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An inverse optimality argument to improve robustness in constrained control

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Abstract: This paper presents a new approach to synthesize a nominal constrained model-based predictive control law which can avoid the ”non-robustness” curse in the presence of model uncertainty. This approach builds on inverse optimality arguments and shows that the unconstrained control law and the choice of nominal system are design elements to synthesize a piecewise affine regulator that guarantees the stability of the polytopic system in a non-degenerate region containing the origin in its interior. Some advantages of this approach are presented and a series of related problems are discussed.

1. INTRODUCTION

Robust constrained control design remains a challenging problem, despite having received significant attention over the last two decades, especially in the so-called robust constrained model predictive control (RCMPC). Consequently, RCMPC is still a very active research direction on topics related to the complexity of solution. It has to be mentioned that there are well established methodologies in order to solve this problem. The principle proposed in Kothare et al. [1996] and subsequently improved in a series of papers Falugi et al. [2010], Cuzzola et al. [2002], Ding et al. [2004] leads to a robust constrained MPC problem in the presence of uncertainties represented in two different ways: polytopic uncertainty and structured feedback uncertainty. The solution is obtained using linear matrix inequalities (LMI). The resulting control law is linear time-varying and guarantees the robust stability of uncertain systems, but the state feedback controller synthesis is solved on-line, so it implies heavy online computation. As a consequence, this difficulty gives rise to a problem in implementing the regulator. Another less attractive approach from the computational point of view is the min-max formulation as proposed early in Campo and Morari [1987] based on the minimization of the worst-case tracking error for a linear nominal model affected by additive disturbance. It was known that this approach in its basic formulation has restricted stability properties Bemporad and Morari [1999]. The closed loop min-max optimal control problem Scokaert and Mayne [1998], Kerrigan and Maciejowski [2003] was subsequently proposed to cope with fundamental shortcomings via the dynamic programming principles. These enhancements require an even more complicated on-line computation. Recently, the tube model predictive control (TMPC) approach in Raković et al. [2005] and Mayne et al. [2005] considers the minimal positively invariant set (or disturbance invariant set) to confine the state to it along the trajectory of reference model. This approach is very attractive for linear time-invariant (LTI) discrete system in the presence of bounded additive disturbances. Its complexity grows however for model uncertainties as long as a parameterization of the tube is needed Raković et al. [2012].

In the present paper we want to answer a related problem to these RCMPC strategies: *How to design a nominal MPC for a linear system affected by polytopic uncertainties which ensures robust asymptotic stability in a region containing the origin?* The idea is to avoid unwieldy control law synthesis by exploiting inverse optimality. This will point to two important aspects: the importance of the nominal model used for the prediction and secondly the systematic procedure of tuning the nominal MPC parameters.

Inverse optimality was for the first time proposed in the continuous case in the classical works of R. E. Kalman in the middle of the 1960s. Its applications were limited, being seen as theoretical results. There are however studies pointing to the use of inverse optimality in connection with MPC. In Rowe and Maciejowski [2000b] for example, the authors used inverse problem as a useful tool to replace the terminal state constraint by choosing an appropriate weighting matrix pair in the cost function while still guaranteeing the stability of the system. Following this idea, the same authors built a bridge between H_∞ loop-shaping procedure and MPC in Rowe and Maciejowski [2000a] so that the same robust stability was guaranteed. In Lovaas et al. [2006] the authors proposed a link between the minimax optimal control and MPC for a LTI model perturbed by bounded additive disturbance through inverse minimax optimality, which required an online optimization. Some existence conditions were presented and solved through an online LMI problem.

Through the inverse optimality, we would like to make an attempt to establish a bridge between RCMPC and nominal constrained MPC, exploiting among other properties its associated piecewise affine solution as detailed in the next section.

Paper structure: Section II presents the main idea of our approach with details of the implementation for each step. Section III will be reserved to present some propositions and discussions around the resulting controller. Finally, section IV proposes some open problems and connections.

2. METHODOLOGY OF DESIGN

To begin, we need to define the problem by considering polytopic uncertain systems of the following form:

$$\begin{aligned} x(k+1) &= A_{\triangle}x(k) + B_{\triangle}u(k), \\ y(k) &= Cx(k), \\ \left[A_{\triangle}\ \ B_{\triangle}\right] &\in \Omega, \end{aligned} \tag{1}$$

where $u(k) \in \mathbb{R}^{n_u}$ is the control input, $x(k) \in \mathbb{R}^{n_x}$ is the state variable and $y(k) \in \mathbb{R}^{n_y}$ is the output signal. The set Ω defines a polytopic set in the parameters' space:

$$\Omega = \mathbf{Co}\{[A_1\ \ B_1], [A_2\ \ B_2], \cdots, [A_L\ \ B_L]\}, \tag{2}$$

where $\mathbf{Co}$ denotes the convex hull, if $[A\ \ B] \in \Omega$ there exists non negative $\lambda_1, ...\lambda_L$ satisfying $[A\ \ B] = \sum_{i=1}^{L} \lambda_i [A_i\ \ B_i]$ and $\sum_{i=1}^{L} \lambda_i = 1$. Our idea can be visualized by the block diagram shown in Figure 1 and can be summarized as follows:

Phase I: Consider the polytopic model $[A_\Delta\ \ B_\Delta]$ and a pair of weighting terms (Q_0, R_0) to obtain an unconstrained linear feedback $u_k = K_{rob}x_k$ with robustly stabilizing properties via RCMPC based on LMIs.

Phase II: Select a nominal model $\left[\bar{A}\ \ \bar{B}\right] \in \Omega$ - a series of conditions for this selection will be provided in Section III.

Phase III: Use $\left[\bar{A}\ \ \bar{B}\right]$ and K_{rob} to solve an inverse optimality problem. The solution will be a weighting pair (Q, R).

Phase IV: Use $\left[\bar{A}\ \ \bar{B}\right]$ and the weighting terms (Q, R) to solve a nominal MPC problem and obtain a piecewise affine control law.

Phase V: Analyze the robustness of the obtained solution.

The dotted connections in Figure 1 represent the steps for which only partial constructive solutions exist. We will detail the mathematical description of the steps denoted by the full arrows in Figure 1 in the next sections.

2.1 Linear state feedback control for polytopic systems based on LMI

Starting from the uncertain system description in (1), with its uncertainty described by (2), the first problem will be the design of unconstrained robust stabilizing linear feedback control law. We recall here a classical approach in this direction, Kothare et al. [1996] address this objective by the optimization of the quadratic criterion (3) and guarantees the stability of uncertain systems. Furthermore, the constraints can be specified leading to an adjustment of the robust stabilizing control law with respect to the component-wise limitations (4).

$$\min_{u(k+i|k),\ i\geq 0}\ \ \max_{[A(k+i)\,B(k+i)]\in\Omega} J_\infty(k)$$

$$J_\infty(k) = \sum_{i=0}^{\infty}[x^T(k+i|k)Q_0x(k+i|k) + u^T(k+i|k)R_0u(k+i|k)], \tag{3}$$

$$\begin{aligned} \|u_l(k+i|k)\|_2 &\leq u_{l,max}, \quad l = \overline{1..n_u}, \\ \|y_r(k+i|k)\|_2 &\leq y_{r,max}, \quad r = \overline{1..n_y}, \quad i \geqslant 0, \end{aligned} \tag{4}$$

where $u(k+i|k) \in \mathbb{R}^{n_u}, x(k+i|k) \in \mathbb{R}^{n_x}, y(k+i|k) \in \mathbb{R}^{n_y}$ represent, respectively, the values at time $k+i$ of control, state variables and measured output as predicted at time k. $\|.\|_2$ presents the Euclidean squared norm. $Q_0 \succeq 0, R_0 \succ 0$ are given symmetric matrices.

According to Kothare et al. [1996], this problem can be parameterized by the current state and solved using an on-line LMI approach leading to a linear feedback control law, but due to the complexity of the on-line computation, we do not pursue this direction and only retain the off-line LMI approach. Practically, the control law is obtained by solving the following standard LMI problem:

$$\min_{\gamma,Z,Y} -\operatorname{logdet}(Z), \tag{5}$$

subject to

$$\begin{gathered} \begin{bmatrix} Z & ZA_j^T + Y^TB_j^T & ZQ_0^{1/2} & Y^TR_0^{1/2} \\ A_jZ + B_jY & Z & 0 & 0 \\ Q_0^{1/2}Z & 0 & \gamma I & 0 \\ R_0^{1/2}Y & 0 & 0 & \gamma I \end{bmatrix} \succeq 0 \\ \begin{bmatrix} u_{l,max}^2 I & Y_l \\ Y_l^T & Z \end{bmatrix} \succeq 0 \\ \begin{bmatrix} Z & (A_jZ + B_jY)^TC_r^T \\ C_r(A_jZ + B_jY) & y_{r,max}^2 I \end{bmatrix} \succeq 0 \\ j = \overline{1...L},\ l = \overline{1...n_u},\ Y_l = Y(l,\cdot),\ r = \overline{1...n_y},\ C_r = C(r,\cdot). \end{gathered} \tag{6}$$

The state feedback controller gain is obtained explicitly as:

$$K_{rob} = YZ^{-1}. \tag{7}$$

2.2 Inverse problem for linear discrete-time system

The inverse optimality was presented by Kalman in Kalman [1964] for SISO continuous systems. Some later works focused on the continuous case (a synthesis is presented in Kong et al. [2012]), but our interest is in LTI discrete systems. In Anderson and Moore [1971] (section 5.6), the authors proposed some comments related to inverse optimal control in the discrete case, in Larin [2003] and Ostertag [2011] (section 6.3.7) the authors proposed also algorithms to solve the inverse optimality problem in both the continuous and the discrete cases, but omitted the specification of the necessary and sufficient conditions for the existence of solution. These conditions will be stated in the next section.

Consider the following nominal discrete system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned} \tag{8}$$

where $(A,\ B)$ is controllable. A given control law $u = -Kx$ satisfies the stability of the closed-loop system. We would like to find a symmetric matrix pair $(Q,\ R)$ $(Q \succeq 0, R \succ 0)$ such that the given control minimizes the following quadratic criterion:

$$J_{Q,R} = \sum_{k=0}^{\infty}(x(k)^TQx(k) + u(k)^TRu(k)). \tag{9}$$

Now we review an LMI formulation Larin [2003] to solve the inverse optimality via a LMI standard problem:

$$\min_{\lambda,S,Y,R,P} \lambda,$$

s.t:

$$\begin{gathered} P \succeq 0 \\ A^TPA - P - K^TRK - K^TB^TPBK \preceq 0 \\ \begin{bmatrix} Y & S \\ S^T & I \end{bmatrix} \succeq 0 \\ Y \preceq \lambda I \\ S = RK + B^TPBK - B^TPA. \end{gathered} \tag{10}$$

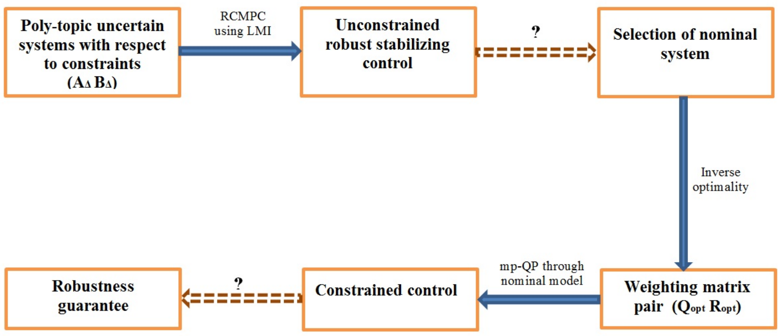

Fig. 1. Block diagram of the proposed control law synthesis

The matrices Q,R,P stem from the Riccati equation:

$$A^T PA - P - K^T RK - K^T B^T PBK + Q = 0.$$

In fact, the resolution of (10) amounts to an approximation of the solution instead of the exact one. The Riccati equation is not solved directly but the error between the left and the right hand side is minimized. The second LMI in (10) relies on the assumption that $Q \succeq 0$. From the solution of Riccati equation:

$$K = (B^T PB + R)^{-1} B^T PA,$$

it follows that S has to be zero, so this LMI problem tries to minimize λ in order to ensure that S is as close as possible to zero by the relationship: $\lambda I \succeq Y \succeq SS^T \succeq 0$.

2.3 Multi-parametric quadratic programming

Consider the nominal system (8), the objective is to find a stabilizing control law by:

$$\begin{aligned} &\min_{\mathbf{u}} J(x(k),\mathbf{u}) = \min_{\mathbf{u}} x^T(k+N|k)Px(k+N|k) + \\ &\sum_{i=0}^{N-1} x^T(k+i|k)Qx(k+i|k) + u^T(k+i|k)Ru(k+i|k), \end{aligned} \tag{11}$$

subject to:

$$\begin{aligned} u_{min} \leq u(k+i|k) \leq u_{max}, \\ y_{min} \leq y(k+i|k) \leq y_{max}, \\ x(k+N|k) \in \mathbb{X}_f, \end{aligned} \tag{12}$$

where $\mathbb{X}_f$ is the terminal constraint set. It is supposed that (A,B) is stabilizable and $(Q^{1/2},A)$ is detectable. The cost function minimization (11) can be easily rewritten as follow:

$$\begin{aligned} \min_{\mathbf{u}} J(x(k),\mathbf{u}) = \min_{\mathbf{u}} \mathbf{u}^T H\mathbf{u} + x(k)^T Yx(k) + x^T(k)F\mathbf{u}, \\ \mathbf{u} = [u^T(k) \quad \dots \quad u^T(k+N-1|k)]^T. \end{aligned} \tag{13}$$

The solution to the above problem through mp-QP is a piecewise affine function $\mathbf{u}^*$ defined over a state space polyhedral partition $\mathscr{X}$ (a set of polytopes) Bemporad et al. [2002]:

$$\begin{aligned} \mathscr{X} = \bigcup_{i=1}^{N_p} \mathscr{X}_i, \\ u_{pwa}(x) = \mathbf{u}^*(1:n_u) = F_i x + G_i, \quad \text{for} \quad x \in \mathscr{X}_i. \end{aligned} \tag{14}$$

Remark: Note that the constraint set (12) needs to be symmetric $u_{max} = -u_{min}, \; y_{max} = -y_{min}$ for the adaptation to constraints (4).

3. DISCUSSION AND RELATED PROBLEMS

Along with the discussions and developments of this section, all the numerical results are carried out with:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0.9 & 0.5 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 1 \\ 3.8 & 0.5 \end{bmatrix} \quad B_1 = B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad C = [1 \; 0].$$

The nominal system is chosen: $A = \begin{bmatrix} 1 & 1 \\ 1.48 & 0.5 \end{bmatrix}$. The weighting matrices $Q_0 = I_2, \;\; R_0 = 1$ and the prediction horizon $N = 5$. Finally, the constraints are chosen as: $y_{max} = -y_{min} = 0.5, \; u_{max} = -u_{min} = 0.2$.

3.1 Existence condition

Through the methodology, we see that a solution exists if and only if the existence conditions of inverse optimality are satisfied. The algorithm for inverse optimality computation was presented above (the detail is available in Larin [2003]) as an LMI problem, but the selection of a nominal model was not discussed. As mentioned in Section II, this selection allows a certain degree of freedom but it is still open in the present paper. We state now the existence conditions (the so-called Kalman conditions) for the discrete case (the continuous counterpart was presented in Kalman [1964]).

Proposition 1: Consider a nominal system (8) *and a stabilizing state feedback controller $u_k = -Kx_k$, the necessary and sufficient conditions for the existence of a matrix $Q = Q^T \succeq 0$ such that $u_k = -Kx_k$ is the optimal solution to the minimization of the objective function* (9) *as $J_{Q,1}$ are:*

i) $K(A-BK)^{-1}B > 0,$

ii) $|1+K(e^{j\omega}I-A)^{-1}B| > \dfrac{1}{(1+K(A-BK)^{-1}B)^{1/2}},$

iii) $(1+K(zI-A)^{-1}B)(1+K(A-BK)^{-1}B)(1+K(z^{-1}I-A)^{-1}B) = 1+B^T(z^{-1}I-A^T)^{-1}Q(zI-A)^{-1}B,$

here z is the forward shift operator.

Proof:

$\rightarrow$ K is the optimal control law. We have to prove all three conditions. Indeed, from the optimality of K, the Riccati equation needs to be satisfied:

$$A^T PA - P - K^T RK - K^T B^T PBK + Q = 0, \tag{15}$$

$$K = (B^T PB + R)^{-1} B^T PA. \tag{16}$$

From (16) we can find after some algebraic transformations that $K(A-BK)^{-1}B = B^T PB > 0$ (recall here that $R = 1$), as P is positive definite, this corresponds to condition i). On the other hand, (15) is equivalent to:

$$\begin{aligned} (z^{-1}I - A^T)P(zI-A) + (z^{-1}I - A^T)PA + A^T P(zI-A) + \\ A^T PB(B^T PB + R)^{-1} B^T PA = Q. \end{aligned}$$

By pre-multiplying and post-multiplying respectively the above equation with $B^T(z^{-1}I - A^T)^{-1}$ and $(zI-A)^{-1}B$, we obtain:

$$(I+B^T(z^{-1}I-A^T)^{-1}K^T)(R+B^TPB)(I+K(zI-A)^{-1}B) = R+B^T(z^{-1}I-A^T)^{-1}Q(zI-A)^{-1}B.$$

Through condition i) and the above equation, condition iii) is proved with the assignment $R = 1$. Moreover, with $z = e^{j\omega}$ the second condition is proved.
$\leftarrow$ From three conditions presented above, we have to prove that the control law K is optimal. Due to the stability of the closed loop dynamics, with respect to a positive semi-definite matrix Q, there exists a unique positive definite matrix P_{opt} and the associated optimal control law K_{opt} satisfying the Riccati equation. K_{opt} satisfies then the three properties above by the necessary condition.

For an ease of representation, let's impose:
$\varepsilon = 1+K(A-BK)^{-1}B$,
$\varepsilon_{opt} = 1+K_{opt}(A-BK_{opt})^{-1}B$.
Invoking Lemma 2 of Kalman [1964], one can find a controllable pair (A,B) in the following basic form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ . & . & . & . & . \\ . & . & . & . & \\ 0 & . & . & 0 & 1 \\ -\alpha_1 & . & . & -\alpha_{n-1} & -\alpha_n \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ . \\ . \\ 0 \\ 1 \end{bmatrix}.$$

We can calculate

$$1+K(zI-A)^{-1}B = 1+\frac{k_n z^{n-1}+k_{n-1}z^{n-2}+\dots+k_1}{z^n+\alpha_n z^{n-1}+\dots+\alpha_1}, \quad (17)$$

similarly,

$$1+K_{opt}(zI-A)^{-1}B = 1+\frac{k_n^* z^{n-1}+k_{n-1}^* z^{n-2}+\dots+k_1^*}{z^n+\alpha_n z^{n-1}+\dots+\alpha_1}, \quad (18)$$

where $K = [k_1 \dots k_n]^T$ and $K_{opt} = [k_1^* \dots k_n^*]^T$.
Suppose that $\sigma(z)$ and $\sigma^*(z)$ are respectively the numerator of the right hand side in (17) and (18). From the third condition we obtain:

$$\varepsilon\sigma(z)\sigma(z^{-1}) = \varepsilon_{opt}\sigma^*(z)\sigma^*(z^{-1}).$$

One can see that $\sigma(z)$ has n zeros $z_1, z_2, \dots, z_n$. Then,

$$\begin{aligned} \varepsilon_{opt}\sigma^*(z)\sigma^*(z^{-1}) &= \varepsilon\sigma(z)\sigma(z^{-1}) \\ &= \varepsilon\prod_{i=1}^{n}(z-z_i)\prod_{i=1}^{n}(z^{-1}-z_i). \end{aligned} \quad (19)$$

If z_i $(i = \overline{1..n})$ is a zero of $\sigma(z)$ then it follows from (19):

$$\sigma^*(z_i) = 0 \text{ or } \sigma^*(z_i^{-1}) = 0.$$

If $\sigma^*(z_i^{-1}) = 0$ then z_i is a root of $\sigma^*(z^{-1}) = 0$. That means

* $z - z_i$ is a factor of $\sigma^*(z^{-1})$,
* $z^{-1} - z_i$ is a factor of $\sigma^*(z)$.

It can be observed that $deg(\sigma^*(z)) = n$, so the case $\sigma^*(z_i^{-1}) = 0$ can not happen, or $\sigma^*(z_i) = 0$ for $i = \overline{1..n}$.

As a consequence, we can obtain:

$$\sigma(z) = \prod_{i=1}^{n}(z-z_i) = \sigma^*(z), \quad (20)$$

and finally it follows that $K = K_{opt}$. ■
Remark: The conditions summed up in Proposition 1 are only applicable for SISO and SIMO systems as the proof relies on the input-output transfer function relationships.

3.2 Robustness analysis

From the design approach proposed in Figure 1 and detailed in Section II, if a solution exists, the state space partition associated with the piecewise affine control law will contain a critical region corresponding with the unconstrained controller (denoted $P_0, 0 \in int(P_0)$), so the control in this region is linear ($u = Kx$) and its state feedback controller gain is equal to this one computed by the first step ($K = K_{rob}$). This has some implications:

* The stability of the nominal system is guaranteed through the terminal constraint set. So there exists at least a trajectory which returns to the origin. That means our approach is viable (which will be stated in the next subsection).
* The region P_0 contains the (non-empty) maximal robustly positively invariant set associated with the unconstrained robust stabilizing control law.

The LMI unconstrained robust control synthesis provides an ellipsoidal invariant set defined by $\mathscr{E} = \{x | x^T Z^{-1} x \le 1\}$ where Z is computed through (5)-(6). This ellipsoid is an invariant set for the polytopic system, so it is contained in the maximal invariant ellipsoid for the nominal system (which is defined by the nominal system and the same unconstrained control), and included in P_0 as seen in Figure 2.

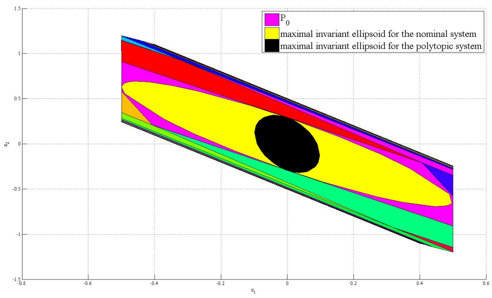

Fig. 2. Maximal invariant ellipsoid for the polytopic system

3.3 Strict robustness guarantee

Let us review a result (Theorem 3.3 from Lombardi et al. [2012]) which provides a basic *local* robustness guarantee for the obtained piecewise control law.
Theorem 1: *The nominal MPC control law designed upon a performance index obtained by inverse optimality with respect to an unconstrained robust linear feedback is robustly stabilizing for the system* (1) *and fulfill the constraints in a non-degenerate neighborhood of the origin.*

Two examples to illustrate the implications of this theorem are the maximal ellipsoidal invariant set computed through the first step of our approach and the solution of Algorithm 1. This algorithm will be introduced below to prove also the advantage of our approach for the polytopic system through the construction of a larger polyhedral robust invariant set (denoted as $\mathscr{X}_{rob}$) than the maximal ellipsoidal one. This one is clearly in the interior of P_0 as seen in Figure 3.
Algorithm 1
1: $\mathscr{X}_{rob} \leftarrow P_0$.
2: $\mathscr{X}_{rob}^+ = \left\{x \in \mathscr{X}_{rob} \mid x^+ = (A_i + B_i K_{rob})x \in \mathscr{X}_{rob}, \forall i = \overline{1 \dots L}\right\}$.
3: **If** $\mathscr{X}_{rob} = \mathscr{X}_{rob}^+$, stop the loop.

4: **Else** $\mathscr{X}_{rob} \leftarrow \mathscr{X}^+_{rob}$, return to Step 2.
5: **End**.

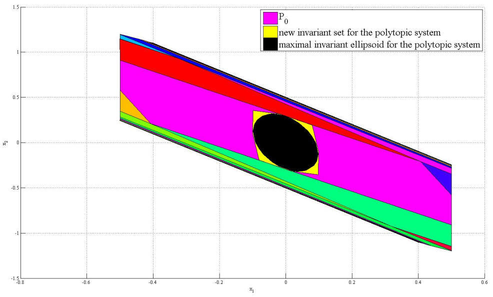

Fig. 3. New invariant set for polytopic system

The above algorithm builds only on the closed loop corresponding to the unconstrained robust control law. Thus, this set is always a subset of P_0. With the aim to enlarge the invariant set for the polytopic systems beyond the unconstrained region of the state space, we present the next two different approaches for the computation of the maximal robustly positively invariant (MRPI) set with respect to the PWA controller obtained from our procedure $u = u_{pwa}(x)$ in (14). They show that such a set is a polyhedral partition of the state space instead of a polytope.

For the simplification of representation, the MRPI set is denoted as $\mathscr{P}_{rob}$ and $\mathscr{X}$ represents the state space partition as the solution of the proposed procedure.

Algorithm 2 Extensive approach

1: $\mathscr{P}_{rob} \leftarrow \mathscr{X}_{rob}$.
2: N_x: the number of regions in the state space partition $\mathscr{X}$.
3: N_p: the number of regions in the partition $\mathscr{P}_{rob}$.
4: $\mathscr{P}^+_{rob} \leftarrow \emptyset$.
5: **For** i_1=*1:N_p*
6: **For** i_2=*1:N_x*
7: $\mathscr{P}^*_{rob} = \{x \in \mathscr{X}(i_2) \mid A_j x + B_j u \in \mathscr{P}_{rob}(i_1), \forall j = \overline{1...L}\}$.
8: $\mathscr{P}^+_{rob} \leftarrow \mathscr{P}^+_{rob} \bigcup \mathscr{P}^*_{rob}$.
9: **End**
10: **End**
11: **If** $\mathscr{P}_{rob} \backslash \mathscr{P}^+_{rob} = \emptyset$ & $\mathscr{P}^+_{rob} \backslash \mathscr{P}_{rob} = \emptyset$ [1]
12: Stop the loop.
13: **Else** $\mathscr{P}_{rob} \leftarrow \mathscr{P}^+_{rob}$. Return to Step 3.
14: **End**.

The numerical result is shown in Figure 4. Another construction

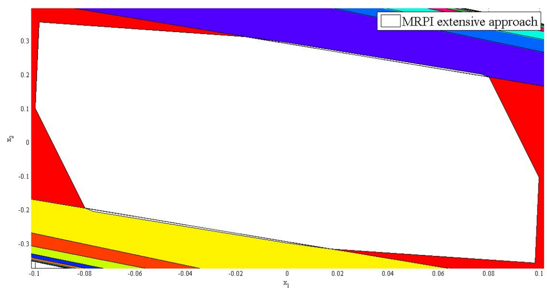

Fig. 4. MRPI set through extensive approach

[1] Recall that $\backslash$ denotes the difference operator of two sets. If P, Q are two arbitrary sets, $P \backslash Q := \{x \mid x \in P,\ x \notin Q\}$. This notion is extended for two polyhedral partitions $P = \bigcup_{i=1}^{N_1} P_i$ and $Q = \bigcup_{j=1}^{N_2} Q_j$ as $P \backslash Q = \cup_{i=1}^{N_1} (\cap_{j=1}^{N_2} P_i \backslash Q_j)$.

based on a so-called the contractive approach is interpreted through Algorithm 3.

Algorithm 3 Contractive approach

1: $\mathscr{P}_{rob} \leftarrow \mathscr{X}$.
2: N_p: the number of regions in the partition $\mathscr{P}_{rob}$.
3: $\mathscr{P}^+_{rob} \leftarrow \emptyset$.
4: **For** i_1=*1:N_p*
5: **For** i_2=*1:N_p*
6: $\mathscr{P}^*_{rob} = \{x \in \mathscr{P}_{rob}(i_2) \mid A_j x + B_j u \in \mathscr{P}_{rob}(i_1), \forall j = \overline{1...L}\}$.
7: $\mathscr{P}^+_{rob} \leftarrow \mathscr{P}^+_{rob} \bigcup \mathscr{P}^*_{rob}$.
8: **End**
9: **End**
10: **If** $\mathscr{P}_{rob} \backslash \mathscr{P}^+_{rob} = \emptyset$ & $\mathscr{P}^+_{rob} \backslash \mathscr{P}_{rob} = \emptyset$.
11: Stop the loop.
12: **Else** $\mathscr{P}_{rob} \leftarrow \mathscr{P}^+_{rob}$. Return to Step 2.
13: **End**.

The result with respect to the same numerical example is shown in Figure 5.

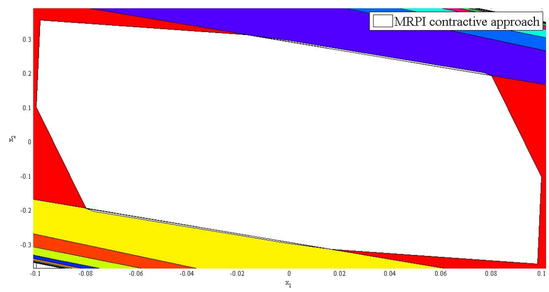

Fig. 5. MRPI set through contractive approach

Remark: Algorithms 2 and 3 are sensitive with respect to the description of the polyhedral partition (possible convex region reduction, degeneracy, etc) and robust numerical procedures should enable the polytopic computations Baotić [2009].

To conclude this part, as presented early, we state a theorem about the viability of our approach.

Theorem 2: *Given an uncertain system* (1), $u_{pwa}(x)$ *is the piecewise affine regulator, obtained as the solution of a nominal MPC:*

- *for a nominal prediction model satisfying the conditions in Proposition 1 with respect to a stabilizing controller* K_{rob},
- *with weighting parameters obtained via inverse optimality,*

then there exists at least one *realization* $[A_k \;\; B_k] \in \Omega$ *(*Ω *in* (2)*) such that the feasible set* $\mathscr{X}$ *is positively invariant in closed loop with* $u_{pwa}(x)$ *and it contains a local region* $\mathscr{X}_{rob}$ *(or* $\mathscr{P}_{rob}$*) which is robustly positively invariant for all* $[A_k \;\; B_k] \in \Omega$.

Proof $(\overline{A}, \overline{B})$ describes the nominal system by which the PWA control law $u_{pwa}(x)$ is synthesized. It is easy to conclude that at least for $A_k = \overline{A}, B_k = \overline{B}$, the feasible set $\mathscr{X}$ is positively invariant while the region P_0 contains a robustly positively invariant set as theoretically stated in Theorem 1 and practically shown through Algorithms 1, 2, 3. ■

Remark: Theoretically, one can easily check the convergence of Algorithms 1, 2, 3. In fact, after each loop $\mathscr{X}_{rob} \supset \mathscr{X}^+_{rob} \supset \{0\}$ for Algorithm 1, the algorithm thus has to be convergent. The same conclusion can be obtained for the other algorithms due to the fact that $\mathscr{X} \supset \mathscr{P}^+_{rob} \supset \mathscr{P}_{rob}$ for Algorithm 2 and

$\{0\} \subset \mathscr{P}_{rob}^{+} \subset \mathscr{P}_{rob}$ for Algorithm 3.

Remark: A margin of robustness can be found for the nominal MPC as a region in parameter space around the nominal $[\overline{A}\ \ \overline{B}]$. A generic method for such a calculation has been recently proposed in Olaru et al. [2013]. We consider for the moment a simpler computation of such a robustness margin.

For each extreme $[A_i\ B_i]$ of the polytopic set Ω in (2), consider the nominal system $[\overline{A}\ \overline{B}]$ and its PWA control law $u_{pwa}(x)$ obtained via our approach, we try to find the maximal value α_i such that the controller $u_{pwa}(x)$ stabilizes also systems $[A\ \ B] = [\overline{A}\ \ \overline{B}] + \alpha_i[A_i - \overline{A}\ \ B_i - \overline{B}]$. As a consequence, any system inside the set: $[\overline{A}\ \ \overline{B}] + \mathbf{Co}\{\alpha_1[A_1 - \overline{A}\ \ B_1 - \overline{B}], \ldots, \alpha_L[A_L - \overline{A}\ \ B_L - \overline{B}]\}$ is also stabilized by $u_{pwa}(x)$ in the sense that the feasible region $\mathscr{X}$ is robustly invariant. We call this set *a robustness margin* associated with the PWA control law $u_{pwa}(x)$ obtained in (14). The above problem can be solved through a series of L linear programming problems in which $(A + BF_j)v + G_j \subset \mathscr{X}$ form the set of linear constraints, here v denotes the vertices of $\mathscr{X}_j$ related to the PWA control law $u_k = F_j x_k + G_j$.

To illustrate this remark, with the weighting matrices obtained by inverse optimality, a robustness margin is shown in Figure 6

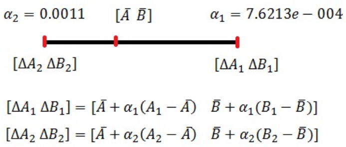

Fig. 6. An example for the calculation of robustness margin for the nominal PWA control law

4. CONCLUSION AND CONNECTIONS WITH OPEN PROBLEMS

This note covers only the first steps toward the development of a new nominal design method with robustness guarantees in a subset of the feasible region $\mathscr{X}$. There are two important problems to be solved in order to make it complete:

- Which feasible solution of this approach guarantees the largest robust stability domain for the polytopic system?
- Suppose a given nominal system with control law K does not satisfy the Kalman inverse optimality condition. Are there other ways to chose the weighting pair (Q, R) such that the control law synthesized by the our procedure can guarantee the stability of the polytopic system?

To guarantee the robust stability over Ω, a relaxed formulation with the nominal cost function computed from the inverse optimality and the set of constraints built on their standard formulation over all vertex realizations of Ω needs to be explored.

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