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# On Infinite Time Performance of Nonlinear Model Predictive Controllers

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**Abstract:** This paper addresses the problem of infinite time performance of model predictive controllers applied to constrained nonlinear systems. The total performance is compared with a finite horizon optimal cost to reveal performance limits of closed-loop model predictive control systems. Based on the Principle of Optimality, an upper and a lower bound of the ratio between the total performance and the finite horizon optimal cost are obtained explicitly expressed by the optimization horizon. The results also illustrate, from viewpoint of performance, how model predictive controllers approaches to infinite optimal controllers as the optimization horizon increases.

*Keywords:* Model Predictive Control; Finite Horizon Optimal Control; Value Iteration; Principle of Optimality.

## 1. INTRODUCTION

Model predictive control (MPC), refers to a kind of control algorithms that measure the current dynamic state and output of the process; solve a finite horizon optimal control problem online iteratively; only implement the first input signal onto the plant and repeat the process on the next iteration. It has attracted much attention from industrial and academic community since 1970s [Richalet et al. (1978), Kwon et al. (1983), Poubelle et al. (1988), and Garcia et al. (1989)].

The unique "receding horizon optimization" of MPC, albeit a great success in industrial application [Qin et al. (1997), Garcia et al. (1989), Maciejowski (2002) and Zheng et al. (2009)], has imposed arduous difficulties on stability and performance analysis of closed-loop systems [Mayne et al. (1990), Mayne et al. (2000), Palma et al. (2007), and Grüne et al. (2008)].

Performance of closed-loop MPC systems has been investigated in a number of papers. For example, [Poubelle et al. (1988)] considers moving-horizon approximation of an infinite horizon optimal control problem of a constrained discrete-time nonlinear systems and shows that the infinite-horizon cost of MPC approaches the infinite horizon optimal cost as the moving horizon extends. [Shamma et al. (1997)] investigates linear non-quadratic optimal control problem of discrete-time linear systems and proposes a receding horizon optimal control law which guarantees the total performance is within a specified bound of infinite-horizon performance. [Palma et al. (2007)] shows a counter-intuitive fact on optimality property of MPC by means of a counterexample that increasing the optimization horizon may not lead to the optimality improvement. Notably, [Grüne et al. (2008)] studies the infinite horizon performance of MPC and provides an upper bound compared with infinite horizon optimal cost. But the performance analysis of closed-loop MPC system is far from thoroughly solved.

The fact that MPC essentially evolves from finite horizon optimal control problem determines that the performance of both controllers are closed-related. However, MPC solves this finite horizon optimal control problem in a completely different way, i.e. "receding horizon optimization", which blurs the relationship between the performance of both controllers.

The main contributions of the present work are the explicit expressions of both an upper and a lower bound of the ratio between performance of MPC and finite horizon optimal controllers in terms of the optimization horizon. Firstly, based on the Principle of Optimality, evolutionary convergent properties of optimal cost sequence are exploited; secondly, quantitative relationships between performance of MPC and finite horizon optimal controllers are revealed; finally, detailed performance analysis of closed-loop MPC systems are given.

This paper is organized as follows. Section 2 provides some preliminaries and problem formulation. Formulation includes value-iteration-based finite horizon optimal control problem, MPC design and objective of this paper while preliminaries list some properties of finite horizon optimal control. Section 3 presents a sufficient condition such that stability of closed-loop MPC systems is guaranteed, followed by the main results on the relationship between infinite time performance of MPC systems and finite horizon optimal cost. Section 4 concludes this paper.

*Terminology:* The set of real numbers and positive, nonnegative real numbers are denoted as $\mathcal{R}$, $\mathcal{R}^+$ and $\mathcal{R}_0^+$ respectively. The set of positive integers is denoted as $\mathcal{N}^+$.

## 2. PRELIMINARIES AND PROBLEM FORMULATION

### 2.1 Problem formulation

Let $\mathcal{X}$ and $\mathcal{U}$ be arbitrary sets. Given $f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$, consider the nonlinear discrete time system

$$\mathbf{x}(t+1) = f(\mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{x}(0) = \mathbf{x}_0 \neq \mathbf{0}, \tag{1}$$

where $\mathbf{x}(t) \in \mathcal{X}$, $\mathbf{u}(t) \in \mathcal{U}$ are the state and control variables, respectively, with $t = 0, 1, 2, \cdots$. It is assumed that $(0, 0)$ is the equilibrium of system. Denote the control law by $\boldsymbol{\mu} : \mathcal{X} \to \mathcal{U}$, which yields the closed-loop dynamics

$$\mathbf{x}(t+1) = f(\mathbf{x}(t), \boldsymbol{\mu}(\mathbf{x}(t))). \tag{2}$$

In essence, MPC design shares many properties with the problem of finite horizon optimal control. In the following subsections, value-iteration based description on design process of finite horizon optimal controllers and MPC controllers with the same performance objective function are stated respectively.

*Finite horizon optimal control problem* The following finite horizon value function is introduced to evaluate performance of the system

$$V_N^{\boldsymbol{\mu}}(\mathbf{x}_0) = V_0(\mathbf{x}(N)) + \sum_{t=0}^{N-1} l(\mathbf{x}(t), \boldsymbol{\mu}(\mathbf{x}(t))), \tag{3}$$

where $\mathbf{x}(t)$, $\boldsymbol{\mu}(\mathbf{x}(t))$ come from system (2); $N \in \mathcal{N}^+$ is the optimization horizon; $V_0(\mathbf{x}(N)) > 0$ is the terminal cost and $l : \mathcal{X} \times \mathcal{U} \to \mathcal{R}_0^+$ is the running cost function and $l(\mathbf{x}, \boldsymbol{\mu}) = 0$ only if $(\mathbf{x}, \boldsymbol{\mu}) = (0, 0)$.

The optimal cost function is thus defined as

$$V_N^*(\mathbf{x}_0) := \inf_{\boldsymbol{\mu} \in \mathcal{U}} V_N^{\boldsymbol{\mu}}(\mathbf{x}_0). \tag{4}$$

A well-known method to get this optimal cost function is value iteration

$$V_{k+1}^*(\mathbf{x}) = \min_{\boldsymbol{\mu} \in \mathcal{U}} \{V_k^*(f(\mathbf{x}, \boldsymbol{\mu})) + l(\mathbf{x}, \boldsymbol{\mu})\}, \tag{5}$$

which starts from initial condition $V_0(\mathbf{x}(N))$ and updates iteratively with $k = 0, 1, 2 \cdots N-1$.

For simplicity, it is assumed that throughout this paper the minimum with respect to $\boldsymbol{\mu} \in \mathcal{U}$ is attained, denoted by $\boldsymbol{\mu}_k$, $k = 1, 2 \cdots N$, which can be described as follows

$$\boldsymbol{\mu}_k(\mathbf{x}) = \arg\min_{\boldsymbol{\mu} \in \mathcal{U}} \{V_{k-1}(f(\mathbf{x}, \boldsymbol{\mu})) + l(\mathbf{x}, \boldsymbol{\mu})\}. \tag{6}$$

Thus the finite optimal control sequence for fixed optimization horizon $N$ is constituted by $\boldsymbol{\mu}_k$ arranged in the reversed order $\boldsymbol{\mu}^* = \{\boldsymbol{\mu}_N, \boldsymbol{\mu}_{N-1}, \cdots \boldsymbol{\mu}_1\}$. Substituting (6) into value iteration equation (5) yields

$$V_{k+1}^*(\mathbf{x}) = V_k^*(f(\mathbf{x}, \boldsymbol{\mu}_{k+1}(\mathbf{x}))) + l(\mathbf{x}, \boldsymbol{\mu}_{k+1}(\mathbf{x})), \tag{7}$$

where $k = 0, 1, 2 \cdots N-1$.

Assume the following inequalities hold throughout this paper

$$l(\mathbf{x}, \boldsymbol{\mu}_k(\mathbf{x})) > 0, \; k = 1, 2, \cdots, N. \tag{8}$$

*Remark 1.* According to [Lincoln et al. (2006)], value iteration equation (7) is convergent under mild conditions with $l(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x})) \to 0$ as $N \to \infty$. For the finite optimization horizon $N$, the above assumption make sense. Further convergent properties are exploited in this paper.

*Model predictive controller design* Now consider the MPC design for the receding horizon optimization problem which has the same optimization horizon $N$ with the same system and objective function as in finite optimal control problem mentioned above.

At each time instant $t \in \mathcal{N}^+$, MPC solves the finite horizon optimal problem with a receding horizon

$$\bar{V}_N^{\boldsymbol{\mu}}(\mathbf{x}_0) = V_0(\mathbf{x}(t+N)) + \sum_{k=t}^{t+N-1} l(\mathbf{x}(k), \boldsymbol{\mu}(\mathbf{x}(k))),$$

subject to system (2) where $V_0(\mathbf{x})$ and $l(\mathbf{x}, \boldsymbol{\mu})$ are the same as in (3).

It is well-known that MPC only implement the first control signal of optimal control sequence onto the system. According to the value iteration approach, MPC law for the fixed receding horizon $N$ is constructed as

$$\boldsymbol{\mu}_N(\mathbf{x}) = \arg\min_{\boldsymbol{\mu} \in \mathcal{U}} \{V_{N-1}(f(\mathbf{x}, \boldsymbol{\mu})) + l(\mathbf{x}, \boldsymbol{\mu})\}. \tag{9}$$

The closed-loop MPC systems turn into

$$\mathbf{x}_{\boldsymbol{\mu}_N}(t+1) = f(\mathbf{x}_{\boldsymbol{\mu}_N}(t), \boldsymbol{\mu}_N(\mathbf{x}_{\boldsymbol{\mu}_N}(t))), \; \mathbf{x}_{\boldsymbol{\mu}_N}(0) = \mathbf{x}_0, \tag{10}$$

where $\mathbf{x}_{\boldsymbol{\mu}_N}$ denotes the solution of the closed-loop MPC system.

The infinite time cost of nonlinear model predictive controllers with receding horizon $N$ is defined as

$$V_\infty^{\boldsymbol{\mu}_N}(\mathbf{x}_0) := \sum_{k=0}^{\infty} l(\mathbf{x}_{\boldsymbol{\mu}_N}(k), \boldsymbol{\mu}_N(\mathbf{x}_{\boldsymbol{\mu}_N}(k))). \tag{11}$$

Note that the definition make sense only when the closed-loop system is stable.

*Objective* The objective of this paper is focused on derivation of explicit upper and lower bounds of comparing infinite time performance of closed-loop MPC system $V_\infty^{\boldsymbol{\mu}_N}$ defined as (11) with the finite horizon optimal value function $V_N^*$ described as in (4).

### *2.2 Preliminaries*

The following assumption is introduced for $V_1^*(\mathbf{x}), V_2^*(\mathbf{x}) > 0$ which come from value iteration (7):

*Assumption 1.* There exist constants $c_1,\ c_2 > 0$ such that the following inequality

$$c_1 V_1^*(\mathbf{x}) \leq V_2^*(\mathbf{x}) \leq c_2 V_1^*(\mathbf{x}) \quad (12)$$

holds for all $\mathbf{x} \in \mathcal{X}$.

The following proposition is needed in the proof of main results.

*Proposition 1.* Let $\gamma > 1$ be arbitrary, $V_1^*(\mathbf{x}), V_2^*(\mathbf{x})$ come from value iteration equation (7) for dynamic system (2) with objective function (3) satisfying Assumption 1. There exists nonnegative constants $\underline{c} = \underline{c}(\gamma, c_1)$ and $\overline{c} = \overline{c}(\gamma, c_2)$ such that the following inequality holds for all $\mathbf{x} \in \mathcal{X}$

$$\frac{\gamma - 1}{\gamma - 1 + \underline{c}} V_1^*(\mathbf{x}) \leq V_2^*(\mathbf{x}) \leq \frac{\gamma - 1 + \overline{c}}{\gamma - 1} V_1^*(\mathbf{x}). \quad (13)$$

**Proof:** By selecting $\underline{c}$ and $\overline{c}$ as follows

$$\underline{c} = \max\{l_1, 0\}, \quad l_1 = (c_1^{-1} - 1)(\gamma - 1),$$
$$\overline{c} = \max\{l_2, 0\}, \quad l_2 = (c_2 - 1)(\gamma - 1),$$

it can be verified that

$$\frac{\gamma - 1}{\gamma - 1 + l_1} = c_1 > 0, \quad \frac{\gamma - 1 + l_2}{\gamma - 1} = c_2 > 0.$$

Thus, the following inequalities hold for all $\mathbf{x} \in \mathcal{X}$

$$\frac{\gamma - 1}{\gamma - 1 + \underline{c}} V_1^*(\mathbf{x}) \leq \frac{\gamma - 1}{\gamma - 1 + l_1} V_1^*(\mathbf{x}) = c_1 V_1^*(\mathbf{x}),$$
$$\frac{\gamma - 1 + \overline{c}}{\gamma - 1} V_1^*(\mathbf{x}) \geq \frac{\gamma - 1 + l_2}{\gamma - 1} V_1^*(\mathbf{x}) = c_2 V_1^*(\mathbf{x}),$$

which completes the proof from Assumption 1. □

*Remark 2.* Proposition 1 is to introduce nonnegative constants $\underline{c}$ and $\overline{c}$, which is related to $\gamma > 1$, lower bound $c_1$ and the upper bound $c_2$, such that the equation (13) holds. The equation (13) is necessary to start proof of the main results. The $\gamma$ is an index quantifying the convergence speed of value iteration process.

As MPC is close-related to finite horizon optimal control problem, some properties are listed in the following part, which will be utilized in the main results.

*Proposition 2.* Let $V_1^*(\mathbf{x})$ and $V_2^*(\mathbf{x})$ come from the value iteration equation (7) for the dynamic system (2) with the objective function (3) satisfying Assumption 1. There exist a constant $\gamma > 1$ and two constants $\underline{c} = \underline{c}(\gamma, c_1)$ and $\overline{c} = \overline{c}(\gamma, c_2)$ from Proposition 1 such that for any given finite horizon $N \in \mathcal{N}^+$ the following inequality holds for all $\mathbf{x} \in \mathcal{X}$ and $k = 1, \ldots, N - 1$:

$$\underline{\tau}_k V_k^*(\mathbf{x}) \leq V_{k+1}^*(\mathbf{x}) \leq \overline{\tau}_k V_k^*(\mathbf{x}), \quad (14)$$

where

$$\underline{\tau}_k := \frac{\gamma^{k-1}(\gamma - 1)}{\gamma^{k-1}(\gamma - 1) + \underline{c}}; \ \overline{\tau}_k := \frac{\gamma^{k-1}(\gamma - 1) + \overline{c}}{\gamma^{k-1}(\gamma - 1)}. \quad (15)$$

**Proof:** The proof is divided into the following three steps:

Step 1 shows the existence of $\gamma$

From (7), the following inequality holds for all $k = 0, \cdots, N - 1$

$$V_{k+1}^*(\mathbf{x}) - V_k^*(f(\mathbf{x}, \boldsymbol{\mu}_{k+1}(\mathbf{x}))) = l(\mathbf{x}, \boldsymbol{\mu}_{k+1}(\mathbf{x})) > 0.$$

Thus there exists constants $\gamma_k > 1$ such that inequalities

$$V_{k+1}^*(\mathbf{x}) \geq \gamma_k V_k^*(f(\mathbf{x}, \boldsymbol{\mu}_{k+1}(\mathbf{x}))) \quad (16)$$

holds for all $\mathbf{x} \in \mathcal{X}$ and $k = 0, \cdots N - 1$.

Denote $\gamma := \min\{\gamma_0, \cdots \gamma_{N-1}\}$. It follows that there exists $\gamma > 1$ such that inequality

$$V_{k+1}^*(\mathbf{x}) \geq \gamma V_k^*(f(\mathbf{x}, \boldsymbol{\mu}_{k+1}(\mathbf{x}))) \quad (17)$$

holds uniformly, which proves the existence of $\gamma$.

Step 2 shows inequality $V_{k+1}^*(\mathbf{x}) \geq \underline{\tau}_k V_k^*(\mathbf{x})$ holds

According to Proposition 1, for any fixed $\gamma > 1$, there are two positive constants $\underline{c} = \underline{c}(\gamma, c_1)$, $\overline{c} = \overline{c}(\gamma, c_2)$ such that the inequality (13) holds. Combining (7) and (17) yields the following inequality

$$(\gamma - 1)V_k^*(f(\mathbf{x}, \boldsymbol{\mu}_{k+1}(\mathbf{x}))) - l(\mathbf{x}, \boldsymbol{\mu}_{k+1}) \leq 0 \quad (18)$$

holds for all $\mathbf{x} \in \mathcal{X}$ with $k = 1, \cdots, N - 1$.

On the other hand, a simple calculation shows that $0 < \underline{\tau}_k \leq 1$ and $\overline{\tau}_k \geq 1$. Moreover they satisfy the following relationships

$$\frac{\gamma}{\overline{\tau}_k + \gamma - 1} = \frac{1}{\overline{\tau}_{k+1}}, \qquad \frac{\gamma \underline{\tau}_k}{1 + (\gamma - 1)\underline{\tau}_k} = \underline{\tau}_{k+1}. \quad (19)$$

The remainder part of proof is completed by induction.

The choice of parameter $(\gamma, \underline{c}, \overline{c})$ and Proposition 1 show that the inequality (14) holds for $k = 1$.

Assume that for any fixed $N$, the inequality (14) holds when $k = m$, where $m = 1, 2, \cdots, N - 2$. The following two steps shows that the inequality holds when $k = m+1$.

At $k = m + 1$, there holds

$$V_{m+2}^*(\mathbf{x}) = V_{m+1}^*(f(\mathbf{x}, \boldsymbol{\mu}_{m+2}(\mathbf{x}))) + l(\mathbf{x}, \boldsymbol{\mu}_{m+2}(\mathbf{x})). \quad (20)$$

Noting the fact that $\frac{1 - \underline{\tau}_m}{1 + (\gamma - 1)\underline{\tau}_m} \geq 0$, multiplying both sides of the inequality (18) by $\frac{1 - \underline{\tau}_m}{1 + (\gamma - 1)\underline{\tau}_m}$ and adding it to the equality (20) yields

$$\begin{aligned} V_{m+2}^*(\mathbf{x}) \geq\ & V_{m+1}^*(f(\mathbf{x}, \boldsymbol{\mu}_{m+2}(\mathbf{x}))) + l(\mathbf{x}, \boldsymbol{\mu}_{m+2}(\mathbf{x})) \\ & + \frac{(1 - \underline{\tau}_m)(\gamma - 1)}{1 + (\gamma - 1)\underline{\tau}_m} V_{m+1}^*(f(\mathbf{x}, \boldsymbol{\mu}_{m+2}(\mathbf{x}))) \\ & - \frac{(1 - \underline{\tau}_m)}{1 + (\gamma - 1)\underline{\tau}_m} l(\mathbf{x}, \boldsymbol{\mu}_{m+2}(\mathbf{x})) \end{aligned}$$

$$= \frac{\gamma}{1+(\gamma-1)\underline{\tau}_m} V_{m+1}^*(f(\mathbf{x},\boldsymbol{\mu}_{m+2}(\mathbf{x}))) + \frac{\gamma\underline{\tau}_m}{1+(\gamma-1)\underline{\tau}_m} l(\mathbf{x},\boldsymbol{\mu}_{m+2}(\mathbf{x})) \quad (21)$$

Noting that $V_{m+1}^*(\mathbf{x}) \geq \underline{\tau}_m V_m^*(\mathbf{x})$ holds for all $\mathbf{x} \in \mathcal{X}$, it follows that

$$V_{m+2}^*(\mathbf{x}) \geq \frac{\gamma\underline{\tau}_m}{1+(\gamma-1)\underline{\tau}_m} \left[V_{m+1}^*(f(\mathbf{x},\boldsymbol{\mu}_{m+2}(\mathbf{x}))) + l(\mathbf{x},\boldsymbol{\mu}_{m+2}(\mathbf{x}))\right]$$

Using the Principle of Optimality, it follows

$$V_{m+1}^* = \min_{\boldsymbol{\mu}} \left\{V_m^*(f(\mathbf{x},\boldsymbol{\mu}_{m+1}(\mathbf{x}))) + l(\mathbf{x},\boldsymbol{\mu}_{m+1}(\mathbf{x}))\right\}$$

Using equalities (19) leads to

$$V_{m+2}^*(\mathbf{x}) \geq \frac{\gamma\underline{\tau}_m}{1+(\gamma-1)\underline{\tau}_m} V_{m+1}^*(\mathbf{x}) = \underline{\tau}_{m+1} V_{m+1}^*(\mathbf{x}),$$

which implies that inequality $V_{k+1}^*(\mathbf{x}) \geq \underline{\tau}_k V_k^*(\mathbf{x})$ holds when $k = m+1$. This completes the proof of Step 2.

<u>Step 3 shows inequality $V_{k+1}^*(\mathbf{x}) \leq \overline{\tau}_k V_k^*(\mathbf{x})$ holds</u>

Noting that the equality (7), the inequality (18) as well as the fact $\frac{\overline{\tau}_m - 1}{\overline{\tau}_m+\gamma-1} > 0$, it leads to

$$\begin{aligned} V_{m+1}^*(\mathbf{x}) &= V_m^*(f(\mathbf{x},\boldsymbol{\mu}_{m+1}(\mathbf{x}))) + l(\mathbf{x},\boldsymbol{\mu}_{m+1}(\mathbf{x})) \\ &\geq V_m^*(f(\mathbf{x},\boldsymbol{\mu}_{m+1}(\mathbf{x}))) + l(\mathbf{x},\boldsymbol{\mu}_{m+1}(\mathbf{x})) \\ &\quad + \frac{(\overline{\tau}_m - 1)(\gamma-1)}{\overline{\tau}_m+\gamma-1} V_m^*(f(\mathbf{x},\boldsymbol{\mu}_{m+1}(\mathbf{x}))) \\ &\quad - \frac{\overline{\tau}_m - 1}{\overline{\tau}_m+\gamma-1} l(\mathbf{x},\boldsymbol{\mu}_{m+1}(\mathbf{x}) \\ &= \frac{\gamma\overline{\tau}_m}{\overline{\tau}_m+\gamma-1} V_m^*(f(\mathbf{x},\boldsymbol{\mu}_{m+1}(\mathbf{x}))) \\ &\quad + \frac{\gamma}{\overline{\tau}_m+\gamma-1} l(\mathbf{x},\boldsymbol{\mu}_{m+1}(\mathbf{x}). \end{aligned} \quad (22)$$

Since the inequality (14) holds at $k = m$, it follows that

$$V_{m+1}^*(\mathbf{x}) \leq \overline{\tau}_m V_m^*(\mathbf{x}),$$

holds for all $\mathbf{x} \in \mathcal{X}$, which implies

$$\overline{\tau}_m V_m^*(f(\mathbf{x},\boldsymbol{\mu}_{m+1}(\mathbf{x}))) \geq V_{m+1}^*(f(\mathbf{x},\boldsymbol{\mu}_{m+1}(\mathbf{x}))).$$

Substituting above inequality into (22) yields

$$V_{m+1}^*(\mathbf{x}) \geq \frac{\gamma}{\overline{\tau}_m+\gamma-1} \left[V_{m+1}^*(f(\mathbf{x},\boldsymbol{\mu}_{m+1}(\mathbf{x}))) + l(\mathbf{x},\boldsymbol{\mu}_{m+1}(\mathbf{x}))\right].$$

Using the Principle of Optimality, it follows

$$V_{m+2}^*(\mathbf{x}) = \min_{\boldsymbol{\mu}} \left\{V_{m+1}^*(f(\mathbf{x},\boldsymbol{\mu}_{m+2}(\mathbf{x}))) + l(\mathbf{x},\boldsymbol{\mu}_{m+2}(\mathbf{x}))\right\}.$$

Using equalities (19), inequality (22) can be simplified into the following inequality

$$V_{m+1}^*(\mathbf{x}) \geq \frac{\gamma}{\overline{\tau}_m+\gamma-1} V_{m+2}^*(\mathbf{x}) = \frac{1}{\overline{\tau}_{m+1}} V_{m+2}^*(\mathbf{x}),$$

which implies that inequality $V_{k+1}^*(\mathbf{x}) \leq \overline{\tau}_k V_k^*(\mathbf{x})$ holds when $k = m+1$. This completes the proof of Step 3.

The proof is completed by combining the results of Step 1-3. □

*Remark 3.* Proposition 2 shows quantitative properties of the sequence $\{V_1^*(\mathbf{x}), V_2^*(\mathbf{x}), \cdots V_N^*(\mathbf{x})\}$. Both upper and lower bounds of the ratio between two adjacent optimal cost are provided. The results are useful in analyzing the performance limits of closed-loop MPC systems since MPC shares some properties with finite horizon optimal controller.

As a affiliated result, monotonicity of the optimal cost sequence is derived.

*Proposition 3.* Let $V_1^*(\mathbf{x})$ and $V_2^*(\mathbf{x})$ come from the value iteration equation (7) for the dynamic system (2) with the objective function (3). For any given finite horizon $N \in \mathcal{N}^+$,

(1) if $V_1^*(\mathbf{x}) \leq V_2^*(\mathbf{x})$ holds for all $\mathbf{x} \in \mathcal{X}$, then $V_k^*(\mathbf{x}) \leq V_{k+1}^*(\mathbf{x})$ holds for all $\mathbf{x} \in \mathcal{X}$ and $k = 1, \cdots, N-1$;
(2) if $V_1^*(\mathbf{x}) \geq V_2^*(\mathbf{x})$ holds for all $\mathbf{x} \in \mathcal{X}$, then $V_k^*(\mathbf{x}) \geq V_{k+1}^*(\mathbf{x})$ holds for all $\mathbf{x} \in \mathcal{X}$ and $k = 1, \cdots, N-1$.

**Proof:** The proof is straightforward.

If $V_1^*(\mathbf{x}) \leq V_2^*(\mathbf{x})$ holds, it follows from Proposition 1 that $\underline{c}$=0. Substituting $\underline{c}$ into Proposition 2 yields $V_k^*(\mathbf{x}) \leq V_{k+1}^*(\mathbf{x})$ holds for all $\mathbf{x} \in \mathcal{X}$ and $k = 1, \cdots, N-1$.

The proof of (2) is similar and thus has been omitted. □

*Remark 4.* Proposition 3 provide monotonic properties of the optimal cost sequence, which has included the monotonic properties of Riccati Difference Equation [Poubelle et al. (1988)] as a special case corresponding to linear quadratic control of linear-time-invariant (LTI) systems.

## 3. MAIN RESULTS

This section provides main results of the relationship between infinite time performance of nonlinear MPC systems and finite horizon optimal cost. At first, a sufficient condition is given to ensure the stability of the closed-loop system.

*Proposition 4.* Let $N \geq 2$ be a fixed integer, $\gamma$, $\overline{c}$ come from Proposition 2, $V_1^*(\mathbf{x})$ and $V_2^*(\mathbf{x})$ come from the value iteration equation (7) for the dynamic system (2) with the objective function (3) satisfying Assumption 1. If for the given $N$, the following holds

$$\gamma^{N-2}(\gamma-1)^2 - \overline{c} > 0, \quad (23)$$

then the closed-loop MPC system (10) is stable.

**Proof:** If $\gamma^{N-2}(\gamma-1)^2 - \overline{c} > 0$ holds, applying Proposition 2 leads to

$$\begin{aligned} V_N^*(\mathbf{x}(t)) &\leq \frac{\gamma^{N-2}(\gamma-1)+\overline{c}}{\gamma^{N-2}(\gamma-1)} V_{N-1}^*(\mathbf{x}(t)) \\ &< \frac{\gamma^{N-2}(\gamma-1)+\gamma^{N-2}(\gamma-1)^2}{\gamma^{N-2}(\gamma-1)} V_{N-1}^*(\mathbf{x}(t)) \\ &= \gamma V_{N-1}^*(\mathbf{x}(t)). \end{aligned} \quad (24)$$

On the other hand, from (17) the following inequality holds for $t \in \mathcal{N}^+$

$$V_N^*(\mathbf{x}(t)) \geq \gamma V_{N-1}^*(\mathbf{x}(t+1)). \qquad (25)$$

Combining the two inequalities yields

$$V_{N-1}^*(\mathbf{x}(t)) - V_{N-1}^*(\mathbf{x}(t+1)) > 0 \qquad (26)$$

holds for all $t \in \mathcal{N}^+$, which prove the stability of closed-loop MPC system (10) by Lyapunov direct method. □

*Remark 5.* Proposition 4 presents a sufficient condition to the stability of closed-loop systems, which is a prerequisite in calculating the infinite time performance of closed-loop systems. The stability condition (23) is closely related to the selection of $\gamma$ and $N$. The larger $\gamma$, the shorter $N$ is needed to ensure the stability.

The relationships between infinite time performance of closed-loop MPC system and finite optimal cost is revealed in the following theorem.

*Theorem 1.* Let the optimization horizon $N$ be fixed, $\gamma$, $\underline{c}$, $\overline{c}$ come from Proposition 2 respectively, $V_1^*(\mathbf{x})$ and $V_2^*(\mathbf{x})$ come from the value iteration equation (7) for the dynamic system (2) with the objective function (3) satisfying Assumption 1. If condition (23) holds for the given $N$, the following inequality holds

$$\begin{aligned}\underline{\zeta} V_N^*(\mathbf{x}) \leq V_\infty^{\boldsymbol{\mu}_N}(\mathbf{x}) &\leq \overline{\zeta} V_N^*(\mathbf{x}) \\ \underline{\zeta} &= \frac{\gamma^{N-2}(\gamma-1)^2 + \underline{c}(\gamma-1)}{\gamma^{N-2}(\gamma-1)^2 + \underline{c}(\gamma-1) + \underline{c}} \\ \overline{\zeta} &= \frac{\gamma^{N-2}(\gamma-1)^2}{\gamma^{N-2}(\gamma-1)^2 - \overline{c}} \end{aligned} \qquad (27)$$

for all $\mathbf{x} \in \mathcal{X}$.

**Proof:** The proof is divided into two parts.

<u>Part 1 shows that $V_\infty^{\boldsymbol{\mu}_N}(\mathbf{x}) \leq \overline{\zeta} V_N^*(\mathbf{x})$ holds</u>

By applying Proposition 2, the following inequality holds

$$\begin{aligned}V_N^*(f(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x}))) &- V_{N-1}^*(f(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x}))) \\ &\leq \frac{\overline{c}}{\gamma^{N-2}(\gamma-1)} V_{N-1}^*(f(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x}))).\end{aligned} \qquad (28)$$

Rearranging inequality (18) yields $V_{N-1}^*(f(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x}))) \leq \frac{1}{\gamma-1} l(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x}))$. Thus,

$$\begin{aligned}V_N^*(f(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x}))) &- V_{N-1}^*(f(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x}))) \\ &\leq \frac{\overline{c}}{\gamma^{N-2}(\gamma-1)^2} l(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x})).\end{aligned}$$

Noting that $V_{N-1}^*(f(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x}))) = V_N^*(\mathbf{x}) - l(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x}))$, it has

$$\begin{aligned}V_N^*(f(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x}))) &- (V_N^*(\mathbf{x}) - l(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x}))) \\ &\leq \frac{\overline{c}}{\gamma^{N-2}(\gamma-1)^2} l(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x})).\end{aligned}$$

Thus, it follows that the following inequality holds for all $\mathbf{x} \in \mathcal{X}$

$$l(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x})) \leq \frac{\gamma^{N-2}(\gamma-1)^2}{\gamma^{N-2}(\gamma-1)^2 - \overline{c}} \left(V_N^*(\mathbf{x}) - V_N^*(f(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x})))\right).$$

According to the dynamics of closed-loop MPC system (10) and definition on the infinite time performance of the nonlinear model predictive controllers and noting the stability of the closed-loop system is guaranteed under condition (23),

$$\begin{aligned}V_\infty^{\boldsymbol{\mu}_N}(\mathbf{x}_0) &= \sum_{k=0}^{\infty} l(\mathbf{x}_{\boldsymbol{\mu}_N}(k), \boldsymbol{\mu}_N(\mathbf{x}_{\boldsymbol{\mu}_N}(k))) \\ &\leq \frac{\gamma^{N-2}(\gamma-1)^2}{\gamma^{N-2}(\gamma-1)^2 - \overline{c}} \sum_{k=0}^{\infty} \big[V_N^*(\mathbf{x}_{\boldsymbol{\mu}_N}(k)) \\ &\qquad - V_N^*(f(\mathbf{x}_{\boldsymbol{\mu}_N}(k), \boldsymbol{\mu}_N(\mathbf{x}_{\boldsymbol{\mu}_N}(k))))\big] \\ &\leq \frac{\gamma^{N-2}(\gamma-1)^2}{\gamma^{N-2}(\gamma-1)^2 - \overline{c}} V_N^*(\mathbf{x}_0)\end{aligned} \qquad (29)$$

Let $\overline{\zeta} = \frac{\gamma^{N-2}(\gamma-1)^2}{\gamma^{N-2}(\gamma-1)^2 - \overline{c}}$, which completes proof of Part 1.

<u>Part 2 shows that $V_\infty^{\boldsymbol{\mu}_N}(\mathbf{x}) \geq \underline{\zeta} V_N^*(\mathbf{x})$ holds</u>

Similar to the proof of Part 1, noting inequality (18) and applying Proposition 2 yields

$$\begin{aligned}V_N^*(f(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x}))) &- V_{N-1}^*(f(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x}))) \\ &\geq -\frac{\underline{c}}{\gamma^{N-2}(\gamma-1) + \underline{c}} V_{N-1}^*(f(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x}))) \\ &\geq -\frac{\underline{c}}{(\gamma^{N-2}(\gamma-1) + \underline{c})(\gamma-1)} l(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x})).\end{aligned} \qquad (30)$$

Substituting $V_{N-1}^*(f(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x}))) = V_N^*(\mathbf{x}) - l(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x}))$ into the above inequality, it follows that

$$\begin{aligned}l(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x})) \geq &\frac{\gamma^{N-2}(\gamma-1)^2 + \underline{c}(\gamma-1)}{\gamma^{N-2}(\gamma-1)^2 + \underline{c}(\gamma-1) + \underline{c}} \\ &\cdot (V_N^*(\mathbf{x}) - V_N^*(f(\mathbf{x}, \boldsymbol{\mu}_N(\mathbf{x}))))\end{aligned} \qquad (31)$$

holds for all $\mathbf{x} \in \mathcal{X}$. Thus the infinite time performance of nonlinear MPC controllers

$$\begin{aligned}V_\infty^{\boldsymbol{\mu}_N}(\mathbf{x}_0) &= \sum_{k=0}^{\infty} l(\mathbf{x}_{\boldsymbol{\mu}_N}(k), \boldsymbol{\mu}_N(\mathbf{x}_{\boldsymbol{\mu}_N}(k))) \\ &\geq \frac{\gamma^{N-2}(\gamma-1)^2 + \underline{c}(\gamma-1)}{\gamma^{N-2}(\gamma-1)^2 + \underline{c}(\gamma-1) + \underline{c}} \sum_{k=0}^{\infty} \big[V_N^*(\mathbf{x}_{\boldsymbol{\mu}_N}(k)) \\ &\qquad - V_N^*(f(\mathbf{x}_{\boldsymbol{\mu}_N}(k), \boldsymbol{\mu}_N(\mathbf{x}_{\boldsymbol{\mu}_N}(k))))\big] \\ &= \frac{\gamma^{N-2}(\gamma-1)^2 + \underline{c}(\gamma-1)}{\gamma^{N-2}(\gamma-1)^2 + \underline{c}(\gamma-1) + \underline{c}} (V_N^*(\mathbf{x}_0) - V_N^*(\mathbf{x}(\infty))) \\ &= \frac{\gamma^{N-2}(\gamma-1)^2 + \underline{c}(\gamma-1)}{\gamma^{N-2}(\gamma-1)^2 + \underline{c}(\gamma-1) + \underline{c}} V_N^*(\mathbf{x}_0).\end{aligned}$$

Let $\underline{\zeta} = \frac{\gamma^{N-2}(\gamma-1)^2 + \underline{c}(\gamma-1)}{\gamma^{N-2}(\gamma-1)^2 + \underline{c}(\gamma-1) + \underline{c}}$, which completes proof of Part 2.

The proof is finished by combining Part 1 and 2. □

*Remark 6.* Theorem 1 provides quantitative relationship between infinite time performance of closed-loop MPC system and finite horizon optimal cost with both upper and lower bound. Since $\gamma > 1$, these bounds are proved to converge to 1 as the optimization horizon increases to infinity.

*Remark 7.* In the extent literature, only upper bounds are constantly investigated. Theorem 1 provides both upper and lower bounds, which are novel compared with existent literature to the authors' best knowledge.

*Corollary 1.* Let the optimization horizon $N \in \mathcal{N}^+$ be fixed, $V_1^*(\mathbf{x})$ and $V_2^*(\mathbf{x})$ come from the value iteration equation (7) for the dynamic system (2) with the objective function (3). If condition (23) holds for the given $N$ and

(1) if $V_1^*(\mathbf{x}) \leq V_2^*(\mathbf{x})$ holds for all $\mathbf{x} \in \mathcal{X}$, then $V_N^*(\mathbf{x}) \leq V_\infty^{\boldsymbol{\mu}_N}(\mathbf{x})$ holds for all $\mathbf{x} \in \mathcal{X}$;
(2) if $V_1^*(\mathbf{x}) \geq V_2^*(\mathbf{x})$ holds for all $\mathbf{x} \in \mathcal{X}$, then $V_N^*(\mathbf{x}) \geq V_\infty^{\boldsymbol{\mu}_N}(\mathbf{x})$ holds for all $\mathbf{x} \in \mathcal{X}$.

**Proof:** The proof is straightforward.

If $V_1^*(\mathbf{x}) \leq V_2^*(\mathbf{x})$ holds, it follows from Proposition 1 that $\underline{c}$=0. Substituting $\underline{c}$ into Theorem 1 yields that $V_N^*(\mathbf{x}) \leq V_\infty^{\boldsymbol{\mu}_N}(\mathbf{x})$ holds for all $\mathbf{x} \in \mathcal{X}$ and $k = 1, \cdots, N-1$.

The proof of (2) is similar and thus has been omitted. □

*Remark 8.* Corollary 1 shows some interesting results, that is, selection of different terminal costs might result in different relationships between performance of both controllers. The effect of terminal cost in adjusting performance of MPC systems is thus revealed.

## 4. CONCLUSION AND FUTURE WORK

Quantitative relationships between performance of MPC and finite horizon optimal controllers have been investigated for constrained nonlinear systems. Upper and lower bounds of the ratio between both controllers have been derived. The proposed bounds have been shown to be convergent as the optimization horizon increases to infinity. Detailed performance analysis of closed-loop MPC systems has been provided.

Future research includes the following topics:

(1) Adaptive MPC controller design which satisfies both stability and performance requirements for the closed-loop system will be considered based on the obtained results.
(2) Model uncertainty will be take into account to reveal the quantitative relationships between performance of robust MPC and robust optimal controllers.

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