizing law is a constant state feedback $u = kx$.

The case $C \neq I$, $l < n$ is more far difficult. Different feedback strategies were considered in the literature (see references in [Leonov et al., 2012]).

A delay feedback will be used as a stabilization feedback. Consider two manners to introduce such feedback: an ordinary delayed feedback

$$u(t) = ky(t - \tau), \tag{2}$$

and Pyragas' delayed feedback

$$u(t) = k[y(t - \tau) - y(t)], \tag{3}$$

where $k \neq 0$ and $\tau > 0$ are running parameters.

System (1), closed by feedbacks (2) and (3), gives the following systems of delay differential equations:

$$\dot{x} = Ax + kBCy(t - \tau) \tag{4}$$

and

$$\dot{x} = Ax + kBC[y(t - \tau) - y(t)], \tag{5}$$

respectively.

**The Problem.** *It is required to find values of parameters $k \neq 0$ and $\tau > 0$ such that the equilibrium of system (4)/(5) is asymptotically stable.*

Note that in the case when $x \in \mathbb{R}$, $A$ and $B$ are numbers, $k = C = 1$ the question of asymptotic stability of differential equation (4) is considered in [El'sgolts et al., 1973].

Here the stabilization problem for two- and three-dimensional systems is considered.

## 3. FORMULATION OF RESULTS

### *3.1 Two-dimensional systems* ($n = 2$).

Suppose that system (1) can be reduced to the form

$$\begin{cases} \dot{x}_1 = x_2, \\ x_2 = -a_1x_1 - a_2x_2 - u, \\ y = c_1x_1 + c_2x_2, \end{cases} \tag{6}$$

where $a_1, a_2; c_1, c_2$ are real parameters. (The reduction of system (1) to the form (6) is possible, for example, if system (1) is controllable.)

Characteristic equations of closed systems (6), (2) and (6), (3) have the form

$$z^2 + a_2z + a_1 + ke^{-\tau z}(c_2z + c_1) = 0, \tag{7}$$

and

$$z^2 + (a_2 - kc_2)z + (a_1 - kc_1) + ke^{-\tau z}(c_2z + c_1) = 0, \quad z \in \mathbb{C}, \tag{8}$$

respectively.

The main problem can be reformulated in the following way: *To find values of parameters $k \neq 0$ and $\tau > 0$ such that the real parts of all roots of characteristic equation (7)/(8) is negative.*

Three cases are possible:

1) $c_1 \neq 0$, $c_2 = 0$, 2) $c_1 = 0$, $c_2 \neq 0$, 3) $c_1 \neq 0$, $c_2 \neq 0$.

Let us formulate stabilization theorems for two-dimensional system (6).

*Theorem 3.1.* Suppose that in system (6) $c_1 \neq 0$, $c_2 = 0$. Then for stabilizability of system (6) by feedback (2) it is necessary and sufficient that, at least, one of the conditions holds:

$$a_1 \leq 0, a_2 > 0 \quad \text{or} \quad a_1 > 0, a_2 > -\sqrt{2a_1}.$$

*Theorem 3.2.* Suppose that in system (6) $c_1 \neq 0$, $c_2 = 0$. Then system (6) is stabilizable by feedback (3) if and only if $a_1 > 0$.

*Theorem 3.3.* Suppose that in system (6) $c_1 = 0, c_2 \neq 0$. Then for stabilizability of system (6) by feedback (2) it is necessary and sufficient that $a_1 > 0$.

*Theorem 3.4.* Suppose that in system (6) $c_1 = 0, c_2 \neq 0$. Then system (6) is stabilizable by feedback (3) if and only if, at least, one of the conditions holds:

$$a_1 > 0, a_2 > 0 \quad \text{or} \quad a_2 \leq 0, a_1 > \pi^2\sigma^2 a_2^2/16,$$

where $\sigma = \min_{\alpha\in[0,2\pi]}(\cos\alpha + (\sin\alpha)/\alpha)$ $(\sigma \approx -1,0419)$

*Theorem 3.5.* Suppose that in system (6) $c_1 \neq 0, c_2 \neq 0$. Then for stabilizability of system (6) by feedback (2) it is necessary and sufficient that, at least, one of the conditions holds

$$\text{a)} \quad c_1c_2 > 0, \quad \text{b)} \quad c_1c_2 < 0, \quad c_1a_2/c_2 < a_1 \leq 0,$$

$$\text{c)} \quad c_1c_2 < 0, \quad a_1 > 0, \quad c_1a_2/c_2 < \sqrt{a_1(a_1 + 2(c_1/c_2)^2)}$$

*Theorem 3.6.* Suppose that in system (6) $c_1 \neq 0, c_2 \neq 0$. Then if, at least, one of the conditions holds

$$\text{a)} \quad c_1c_2 > 0, a_1 > 0, \quad \text{b)} \quad c_1c_2 < 0, \quad a_1 > 0, \quad a_2 > c_1/c_2$$

then system (6) is stabilized by feedback (3).

**Remark 1.** In the case $c_1c_2 > 0$ the inequality $a_1 > 0$ is also necessary for stabilization of system (6) by feedback (3).

**Remark 2.** Theorems 3.1 – 3.6 illustrate the fact that an introduction of a time-delay in feedback controller extends (except for Theorem 3.4) the possibility of stationary stabilization realized by an ordinary time-invariant output feedback $u = ky$ (without delay).

**Remark 3.** The results concerning Cases 1), 2), and 3) are presented below in Table 1.

**Remark 4.** For system (6), where $a_1 = 1$, $a_2 = -d$, $c_1 = 0$, and $c_2 = 1$, the stabilization region $0 \leq d < 4/\pi \mid \sigma \mid$, obtained in the case $c_1 = 0$, $c_2 \neq 0$ well agrees with the stabilization interval $0 < d < 1,216$, obtained in [Pyragas, 1995], using computer experiment. (Here $\sigma \approx 1,0419$.)

**Remark 5.** In the case $c_1 = 0$, $c_2 \neq 0$ the condition $a_1 > \pi^2\sigma^2a_2^2/16$ slightly improves the corresponding condition (where $\sigma^2 = 1$) of the work [Huijberts et al., 2009], obtained by analytical-numerical methods.

### *3.2 Three-dimensional systems* ($n = 3$).

Suppose that system (1) can be reduced to the form

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = -a_1x_1 - a_2x_2 - a_3x_3 - u, \\ y = c_1x_1 + c_2x_2 + c_3x_3, \end{cases} \tag{9}$$

where $a_i, c_i (i = 1, 2, 3)$ are real parameters. (The reduction to the form (9) is possible, for example, if system (1) is controllable.)

Here the main stabilization problem is also reduced to the problem on the analysis of roots disposition of characteristic equations:

$$z^3 + a_3z^2 + a_2z + a_1 + ke^{-\tau z}(c_3z^2 + c_2z + c_1) = 0, \tag{10}$$

$$\begin{aligned} z^3 + (a_3 - kc_3)z^2 + (a_2 - kc_2)z + (a_1 - kc_1) + \\ + ke^{-\tau z}(c_3z^2 + c_2z + c_1) = 0, \end{aligned} \tag{11}$$

in the left-hand semiplane of complex variable $z$.

Consider the following typical cases:

1) $c_1 \neq 0, c_2 = c_3 = 0$, 2) $c_2 \neq 0, c_1 = c_3 = 0$,
3) $c_3 \neq 0, c_1 = c_2 = 0$.

The results concerning these cases are presented below in Table 2.

## 4. IDEA OF PROOFS OF STABILIZATION THEOREMS

Denote by $F_1, F_2, F_3$, and $F_4$ the quasipolynomials in the left-hands of equations (7), (8), (10), and (11), respectively. The proofs of stabilization theorems are based on the method of $D$-decomposition [Neymark , 1978] of the space of the quasipolynomials $F_i (i = \overline{1,4})$.

Decompose the space of the coefficients $\{a_i\}$ of quasipolynomials $F_j$ $(j = \overline{1,4})$ into regions by lines (in the case $n = 2$) and surfaces (in the case $n = 3$), the points of which correspond to quasipolynomials having, at least, one zero on the imaginary axis. Such a decomposition is called a $D$-decomposition. Evidently, to the points of each domain of such $D$-decomposition correspond quasipolynomials with the same number of zeros (with regard to their multiplicity) with positive real part. Hence to each domain $D^p$ $D$-decomposition may assign the number $p$ being the number of zeros with positive real part of the quasipolynomial $F_j$, defined by the point of this domain. The domains of this decomposition may contain domains $D^0$ (if they exist), to which correspond quasipolynomials having no any root with positive real part. These domains are regions of asymptotical stability for systems, corresponding to the considered quasipolynomial $F_j$.

Since the ideas of proving all theorems on stabilization concerning to all considered cases are similar to each other, only the case of two-dimensional system (6) stabilized by ordinary feedback (2) for $c_1 \neq 0, c_2 = 0$ is considered below.

The boundary $N_{k,\tau}$ of $D$-decomposition of the plane $\mathbb{R}_a^2 = \{(a_1, a_2)\}$ of the quasipolynomials $F_1(z; a)$ from (7) ($c_1 \neq 0, c_2 = 0$) consists of the straight line $K = \{a_1 = -kc_1\}$ and the curve $L_{k,\tau}$, defined by the following parametric equations

$$\begin{cases} a_1 = y^2 - kc_1 \cos\tau y, \\ a_2 = kc_1 \dfrac{\sin\tau y}{y}, \quad y \neq 0. \end{cases} \tag{12}$$

Here $k \neq 0$ and $\tau > 0$ are varied parameters. For the given values of $k$ and $\tau$ the boundary $N_{k,\tau}$ decomposes the plane $\mathbb{R}_a^2$ into the regions $D_{k,\tau}^p$, to the points of which correspond quasipolynomials (7) with the same number $p$ of roots in the right half-plane $Rez > 0$ $(z \in \mathbb{C})$. From the obtained $D$–decomposition the regions $D_{k,\tau}^0$ $(p = 0)$ of asymptotic stability of closed system (6), (2) are to be extracted. By various values of $k \neq 0$ and $\tau > 0$ the join $S$ of all the regions $D^0_{k,\tau}$ will be the stabilization domain of system (6): $S = \bigcup_{k,\tau} D^0_{k,\tau}$. To extract a domain $D^0_{k,\tau}$, it is necessary that, at least, one of its points corresponds to a quasipolynomial, all the zeros of which have negative real parts. For this purpose Rushe's theorem on zeros of analytic function is applied.

It is required to investigate the possible forms of the boundary $N_{k,\tau}$ of the $D$ - decomposition for various $k \neq 0$ and $\tau > 0$. Consider two cases:

$$1)\ |kc_1|\tau^2 \leq 2 \quad \text{and} \quad 2)\ |kc_1|\tau^2 > 2.$$

It can be obtained that in the first case the boundary $N_{k,\tau}$ has no self-intersections and in the second case it has self-intersections if $kc_1 < 0$, and, therefore, if $kc_1 > 0$ for sufficiently large $\tau$. In the last case the boundary $N_{k,\tau}$ contains loops. Consider a change of number $p$ of roots with $Rez > 0$ in crossing the boundary $N_{k,\tau}$ of $D$ - decomposition. The latter is defined by the sign of the derivative $\partial x/\partial a_1$ or $\partial x/\partial a_2$ of real part of the root $z = x + iy$, computed on the boundary $N_{k,\tau}$. One has

$$\partial x/\partial a_1(z; a_1, a_2) = -\xi/(\xi^2 + \eta^2), \qquad (13)$$

where

$$\begin{aligned} \xi &= 2x + a_2 - kc_1\tau \exp(-\tau x)\cos\tau y, \\ \eta &= 2y + kc_1\tau \exp(-\tau x)\sin\tau y. \end{aligned} \qquad (14)$$

By (13), (14) it can be found a number of roots $p$ with $Rez > 0$ in each region of $D$ - decomposition. It is established that $p = 4$ for $kc_1 < 0$ inside single loops. If the loops superpose on one another, then the number $p$ in the region of their intersection increases by $2n$, where $n$ is a number of superpositions of single loops. In the case $kc_1 > 0$ the reasoning is the same. It is easy established that the stabilization domain $S$ includes region $\{a_2 > 0\}$.

Suppose now that $a_2 \leq 0$. Then the feedback stabilization without delay is impossible.

Denote

$$G^0_{k,\tau} := D^0_{k,\tau} \cap \{a_2 \leq 0\}, \quad G := \cup_{k,\tau:|kc_1|\tau^2=2,kc_1<0} G^0_{k,\tau} \qquad (15)$$

It can be proved that the parametric equations of the "lower" boundary of the region $G$ in (15) are the following ($k$ is a prameter):

$$a_1 = -kc_1, \quad a_2 = -\sqrt{-2kc_1}.$$

This implies that a region of stabilization is the region $G = \{-\sqrt{2a_1} \leq a_2 \leq 0, \quad a_1 > 0\}$. This completes the proof of sufficiency of Theorem 3.1.

For the proof of necessity it is established that the region $G$ coincides with the region $S \cap \{a_2 \leq 0\} = \cup_{k,\tau} G^0_{k,\tau}$, i.e. the region $G$ is maximal. This completes the proof of Theorem 3.1.

## 5. CONCLUSION

In the work a stabilization problem of unstable equilibria of two- and three-dimensional dynamical systems by delay feedback is considered. For incorporating a feedback into the system, two approach are used: ordinary and due to Pyragas. Necessary and/or sufficient conditions of stabilizing the systems considered are represented. The possibility of stabilization by delay feedback is shown. For two-dimensional systems, sufficiently complete classification of all possible linear feedbacks is given which can be applied to unstable equilibria stabilization. For three-dimensional systems, it is considered the most often occurring in practice cases when the output of system is a one of coordinates of system state. The comparative analysis shows that the introduction of delay in feedback, in large, extends the opportunities of usual (without delay) stationary stabilization.

The results of this work may be applied to stabilizing linear control systems and also to stabilizing unstable equilibria, embedded in strange attractors of chaotic systems, in particular, of the Lorenz and Rössler systems.

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**Two-dimensional Systems**

<table>
<tr><td colspan="4">Stabilization by Time-Delayed Feedback</td></tr>
<tr><td colspan="3">Time-Delayed Output Feedback</td><td>Time- Delayed<br>State Feedback</td></tr>
<tr><td>Case</td><td>Usual<br>$u(t) = ky(t-\tau)$</td><td>K. Pyragas<br>$u(t) = k[y(t-\tau) - y(t)]$</td><td>$u(t) =$<br>$= K[x(t-\tau) - x(t)],$<br>$x \in \mathbb{R}^2,\ K \in \mathbb{R}^{2\times 2}$</td></tr>
<tr><td>$c_1 \neq 0$<br>$c_2 = 0$</td><td>$a_1 \leq 0, \quad a_2 > 0$<br>or<br>$a_1 > 0, \quad a_2 > -\sqrt{2a_1}$</td><td>$a_1 > 0$</td><td rowspan="3">$a_1 > 0$</td></tr>
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<tr><td>$c_1 \neq 0$<br>$c_2 \neq 0$</td><td>$c_1c_2 > 0$<br>or<br>$c_1c_2 < 0, \quad \frac{c_1a_2}{c_2} < a_1 \leq 0$<br>or<br>$c_1c_2 < 0, \quad a_1 > 0,$<br>$\frac{c_1a_2}{c_2} < \sqrt{a_1\left[a_1 + 2\left(\frac{c_1}{c_2}\right)^2\right]}$</td><td>$c_1c_2 > 0, \quad a_1 > 0$<br>or<br>$c_1c_2 < 0, \quad a_1 > 0$<br>$a_2 > c_1/c_2$</td></tr>
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**Three-Dimensional Systems**

| Stabilization by Time-Delayed Feedback | | | |
|---|---|---|---|
| Time-Delayed Output Feedback | | | Time- Delayed<br>State Feedback |
| Case | Usual<br>$u(t) = ky(t-\tau)$ | K. Pyragas<br>$u(t) = k[y(t-\tau) - y(t)]$ | $u(t) =$<br>$= K[x(t-\tau) - x(t)]$,<br>$x \in \mathbb{R}^3,\ K \in \mathbb{R}^{3\times 3}$ |
| $c_1 \neq 0$,<br>$c_2 = 0$,<br>$c_3 = 0$,<br>$(c_1 := 1)$ | $a_2 > 0,\ \ a_3 > 0$<br>or<br>$a_2 < 0,\ \ a_3 > 0,\ a_2^2 < 2a_1a_3$ | $a_1 > 0,\ \ a_3 > 0,$ | $a_1 > 0,\ \ a_3 > 0$ |
| $c_2 \neq 0$,<br>$c_1 = 0$,<br>$c_3 = 0$,<br>$(c_2 := 1)$ | $a_1 > 0,\ \ a_3 > 0$ | $0 < a_1 < \frac{\pi^2 a_2 a_3}{\pi^2 - 8} +$<br>$+ \frac{8\pi a_2\sqrt{a_2}}{(\pi^2-8)\sqrt{\pi^2-8}}$<br>$a_2 > 0,\ \ a_3 > 0$<br>or<br>$a_1 > \frac{\pi a_2 a_3}{\pi - 4} +$<br>$+ \frac{4\sqrt{\pi} a_2 \sqrt{-a_2}}{(\pi-4)\sqrt{4-\pi}},$<br>$a_2 < 0,\ \ a_3 > 0,$ | |

| | | | |
|---|---|---|---|
| $c_3 \neq 0,$<br>$c_1 = 0,$<br>$c_2 = 0$<br>$(c_3 := 1)$ | $a_1 > 0, \quad a_2 > 0$<br>or<br>$a_1 > \frac{\pi + 3\sqrt{3}}{\pi - 3\sqrt{3}} a_2 \times$<br>$\times \left( a_3 + 6\sqrt{\frac{a_2}{\pi^2 - 27}} \right),$<br>$a_2 < 0, \quad a_3 > 0$<br>or<br>$a_1 > \frac{\pi + 3\sqrt{3}}{\pi - 3\sqrt{3}} a_2 \times$<br>$\times \left( a_3 + 6\sqrt{\frac{a_2}{\pi^2 - 27}} \right),$<br>$a_2 < \frac{\pi^2 - 27}{(6 - \pi\sqrt{3})^2} a_3^2,$<br>$a_3 < 0$ | $0 < a_1 < a_2 \left( a_3 + \frac{4\sqrt{a_2}}{(-\sigma)\pi} \right),$<br>$a_2 > 0, \quad a_3 > 0$<br>$\left( \sigma = \min_{\alpha \in [0, 2\pi]} \left( \cos\alpha + \frac{\sin\alpha}{\alpha} \right) \right)$<br>or<br>$a_2 > \frac{\pi + 12(2 - \sqrt{3})}{\pi} \frac{a_1}{a_3} -$<br>$- \frac{12\sqrt{a_1 a_3}}{\sqrt{\pi[\pi + 12(2 - \sqrt{3})]}},$<br>$a_1 > 0, \quad a_3 > 0$<br>or<br>$a_1 > \frac{\pi + 3\sqrt{3}}{\pi - 3\sqrt{3}} a_2 \times$<br>$\times \left( a_3 - 6\sqrt{\frac{a_2}{\pi^2 - 27}} \right),$<br>$\frac{\pi^2 - 27}{(6 + \pi\sqrt{3})^2} a_3^2 < a_2 < 0$ | |