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Suboptimal Feedback Control of Nonlinear Distributed Parameter Systems by Stable Manifold Method

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Abstract: This paper proposes a suboptimal feedback controller design for nonlinear distributed parameter systems via the stable manifold method. The stable manifold method provides approximate stabilizing solutions of Hamilton-Jacobi (HJ) equations in nonlinear optimal control theory. We apply this method to a reduced-order system obtained from the proper orthogonal decomposition (POD) and Galerkin projection. The feasibility of the design is demonstrated by the numerical example of feedback controls of the viscous Burgers' equation.

Keywords: distributed-parameter systems, nonlinear control systems, quadratic optimal regulators, proper orthogonal decomposition, Burgers' equation

1. INTRODUCTION

In nonlinear optimal control problems, Hamilton-Jacobi (HJ) equations play a significant role (Aliyu (2011)). Recently, the stable manifold method has been developed for approximately solving the HJ equations (Sakamoto and van der Schaft (2008)). Compared with conventional approaches to the HJ equations, such as Taylor expansion method (Lukes (1969)), the method is advantageous in that the computational complexity does not increase with respect to the accuracy of the approximation. Some applications using the stable manifold method have been reported, e.g., a swing up control of a 2-dimensional inverted pendulum (Sakamoto (2013)) and a guidance control of a drifting vehicle (Abe et al. (2013)). However, all of these researches have treated systems described by ordinary differential equations, that is, lumped-parameter systems.

In this paper, we treat nonlinear distributed-parameter systems, which often arise in fluid dynamics or chemical engineering. It is very challenging to solve the optimal feedback control problems of nonlinear distributed-parameter systems due to an infinite number of degree of freedom. Furthermore, even if it can be solved, it leads to infinite dimensional controllers that are not implementable. Thus, in controls of distributed-parameter systems, we usually approximate partial differential equation as ordinary differential equations and synthesis a feedback controller for the approximated model. However, such an approximation like the finite element method (FEM) tends to be high dimensional.

We solve this problem by using of a unified model reduction by the proper orthogonal decomposition (POD) and Galerkin projection (Holmes et al. (1998)). POD, also known as the Karhunen-Loève method, is a statistical analysis technique for obtaining basis functions describing a dominant character of systems from numerical or experimental data. By using Galerkin projection, very low dimensional reduced-order models (ROM) can be derived from the POD basis functions. Then, we solve optimal control problems based on the ROM. Though this strategy can be found in past researches, e.g. Kunisch et al. (2004), we use the stable manifold method to solve the optimal control problem efficiently. To our knowledge, the combination of POD-based model reductions and the stable manifold method is new.

The organization of the paper is as follows. In section 2, we explain a class of distributed parameter systems considered in this work. This class includes some practical equations like the viscous Burgers' equation and two-dimensional incompressible Navier-Stokes equation. Section 3 is devoted to the review of POD and constructing ROM. In section 4, we describe the numerical strategy for the optimal feedback controller design with the stable manifold method. In section 5, we illustrate a numerical example to show the effectiveness of the proposed controller synthesis for the viscous Burgers' equation.

We end this section with some mathematical notations which are used throughout this paper.

Let X be real Hilbert Space. The inner product and the norm in X are denoted by $(\cdot,\cdot)_X$, $\|\cdot\|_X$ respectively. X'

denotes the dual space of X and $(\cdot,\cdot)_{X'\times X}$ is the duality pairing. For $0 < T \leq \infty$, $L^2(0,T;X)$ is defined by $\{w(t) \in X \text{ a.e. } t \in (0,T) : \int_0^T \|w(t)\|_X^2 dt < \infty\}$. We define the space $W(0,T;X) := \{\phi \in L^2(0,T;X) : \phi_t \in L^2(0,T;X')\}$ and we set $W_{loc}(0,\infty;X) := \bigcap_{T>0} W(0,T;X)$.

2. NONLINEAR SYSTEM AND OPTIMAL CONTROL PROBLEM

In this section, we define nonlinear distributed parameter systems and formulate the optimal control problem.

Let V and H be real separable Hilbert spaces with $V \subset H$, V dense in H, the injection of $V \subset H$ being compact. We consider a symmetric bilinear continuous form $a : V \times V \to \mathbb{R}$ and it is coercive, that is, there exist constant $\kappa > 0$ such that $a(v,v) \geq \kappa\|\phi\|_V^2$ for all $v \in V$. Let $N : V \to V'$ be a nonlinear continuous operator mapping satisfied $N(0) = 0$ and it's Frèchet derivative $N'(0) = 0$.

We consider the nonlinear evolution equation

$$\begin{aligned}\frac{d}{dt}(y(t),\varphi)_H + a(y(t),\varphi)\\ +(N(y(t)),\varphi)_{V',V} = (Bu(t),\varphi)_{V',V}\end{aligned} \tag{1a}$$

for all $\varphi \in V$ with initial condition

$$y(0) = y_0 \in H \tag{1b}$$

where $y(t) \in V$ is a state, $u(t) \in \mathbb{R}^m$ a control input and $B : \mathbb{R}^m \to V'$ a continuous linear operator.

Example 1. Let us present an example for (1a), which will be treated in Section 5 as a numerical example. With the domain $\Omega := (0,\ell)$, we consider the viscous Burgers' equation, which is written as

$$\frac{\partial y}{\partial t} = \nu\frac{\partial^2 y}{\partial x^2} - y\frac{\partial y}{\partial x} + B(x)u(t) \tag{2a}$$

with the boundary condition

$$y(0,t) = 0,\ \frac{\partial y}{\partial x}(\ell,t) = 0 \tag{2b}$$

and the initial state

$$y(x,0) = y_0(x). \tag{2c}$$

Here, $y(x,t)$ represents the state at the position x and the time t, $B(x)$ the function for distribution of the control input $u(t) \in \mathbb{R}^m$ and $\nu > 0$ is a positive constant.

To rewrite Burgers' equation in the weak form (1a 1b), we set $\mathcal{V} = \{\phi \in C^\infty(\Omega) : \phi(0) = 0\}$ and define V as the closure of $\mathcal{V}$ in well-known Sobolev space $H^1(\Omega)$ and H as the closure of $\mathcal{V}$ in Lebesgue space $L^2(\Omega)$. We define the bilinear form a as $a(v,w) = \nu(v_x,w_x)_H$ and nonlinear operator N as $N(v) = v_x v$, where $v,w \in V$. Considering $y(x,t)$ with boundary condition satisfy

$$\nu\int_\Omega y_{xx}(x,t)\varphi(x)dx = -a(y,\varphi)$$

for all $\varphi \in V$, the equations (2a-2c) can be written the form as (1a 1b) in abstract. For the functional analytical treatment of Burgers' equation, See Ly et al. (1997) for example.

We consider the control problem with the cost functional to be minimized, which is defined as

$$\mathcal{J}(y_0,u) := \int_0^\infty (\mathcal{Q}y(t),y(t))_H + u^T(t)Ru(t)dt \tag{3}$$

where Q is a positive semi-definite linear operator and R a positive definite matrix.

Problem 2. The optimal control problem is, given a $y_0 \in H$, to find the optimal control input u_{opt} that minimizes the cost functional $\mathcal{J}(y_0,u)$ over all trajectories of (1a) and (1b).

We approximate this problem to the finite-dimensional problem in section 4.

3. POD AND GALERKIN PROJECTION

In this section, we review POD and Galerkin projection. The POD gives, in some sense, an optimal orthogonal basis from a collected data set. The POD and related model reduction techniques have been widely applied in fluid mechanics. For more details, we refer the readers to, e.g., Holmes et al. (1998) and Kunisch et al. (2004).

Let $U = \{u_k : u_k \in X\}_{k=1}^n$ be a given ensemble set, where X is separable Hilbert space and n the number of elements of the set U. As the average of the image $f(U)$, where f is a scalar map, we define $\langle f(u_k)\rangle_k := (1/n)\sum_{k=1}^n f(u_k)$. In order to represent efficiently for the set U, we consider the optimal orthogonal basis for representation of the set U by linear combination.

First, we consider a single function φ for representation of the set U that maximizes the objective function:

$$\max_{\varphi\in X}\frac{\left\langle(u_k,\varphi)^2\right\rangle_k}{\|\varphi\|^2} \tag{4}$$

The optimal solution is the most parallel to the set U in the sense of mean square.

For the following argument, we define some operators. The linear bounded operator $\mathcal{U} : \mathbb{R}^n \to X$ is defined as $\mathcal{U}v = \sum_i v_i u_i$ for $v = [v_1,\ldots,v_n]^T \in \mathbb{R}^n$. Furthermore, the map $\mathcal{R} : X \to X$ and the matrix $\mathcal{K} \in \mathbb{R}^{n\times n}$ are given by $\mathcal{R} = (1/n)\mathcal{U}\mathcal{U}^*$, $\mathcal{K} = (1/n)\mathcal{U}^*\mathcal{U}$ respectively, where $\mathcal{U}^*$ is adjoint operator of $\mathcal{U}$, that is, $\mathcal{U}^*w = [(w,u_1),\ldots,(w,u_n)]^T$ for $w \in X$. $\mathcal{R}$ is bounded, non-negative and self-adjoint operator.

From the perturbation method for the above maximum problem, we get the following eigenvalue problem as necessary conditions for optimal solution.

$$\mathcal{R}\varphi = \lambda\varphi \tag{5}$$

Since the image of $\mathcal{R}$ is finite dimensional, $\mathcal{R}$ is a compact operator and there exist an optimal solution of (4). The maximum value corresponds with the maximum eigenvalue of (5). By Hilbert-Schmidt theorem, there exist orthonormal basis consisting of eigenvectors in (5) equation. The eigenvalues is real and non-negative, So we sort them such $\lambda_1 \geq \lambda_2 \cdots \geq 0$. The orthogonal basis from this procedure is called sometimes *empirical eigenfunctions* or *POD basis*.

Remark 3. The computation for integral eigenvalue problem (5) is, in general, highly time consuming. So it is common to obtain POD basis from not solving (5), but eigenvalue problem of the matrix $\mathcal{K}$. It is called method of Snapshots and we use it in the later computations.

Remark 4. The optimality of POD basis is considered as follow.

Let $u(t) \in X$ be time-dependent signal. We construct ensemble from *snapshots* of the signal u at $t = t_1, \dots, t_n$, that is, we set $U = \{u(t_i)\}_{i=1}^n$. We consider the decomposition of u by *any* orthogonal basis $\{\psi_j : \psi_j \in X\}$. The time coefficient in j-th mode is defined as $b_j(t) = (u(t), \phi_j)$. The average energy is given by

$$\left\langle \|u(t_k)\|_X^2 \right\rangle_k = \sum_{j=1}^{\infty} \left\langle b_j(t_k)^2 \right\rangle_k ,$$

and so the average energy in the j-th mode is given by $\left\langle b_j(t_k)^2 \right\rangle_k$.

Suppose that we have POD basis $\{\varphi_j\}$ and associated eigenvalues $\{\lambda_j\}$ from the ensemble U. Time coefficient in j-th POD mode is defined as $a_j(t) = (u(t), \varphi_j)$. Then POD basis have the following optimality. For every natural number $r \leq n$, we have

$$\sum_{j=1}^{r} \left\langle a_j(t_k)^2 \right\rangle_k = \sum_{j=1}^{r} \lambda_j \geq \sum_{j=1}^{r} \left\langle b_j(t_k)^2 \right\rangle_k .$$

For a proof we refer the reader to Holmes et al. (1998). This optimality means that the POD basis captures most energy than any other basis on average. To total energy, the ratio of the energy captured by from first to r-th POD basis is expressed as

$$\mathcal{E}(r) := \sum_{j=1}^{r} \lambda_j / \sum_{j=1}^{n} \lambda_j. \tag{6}$$

3.1 Galerkin projection and reduced order models

By using Galerkin projection with POD basis, we derive finite-dimensional systems of the infinite-dimensional one (1a, 1b).

First, we set $u(t) \equiv 0$ in (1a) and get the (simulated or experimental) response data of the system in order to extract dominant characteristics of the dynamics, that is, POD basis. Then, we set $U = \{u(t_i) : u(t_i) \in H\}_{i=1}^n$ and obtain POD basis $\{\varphi_i\}$. We determine the reduced-order r to have sufficient modes of the original dynamics, namely $\mathcal{E}(r) \approx 1$.

Then, We express y as a finite summation of time-dependent coefficients multiplied by POD basis:

$$y(t) = \sum_{i=1}^{r} a_i(t) \varphi_i. \tag{7}$$

We substitute (7) into the left-hand side of (1a) and take the inner product with each basis $\varphi_j (j = 1, \dots, r)$. Then the equation (1a) is reduced to the following finite-dimensional equation:

$$\dot{a} = A_r a + N_r(a) + B_r u \tag{8a}$$

where we definite vector a as $a(t) = [a_1(t), \dots, a_r(t)]^T \in \mathbb{R}^r$ and the matrices $A_r \in \mathbb{R}^{r \times r}$, $B_r \in \mathbb{R}^{r \times m}$ and the nonlinear function $N_r(a) \in \mathbb{R}^r$ are defined by $(A_r)_{ij} = -a(\varphi_i, \varphi_j)$, $(B_r)_{ij} = (b_i, \varphi_j)_{V', V}$, $(N_r(a))_i = (N(\sum_{k=1}^r a_k \varphi_k), \varphi_i)_{V', V}$ respectively, where $b_k : \mathbb{R} \to V'$ is the map derived from $Bu = \sum_{k=1}^m b_k u_k$ uniquely. To summarize linear and nonlinear terms, we set $f_r(a) := A_r a + N_r(a)$.

The initial value $a_0 = a(0)$ is determined by

$$(a_0)_i = (y_0, \varphi_i)_H, \; i = 1, \dots, r. \tag{8b}$$

The system consist of (8a) and (8b) is called reduced-order model (ROM). By Galerkin projection, the other orthogonal basis could lead the other reduced-order models. However, using POD basis, we are able to obtain very low dimensional models.

4. OPTIMAL CONTROL FOR REDUCED-ORDER MODEL

It is optional how to synthesis of controllers based on the ROM. However, considering the influence of the modeling error and the disturbances, it is desirable to design a closed-loop controller, not open-loop one.

In this section, we consider the synthesis of nonlinear optimal feedback controller for the ROM using by the stable manifold method.

First, using POD basis we approximate the first term of cost functional (3) as following:

$$\begin{aligned} (\mathcal{Q}y, y) &\simeq \left(\mathcal{Q} \sum_i^r a_i(t) \varphi_i, \sum_j^r a_j(t) \varphi_j \right) \\ &= \sum_{i,j} a_i(t) \left(\mathcal{Q} \varphi_i, \varphi_j \right) a_j(t) \\ &= a(t)^T Q a(t) \end{aligned}$$

where $Q_{ij} = (\mathcal{Q}\varphi_i, \varphi_j)$. Q is semi-positive definite matrix since $\mathcal{Q}$ is semi-positive definite. Thus, we get approximation of cost functional (3) as

$$J(a_0, u) := \int_0^{\infty} a^T(t) Q a(t) + u^T(t) R u(t) dt. \tag{9}$$

We consider the optimal control for the ROM with cost functional (9).

Problem 5. The optimal control problem is, given a a_0 by (8b), to find an optimal control input u_{opt} that minimizes the cost functional $J(a_0, u)$ over all trajectories of (8a) and (8b).

The optimal feedback controller of the above optimal regulator problem is given by

$$u(a) = -1/2 R^{-1} B_r^T p(a)$$

where $p = (\partial V / \partial a)^T$ and V is the stabilizing solution of the Hamilton-Jacobi (HJ) equation:

$$H(a, p) = p^T f_r(a) + a^T Q a - \frac{1}{4} p^T B_r R^{-1} B_r^T p = 0. \tag{10}$$

A solution V of HJ equation is said to be the stabilizing solution if $p(0) = 0$ and 0 is an asymptotically stable equilibrium of the vector field $f_r(a) - 1/4 B_r R^{-1} B_r^T p$.

HJ equation is a first order partial differential equation and it is difficult to derive analytical solutions except for special cases. Although various approximation techniques have been proposed, they are poor in terms of computational cost and accuracy in practical. However, the stable manifold method is the efficient numerical method and has been reported its usefulness in various practical nonlinear systems (Sakamoto (2013)). We obtain the numerical stabilizing solution of HJ equation using the stable manifold method and construct a nonlinear feedback controller. See Sakamoto and van der Schaft (2008) for the detailed of the stable manifold method.

The stable manifold method constructs the partial differential function of the stabilizing solution V from the origin. Therefore the Riccati algebraic equation, which is obtained from linearizing the HJ equation, plays an important role in the stable manifold method. The following theorem gives the relation of HJ equations and Riccati equations.

Theorem 6. (van der Schaft (1991)) We consider the Riccati equation:

$$PA_r + A_r^T P - PB_r R^{-1} B_r^T P + Q = 0 \tag{11}$$

which is obtained by the linearization of HJ equation (10). The stabilizing solution of HJ equation (10) exists in the neighborhood of origin if the stabilizing solution of (11) exist, i.e., there exists the solution P satisfying $A_r - B_r R^{-1} B_r^T P$ be stable.

The stabilizing solution of the Riccati equation is necessary for not only guaranteeing the existence of the stabilizing solution of the HJ equation, but also solving the HJ equation numerically by the stable manifold method. We will ensure the existence of the stabilizing solution of the Riccati equation (11) using the following well-known lemma.

Lemma 7. (Kučera (1973)) The stabilizing solution P of the Riccati equation (11) exists and it is the unique nonnegative solution if and only if (A_r, B_r) is stabilizable and $(Q^{1/2}, A_r)$ detectable.

Theorem 8. There exists the stabilizing solution $P \geq 0$ of the Riccati equation (11) uniquely.

Proof. First, we will show that A_r is stable. For any vector $x = [x_1, \dots, x_r]^T \in \mathbb{R}^r$, the following inequality hold:

$$\begin{aligned} -x^T A_r x &= \sum_{i,j} x_i a(\varphi_i, \varphi_j) x_j \\ &= a\Big(\sum_i x_i \varphi_i, \sum_j x_j \varphi_j\Big) \\ &\geq \kappa \Big\| \sum_i x_i \varphi_i \Big\|_V . \end{aligned}$$

The last inequality comes from the coerciveness of bilinear form a. Since POD basis $\{\varphi_i\}$ are linear independent, $x^T A_r x$ is negative for all $x \neq 0$, which implies that symmetric matrix A_r is negative definite, that is, A_r is stable. Since A_r is stable, for any matrix B_r and $Q^{1/2}$, (A_r, B_r) is stabilizable and $(Q^{1/2}, A_r)$ detectable. Lemma 7 gives the completeness of the proof.

From Theorem 8, we are able to apply the stable manifold method to Problem 5 every time.

5. APPLICATION FOR VISCOUS BURGERS' EQUATION

In this section, we consider the optimal distributed control of the viscous Burgers equation, which was appeared in Example 1. We confirm numerically that the nonlinear controller obtained by POD-Galerkin model reduction and the stable manifold method decreases the cost functional value in comparison to the uncontrolled case. In the latter of this section, we compute the cost functional values with varying initial states. For a comparison, We use a linear optimal controller obtained from the stabilizing solution of the Riccati equation (11).

There are a number of researches on the control of the viscous Burgers' equation. See Efe and Özbay (2004), Glowinski et al. (2008) and referenced literatures in them. The control of Burgers equation using POD is considered in Efe and Özbay (2004); Kunisch and Volkwein (1999); Kunisch et al. (2004); Smaoui (2005). Especially, Kunisch and Volkwein (1999) considers of both open-loop and closed-loop optimal distributed control of Burgers' equation. However, they construct feedback controller interpolating of open-loop controls for each state space point. On the other hand, We solve the HJ equation directly.

We set the length $\ell = 3$ and the positive constant $\nu = 0.1$ in the following computations and the distribution function of the control inputs $B(x) = [b_1(x), b_2(x), b_3(x)]$ is pictured in Fig. 1.

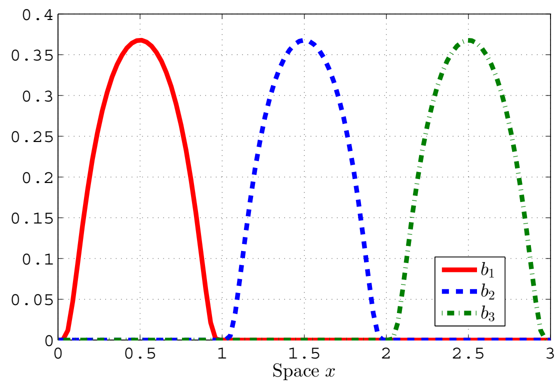

Fig. 1. Distribution of inputs

Then, we give a discontinuous initial condition as following:

$$y_0(x) = \begin{cases} 0 \ (0 \leq x \leq 1) \\ c \ (1 \leq x \leq 3) \end{cases} \tag{12}$$

where $c = 4$, which will be varied latter.

First, we simulate the uncontrolled Burgers' equation by finite element method with first-order splines and 100 elements. Fig. 2 shows the free response and the advection occurs due to the nonlinear term.

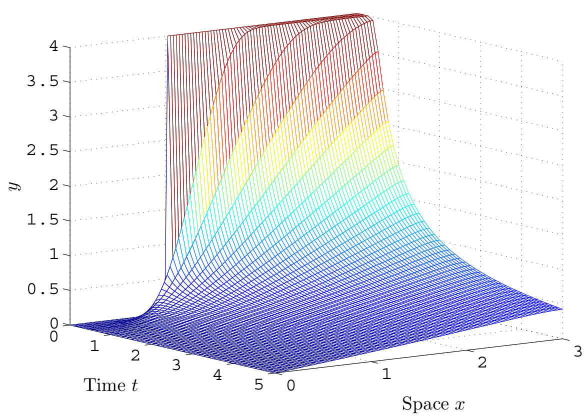

Fig. 2. Response of free Burgers' equation

We carry out POD from the snapshots of the solution and the first three POD basis are displayed in Fig. 3. In terms of the ratio (6), we get $\mathcal{E}(1) = 0.976, \mathcal{E}(2) = 0.994$.

Fig. 4 shows the time coefficients $a_i(t)(i = 1, \ldots, 5)$ corresponding to each POD basis. We confirm that the first POD basis depict the dominant dynamics

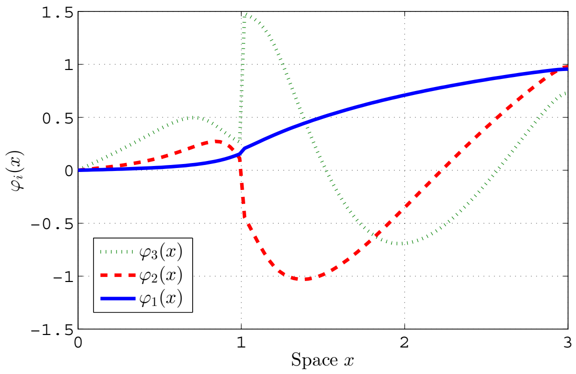

Fig. 3. Primal three POD bases

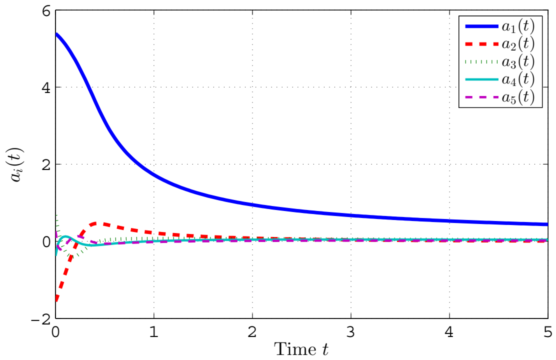

Fig. 4. Response of POD modes in free Burgers' equation

We set the reduced-order $r = 2$ and obtain the ROM of Burgers' equation. The concrete form of the ROM, see Kunisch and Volkwein (1999), Smaoui (2005). Then, we consider the optimal control problem with $\mathcal{Q} = I$ and $R_{i,j} = (b_i, b_j)_H = 0.0665\delta_{i,j}$, where I and $\delta_{i,j}$ represents identity operator and Kronecker delta respectively. The initial state of ROM is given as $a_0 = [5.3892, -1.5593]^T$.

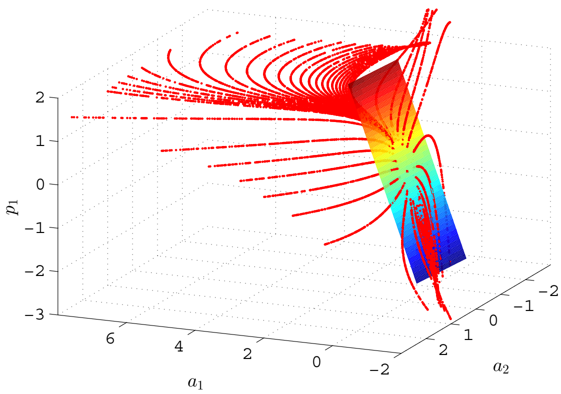

Fig. 5. Stable manifold and linear solution

Fig. 5 shows the part of the stable manifold of associated Hamiltonian system in the optimal control problem and the stabilizing solution of the Riccati equation. In this figure, we can verify the stable manifold contacts the hyperplane, that is the solution of the Riccati equation, at the origin and pass through near the initial state of ROM a_0. By the manifold, we construct the optimal feedback controller using 7th order polynomial fitting. Fig. 6 depicts the time response of the control of ROM and the states converge to zero.

We implement the controller to the original system and simulate the time response by FEM. Fig. 7 shows the response of the state y and Fig. 8 the input u.

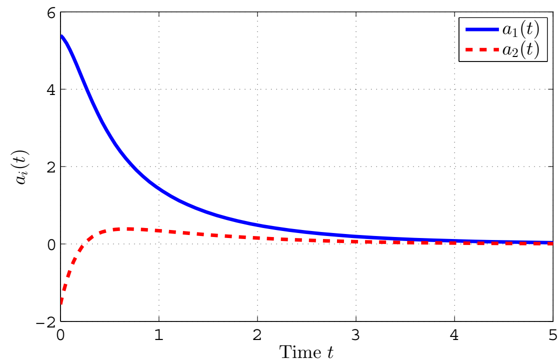

Fig. 6. Response of controlled reduced order model

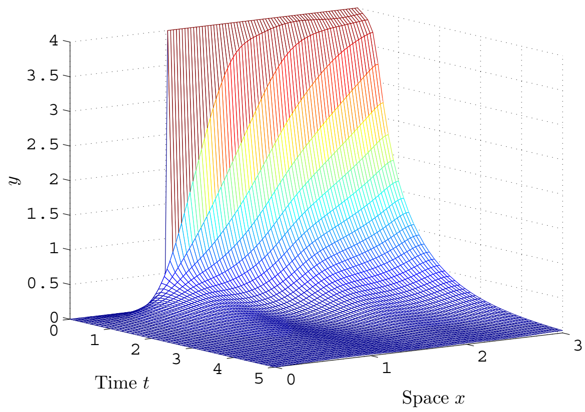

Fig. 7. Response of controlled Burgers' equation

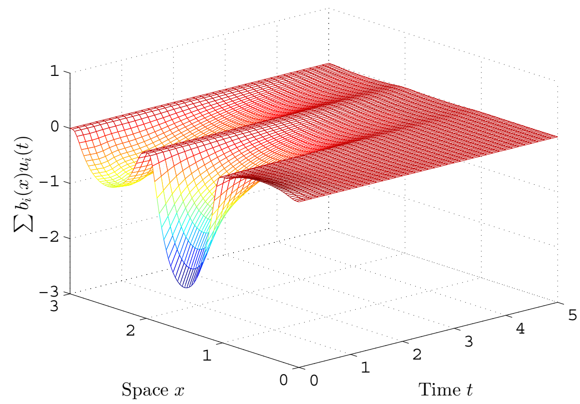

Fig. 8. Inputs of controlled Burgers' equation

In order to analyze the controlled response in Fig. 7, we observe it in each POD modes in Fig. 9. It shows that the first and second POD modes converge to zero similarly for the ROM and the ignored POD modes remain small sufficiently.

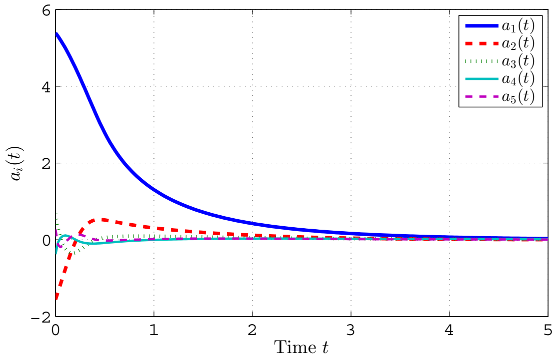

Fig. 9. Response of POD modes in controlled Burgers' equation

We summarize the cost functional values of each models and controllers in the Table 1. As a comparison, we also display the cost functional values of the responses by the linear controller, constructing from the stabilizing solution of the Riccati equation.

It can be seen that, in both linear and nonlinear, the cost functional value with original model is larger than the one with ROM. It is due to the model error between Burgers' equation and the ROM. It can be confirmed, however, that the cost functional value with nonlinear controller for Burgers' equation is smaller than one in uncontrolled case, whereas one with linear controller is larger than it. Thus, in term of the cost functional, linear optimal controller results in lower performance than uncontrolled case.

Table 1. Values of cost functions at $c = 4$.

Model	Controller	Cost funtional	Value
Burgers	None	Eq. (3)	16.894
Burgers	Nonlinear	Eq. (3)	13.383
Burgers	Linear	Eq. (3)	17.955
ROM	Nonlinear	Eq. (9)	13.723
ROM	Linear	Eq. (9)	17.987

Finally, with the controller obtained above, we calculate the cost functional values for varying the c, which is the height of initial state (12).

Fig. 10 depicts the cost functional values for varying c from 0 to 6.5 with 0.1 interval in linear controlled, nonlinear controlled and uncontrolled cases. It can be seen that the nonlinear controller has robust performance for the initial state. When c is small sufficiently, the effect of nonlinear term of Burgers' equation is insignificant and linear controller performs as nonlinear controller. However, As c increases, the cost functional value with linear controller grows rapidly and finally exceeds one in uncontrolled case. On the other hand, it is observed that the cost functional value with nonlinear control is smaller than uncontrolled one nonetheless c increases. Thus, nonlinear controller performs proper on more wide region than linear one.

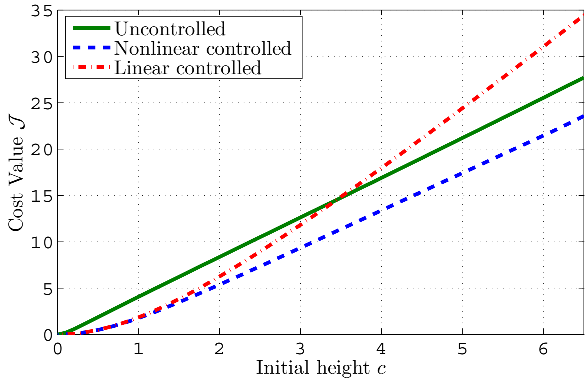

Fig. 10. Values of cost function with varying c.

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