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Improving $\mathcal{L}_2$ Gain Performance of Linear Systems by Reset Control ⋆

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Abstract: In this paper, new Lyapunov-based reset rules are constructed to improve $\mathcal{L}_2$ gain performance of linear-time-invariant (LTI) systems. By using the hybrid system framework, sufficient conditions for exponential and finite gain $\mathcal{L}_2$ stability are presented. It is shown that the $\mathcal{L}_2$ gain of the closed loop system with resets can be improved compared with the base system. Numerical example supports our results.

Keywords: Hybrid system, Lyapunov-based reset rules, $\mathcal{L}_2$ gain, reset controller

1. INTRODUCTION

The $\mathcal{L}_2$ gain is an important performance index in control systems, as it captures the disturbance attenuation ability that is needed in many engineering applications. In this paper, reset controllers are explored to improve this performance, showing the potential of reset control schemes.

Reset controllers were first proposed in Clegg [1958], where the so-called Clegg integrator was introduced. The Clegg integrator was generalized to First Order Reset Element (FORE) in Horowitz and Krishnan [1975]. In this early work, it was suggested that reset controllers can overcome some fundamental limitations or improve transient performance of linear control systems. Beker et al. [2001] shown the first example that non-overshoot performance can be achieved by using a reset controller, whilst it is impossible to do so by any LTI controller. The FORE with zero crossing reset conditions were used in many papers to improve transient performance or stabilize systems, see, for example, Zheng et al. [2000], Beker et al. [2004], Banos et al. [2011] and references therein.

In Zaccarian et al. [2005], a new model of FORE under the hybrid systems framework developed by Goebel and Teel [2006] was proposed, with the hybrid framework, and it allows jumps on more complicated sets. Furthermore, temporal regulation is introduced to avoid Zeno solutions. With this new model, the exponential stability and input-output stability were presented, leading to a systematic design tool for reset controllers. The $\mathcal{L}_2$ stability of reset control systems was first presented in Nešić et al. [2005]. Linear Matrix Inequality(LMI) conditions were established in Zaccarian et al. [2005] to estimate the $\mathcal{L}_2$ gain and piecewise quadratic Lyapunov functions were used to get a tighter $\mathcal{L}_2$ gain estimation. Similar approaches were then used in Loquen et al. [2007] for reset systems with input saturation and in Loquen et al. [2008] for reset systems with nonzero references to establish $\mathcal{L}_2$ stability. In Zaccarian et al. [2011] the analytical and numerical Lyapunov functions were provided to prove stability and $\mathcal{L}_2$ gain performance of FORE control systems. Furthermore, the LMI-based analysis method was used to determine the performance of Single-Input-Single-Output (SISO) reset systems in both $\mathcal{L}_2$ gain and $\mathcal{H}_2$ sense as discussed in Aangenent et al. [2010]. In Prieur et al. [2011], by adding a hybrid loop in the control systems, reset controllers have shown the potential to maximize the decay rate and reduce overshoot of the systems based on full state feedback. In Fichera et al. [2012], the performance improvement results were extended to output feedback case, and these results were summarized in Prieur et al. [2013], with a systematic analysis.

The improved reset rules were proposed in Nešić et al. [2011], by tilting the boundary between the flow set and the jump set, and the strictly decreasing Lyapunov functions during jumps were constructed. For planar reset control systems, necessary and sufficient conditions for the exponential stability and finite $\mathcal{L}_2$ gain were presented, and rigorous proofs about improving $\mathcal{L}_2$ gain performance and $\mathcal{L}_2$ gain trends were given. The obtained results can be used to stabilize high order minimum-phase relative degree-one plants. However, these results cannot be directly applicable for plant with a higher relative degree, because the reset conditions only depend on the input and output of the plants.

This paper extends the $\mathcal{L}_2$ gain performance improvement result in Nešić et al. [2011] to plants with higher relative degree. The Lyapunov-based reset rules motivated from Prieur et al. [2013] are proposed. The main contributions of this work are two-fold. On one hand, with the state

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feedback and Lyapunov-based reset rules, sufficient conditions for exponential and finite $\mathcal{L}_2$ gain stability are provided. On the other hand, it is shown rigorously that the $\mathcal{L}_2$ gain with resets must be less than or equal to the $\mathcal{L}_2$ gain of its *base linear system* (system without resets). This clearly shows the advantage of reset controllers. A simulation example is used to demonstrate our results.

The paper is organized as follows. Section 2 states the problem formulation. The controller design is provided in Section 3 and Section 4 presents the stability and performance improvement results. A numerical example is given in Section 5 and Section 6 concludes this paper.

Notations: The set of real numbers is denoted as $\mathbb{R}$. The set of integers is denoted as $\mathbb{N}$. For any $x \in \mathbb{R}$, $|x|$ denotes the Euclidean norm of x. For a matrix $A \in \mathbb{R}^{n\times n}$, $|A|$ is the induced norm. Given a state variable x of a system with jumps, its derivative with respect to time (which is defined almost everywhere) is denoted as $\dot{x}$. At jump instants, we denote the value of state after the jump by x^+ and the value of the state before the jump simply by x. For any positive definite matrix P ($P > 0$), $\lambda_{\max}(P)$ ($\lambda_{\min}(P)$) denotes the maximum (minimum) eigenvalue of P. I_n (0_n) denotes the n dimensional identity (zero) matrix.

2. PRELIMINARIES AND PROBLEM FORMULATION

Consider the following LTI SISO plant with some disturbances

$$\Sigma_p : \begin{cases} \dot{x}_p = A_p x_p + B_p u + B_d d, \\ y = C_p x_p. \end{cases} \tag{1}$$

where $x_p \in \mathbb{R}^{n_p}$ is the state, y is the output, u is the control input, $d \in \mathbb{R}$ is the disturbance, and (A_p, B_p, C_p, B_d) are matrices with appropriate dimensions.

The controller for the plant (1) takes the following form:

$$\Sigma_c : \begin{cases} \dot{x}_c = A_c x_c + B_c x_p, \\ u_c = C_c x_c. \end{cases} \tag{2}$$

where $x_c \in \mathbb{R}^{n_c}$ is the controller state, x_p is the state from the plant (1), $u_c \in \mathbb{R}$ is the output of the controller, and (A_c, B_c, C_c) are matrices with appropriate dimensions.

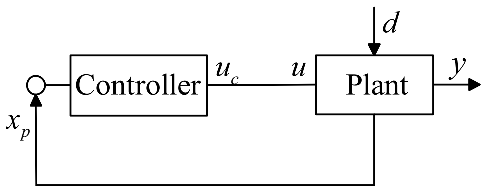

Fig. 1. The control structure

The connection between the plant (1) and the controller (2) is shown in Figure 1, where

$$u = u_c. \tag{3}$$

Hence, the closed loop system can be represented as

$$\Sigma_{cl} : \begin{cases} \dot{x} = Ax + Bd, \\ y = Cx. \end{cases} \tag{4}$$

where $x = [x_p^T \;\; x_c^T]^T \in \mathbb{R}^{n_p+n_c} = \mathbb{R}^n$, and

$$A = \begin{bmatrix} A_p & B_p C_c \\ B_c & A_c \end{bmatrix}, \; B = \begin{bmatrix} B_d \\ 0 \end{bmatrix}, \; C = [C_p \;\, 0]. \tag{5}$$

2.1 Hybrid systems

In this work, the design of reset controllers is based on hybrid systems framework proposed in Goebel et al. [2012]. This section thus gives a brief introduction to hybrid systems. A class of generic nonlinear hybrid systems with temporal regularization can be represented as

$$\left.\begin{aligned} \dot{\tau} &= 1 \\ \dot{x} &= f(x, w) \end{aligned}\right\} \text{ If } x \in \mathcal{F} \text{ or } \tau \le \rho \tag{6a}$$

$$\left.\begin{aligned} \tau^+ &= 0 \\ x^+ &= g(x) \end{aligned}\right\} \text{ If } x \in \mathcal{J} \text{ and } \tau \ge \rho \tag{6b}$$

where $x \in \mathbb{R}^n, w \in \mathbb{R}^m, \tau \in \mathbb{R}_{\ge 0}$. In order to avoid the presence of Zeno solutions (see Zaccarian et al. [2005]), an extra jump rule is added to the hybrid model, where a given time interval ρ has to expire before the next jump occurs. This type of jump rule is called "temporal regularization". With this reset model, the time interval between two jumps is forced to be at least ρ, hence, the Zeno solution is excluded. $\mathcal{F}$ is the flow set and $\mathcal{J}$ is the jump set, $\mathcal{F} \cup \mathcal{J} = \mathbb{R}^n$.

A *hybrid time domain* is defined as a subset of $\mathbb{R}_{\ge 0} \times \mathbb{N}$. It is the union of infinitely many intervals of the form $[t_j, t_{j+1}] \times \{j\}$, where $0 = t_0 \le t_1 \le \cdots$, or of finitely many such intervals, with the last one possibly of the form $[t_j, \infty) \times \{j\}$. A hybrid signal is a function defined on a hybrid time domain. Denote $\xi := (x, \tau)$. A hybrid signal $\xi : \mathrm{dom}(\xi) \to \mathbb{R}^{n+1}$ is a *hybrid arc* if $\xi(t, j)$ is locally absolutely continuous for each j. A hybrid signal $w : \mathrm{dom}(w) \to \mathbb{R}^m$ is called a hybrid input.

Given any function $\eta(\cdot,\cdot)$ defined on the hybrid domain $\mathrm{dom}(\eta)$, and any $(t, j) \in \mathrm{dom}(\eta)$, denote

$$\int_0^t \eta(s)ds := \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} \eta(s, i)ds + \int_{t_j}^{t} \eta(s, j)ds.$$

Given any hybrid signal $\zeta(\cdot,\cdot)$ and $p \in [1, \infty)$, its $\mathcal{L}_p$ norm can be defined as $\|\zeta\|_p := \lim\limits_{t\to\infty} \left(\int_0^t |\zeta(s)|^p \, ds\right)^{1/p}$. We say that $\zeta \in \mathcal{L}_p$ whenever $\|\zeta\|_p < \infty$. Define $\mathcal{L}_\infty$ norm as $\|\zeta\|_\infty := \text{ess.sup}_{(t,j)\in\mathrm{dom}(\zeta)} |\zeta(t, j)|$, and say that $\zeta \in \mathcal{L}_\infty$ whenever $\|\zeta\|_\infty < \infty$. The origin of the x dynamics of system (6) with $w = 0$ is *exponentially stable* if there exist constants $m, l > 0$ such that given any initial condition $(x_0, \tau_0) \in \mathbb{R}^n \times \mathbb{R}_{\ge 0}$, the bound $|x(t, j)| \le \beta(|x_0|, t), \forall (t, j) \in \mathrm{dom}(x)$ holds for all solutions ($\beta(\cdot,\cdot)$ is class $\mathcal{KL}$ function (see Nešić et al. [2011])). The system (6) is *finite gain $\mathcal{L}_p$ stable from w to x* (respectively, *finite gain $\mathcal{L}_p$ to $\mathcal{L}_\infty$ stable from w to x*), if there exist constants $\gamma_p, \gamma_0 > 0$ (respectively, $\gamma_{p,\infty}, \gamma_0 > 0$) such that for any initial condition $(x_0, \tau_0) \in \mathbb{R}^n \times \mathbb{R}_{\ge 0}$ and any $w \in \mathcal{L}_p$

$$\|x\|_p \leq \gamma_0 \cdot |x_0| + \gamma_p \cdot \|w\|_p \tag{7}$$

(respectively, $|x(t,j)| \leq \gamma_0 |x_0| + \gamma_{p,\infty} \cdot \|w\|_p, \forall (t,j) \in \mathrm{dom}(x)$).

2.2 Problem formulation

It is known that introducing resets may improve the performance of linear control systems, even though a rigorous proof of this fact is not available for general linear systems. Some experimental results (see, for example, Zheng et al. [2000], Guo et al. [2009], Fernandez et al. [2011], Zhao et al. [2013]) and theoretical results (such as Nešić et al. [2011]) have been proposed for some classes of systems.

In this paper, we propose the following reset controller with temporal regularization, which corresponds to the linear controller (2) plus reset rules

$$\left.\begin{aligned} \dot{\tau} &= 1 \\ \dot{x}_c &= A_c x_c + B_c x_p \end{aligned}\right\} \text{ If } x \in \mathcal{F} \text{ or } \tau \leq \rho \tag{8a}$$

$$\left.\begin{aligned} \tau^+ &= 0 \\ x_c^+ &= A_{cr} x_c \end{aligned}\right\} \text{ If } x \in \mathcal{J} \text{ and } \tau \geq \rho \tag{8b}$$

where $A_{cr} \in \mathbb{R}^{n_c \times n_c}$, $\mathcal{F}$ and $\mathcal{J}$ are flow set and jump set, $\mathcal{F} \cup \mathcal{J} = \mathbb{R}^n$. Then, the closed loop consisting of plant (1), reset controller (8) and (3) (see Figure 1) can be written in the following form

$$\left.\begin{aligned} \dot{\tau} &= 1 \\ \dot{x} &= Ax + Bd \end{aligned}\right\} \text{ If } x \in \mathcal{F} \text{ or } \tau \leq \rho \tag{9a}$$

$$\left.\begin{aligned} \tau^+ &= 0 \\ x^+ &= A_r x \end{aligned}\right\} \text{ If } x \in \mathcal{J} \text{ and } \tau \geq \rho \tag{9b}$$

where A, B are defined in (5) and

$$A_r = \begin{bmatrix} I_{n_p} & 0 \\ 0 & A_{cr} \end{bmatrix}. \tag{10}$$

This paper focus on solving the following problem:

Problem: Consider the closed loop system (9), design reset rules $(\mathcal{F}, \mathcal{J}, A_{cr})$ in the reset controller (8) such that the $\mathcal{L}_2$ gain γ of the closed loop system is less than or equal to γ_L, where γ_L is a prescribed $\mathcal{L}_2$ gain of the system without resets.

3. RESET RULE DESIGN

In Nešić et al. [2011], a rigorous proof was given to illustrate that introducing resets in linear control systems can improve $\mathcal{L}_2$ gain performance for first order plant. And up to now, there are no other similar results in the literatures. It is expected that the result should hold for higher order plants, however, the proof techniques cannot be directly applicable for higher order plants.

In this paper, we design new reset controllers in order to extend the results in [Nešić et al., 2011, Theorem 4] to a class of more general SISO LTI systems. Consider the closed loop system (9). First, design the matrices (A_c, B_c, C_c) of linear controller (2) for the base linear system (4) such that a prescribed $\mathcal{L}_2$ gain γ_L is satisfied. Then, based on the designed linear controller, we propose a new Lyapunov based reset rule such that the reset controller can improve the $\mathcal{L}_2$ gain performance.

3.1 Linear controller design with a prescribed $\mathcal{L}_2$ gain γ_L

First step is to design a stabilizing controller for the base linear system (4) with a prescribed $\mathcal{L}_2$ gain γ_L. Suppose that $P \in \mathbb{R}^{n \times n}$ is symmetric positive definite. The following block matrices are introduced

$$\begin{aligned} \Sigma &= \begin{bmatrix} I_{n_p} & 0_{n_p \times n_c} \end{bmatrix}, \Sigma_0 = \begin{bmatrix} 0_{n_p \times n_c} \\ I_{n_c} \end{bmatrix}, \\ \Sigma_c &= \begin{bmatrix} 0_{n_c \times n_p} & I_{n_c} \\ I_{n_p} & 0_{n_p \times n_c} \end{bmatrix}. \end{aligned} \tag{11}$$

A simple calculation leads to

$$PA = P\Sigma^T A_p \Sigma + P\Sigma_0 \begin{bmatrix} A_c & B_c \end{bmatrix} \Sigma_c + P\Sigma^T B_p C_c \Sigma_0^T \tag{12}$$

Denote $\Phi_c = [A_c \ B_c]$, the following results show a sufficient condition to ensure that a prescribed $\mathcal{L}_2$ gain is satisfied. The proof is quite simple, thus the proof is omitted.

Theorem 1. Let γ_L be an arbitrary positive constant. For the closed loop of the base linear system (4), if there exists $P > 0$ and (A_c, B_c, C_c) such that the following bilinear matrix inequality (BMI) holds

$$\begin{bmatrix} \Psi & PB \\ B^T P & -\gamma_L^2 \end{bmatrix} < 0 \tag{13}$$

where

$$\begin{aligned} \Psi =& P\Sigma^T A_p \Sigma + P\Sigma_0 \Phi_c \Sigma_c + P\Sigma^T B_p C_c \Sigma_0^T \\ &+ (P\Sigma^T A_p \Sigma + P\Sigma_0 \Phi_c \Sigma_c + P\Sigma^T B_p C_c \Sigma_0^T)^T + C^T C \end{aligned} \tag{14}$$

then, this closed loop system satisfies the prescribed $\mathcal{L}_2$ gain γ_L if the controller gain matrices defined in (2) are as follows

$$A_c = \Phi_c \Sigma^T, B_c = \Phi_c \Sigma_0^T$$

and C_c is obtained from the BMI.

Remark 1. The condition (13) is BMI, it is usually hard to get a global solution to a BMI problem. But according to the path-following algorithm (see Hassibi et al. [1999]), if an appropriate initial guess for P is obtained, then, (13) is transformed into a LMI problem, which can be easily solved by some standard LMI Toolbox. ∘

Remark 2. Note that when the BMI (13) is satisfied, the following conditions hold for all $x \neq 0$:

$$x^T(PA + A^T P)x < 0, \ d = 0 \tag{15a}$$

$$x^T(PA + A^T P)x + x^T(PB + B^T P)d + |y|^2 - \gamma_L^2 |d|^2 < 0, \ d \neq 0. \tag{15b}$$

This indicates that the closed loop matrix A is Hurwitz.

3.2 Reset rule design

Assume that the BMI (13) is satisfied for the base linear system, next, the reset rule will be designed based on stability properties of the base linear system. In order to achieve this objective, the matrix P coming from (13) can be partitioned as:

$$P = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{12}^T & P_{22} \end{array}\right] \tag{16}$$

where $P_{11} \in \mathbb{R}^{n_p \times n_p}, P_{12} \in \mathbb{R}^{n_p \times n_c}, P_{22} \in \mathbb{R}^{n_c \times n_c}$ and (P_{11}, P_{22}) are symmetric positive definite matrices (see Zhou et al. [1996] for more details).

Now, based on the above analysis, we design the following flow set and jump set

$$\begin{aligned} \mathcal{F} &:= \{x \in \mathbb{R}^n : x^T M x \geq 0\} \\ \mathcal{J} &:= \{x \in \mathbb{R}^n : x^T M x \leq 0\} \\ M &= \begin{bmatrix} \kappa I_{n_p} & -P_{12} \\ -P_{12}^T & 0 \end{bmatrix} \end{aligned} \tag{17}$$

where $\kappa > 0$ is a constant, P_{12} is from P in (16). In order to make sure that all states in controller (8) are reset to zero at jumps, the reset map matrix A_{cr} is set to zero so that A_r in (9) can be re-written as

$$A_r = \begin{bmatrix} I_{n_p} & 0 \\ 0 & 0_{n_c} \end{bmatrix}$$

4. STABILITY AND PERFORMANCE

In this section, we characterize the exponential and finite $\mathcal{L}_2$ gain stability properties of the reset system (9), and prove the performance improvement of $\mathcal{L}_2$ gain.

Theorem 2. (Stability conditions) Consider the reset control system (9). Assume that there exist $P > 0$ and (A_c, B_c, C_c) such that BMI (13) is satisfied for the base linear system. Then, the proposed reset control laws (8) and (17) ensues that the following statements hold.

(1) For any $\kappa > 0, \rho > 0$, the origin of the x dynamics with $d = 0$ is exponentially stable.
(2) For any $\kappa > 0, \rho > 0$, the system is finite gain $\mathcal{L}_2$ and $\mathcal{L}_2$ to $\mathcal{L}_\infty$ stable from d to y.

Proof. Item (1): For the purpose of proving exponential stability, consider the disturbance $d = 0$. Let $V(x) = x^T P x$, where P comes from the BMI (13). The stability properties of the base linear system ensures that there must exist $\lambda > 0$ such that $\dot{V}(x) \leq -\lambda V(x)$ holds.

Now, consider reset system (9), note that the system dynamics coincide with its base linear system on the continuous dynamics, which implies that the condition $\dot{V}(x) \leq -\lambda V(x)$ holds for $x \in \mathcal{F}$ or $\tau \leq \rho$. Denote t_k ($k \in \mathbb{N}_{\geq 0}$) as the k^{th} jump instant, and $V(x) = V(x(t,k))$, $(t,k) \in \text{dom}(x) := [t_0, t_1] \times 0 \cup [t_1, t_2] \times 1 \cup \cdots$, then, we have

$$\begin{aligned} &V(x(t,k)) \leq \exp(-\lambda(t - t_k))V(x(t_k, k)) \\ &\quad \forall\, x \in \mathcal{F} \text{ or } \tau \leq \rho. \end{aligned} \tag{18}$$

When jump occurs, noticing that $x^T M x \leq 0$ is satisfied (see (17)) in the jump set, it follows that

$$\begin{aligned} &x^+ = A_r x \\ &\Rightarrow x(t_k, k) = A_r x(t_k, k-1) \\ &\Rightarrow V(x(t_k, k)) - V(x(t_k, k-1)) \\ &\quad = x^T(t_k, k-1)(A_r^T P A_r - P)x(t_k, k-1) \\ &\quad \leq x^T(t_k, k-1)(A_r^T P A_r - P - \mu M)x(t_k, k-1) \end{aligned} \tag{19}$$

for $x \in \mathcal{J}$ and $\tau \geq \rho$. Noting that

$$\begin{aligned} &A_r^T P A_r - P - \mu M \\ &= \begin{bmatrix} 0_{n_p} & -P_{12} \\ -P_{12}^T & -P_{22} \end{bmatrix} - \mu \begin{bmatrix} \kappa I_{n_p} & -P_{12} \\ -P_{12}^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\mu\kappa I_{n_p} & -P_{12}(1-\mu) \\ -P_{12}^T(1-\mu) & -P_{22} \end{bmatrix}. \end{aligned} \tag{20}$$

in which the condition $A_r P A_r - P - \mu M < 0$ is evidently satisfied by selecting $\mu = 1$. Noting that P_{22} is symmetric positive definite matrix, it follows that

$$V(x^+) < V(x), \quad \forall\, x \in \mathcal{J} \text{ and } \tau \geq \rho. \tag{21}$$

From (18) and (21), we can obtain that

$$V(x(t,k)) \leq \exp(-\lambda(t - t_k))V(x(t_k, k)), \forall x \in \text{dom}(x). \tag{22}$$

This leads to the following bound

$$\begin{aligned} &V(x(t,k)) \leq \exp(-\lambda(t - t_0))V(x(t_0, 0)) \\ &\Rightarrow \lambda_{\min}(P)\,|x(t,k)|^2 \leq V(x(t,k)) \\ &\quad \leq \exp(-\lambda(t - t_0))V(x(t_0, 0)) \\ &\quad \leq \exp(-\lambda(t - t_0))\lambda_{\max}(P)\,|x(t_0, 0)|^2 \\ &\Rightarrow |x(t,k)| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \exp\left(-\frac{1}{2}\lambda(t - t_0)\right)|x(t_0, 0)|. \end{aligned} \tag{23}$$

Hence, we conclude that the origin of x dynamics of system (9) is exponentially stable.

Item (2): Consider $d \neq 0$. Note that the system dynamics coincide with its base linear system on continuous dynamics. Hence, by applying Theorem 1, the condition (15b) can be represented in hybrid time domain as follows

$$\dot{V}(x(t,k)) \leq -|y(t,k)|^2 + \gamma_L^2\,|d(t,k)|^2 \tag{24}$$

for almost all $t \in [t_k, t_{k+1}]$. Consider now any $(t,k) \in \text{dom}(x)$, and denote for simplicity $t_{k+1} = t$. Then, integrating (24) gives

$$\begin{aligned} 0 \leq &-V(x(t_{j+1}, j)) + V(x(t_j, j)) \\ &- \int_{t_j}^{t_{j+1}} |y(s,j)|^2\, ds \\ &+ \gamma_L^2 \int_{t_j}^{t_{j+1}} |d(s,j)|^2\, ds, \quad j = 0, 1, \cdots, k \end{aligned} \tag{25}$$

Summing up all the above equations leads to the following bound for all $(t,k) \in \text{dom}(x)$:

$$
\begin{aligned}
V(x(t,k)) \leq V(x(t_0,0)) &- \sum_{i=0}^{k} \int_{t_j}^{t_{j+1}} |y(s,j)|^2 \, ds \\
&+ \gamma_L^2 \sum_{i=0}^{k} \int_{t_j}^{t_{j+1}} |d(s,j)|^2 \, ds \\
\leq V(x(t_0,0)) &- \sum_{i=0}^{k} \int_{t_j}^{t_{j+1}} |y(s,j)|^2 \, ds + \gamma_L^2 \, \|d\|_2^2
\end{aligned} \tag{26}
$$

Because $V(x(t,k)) \geq \lambda_{\min}(P) \, |x(t,k)|^2$, the bound (26) guarantees the finite $\mathcal{L}_2$ to $\mathcal{L}_\infty$ bound with finite gain $\gamma_{2,\infty} = (\gamma_L^2/(\lambda_{\min}(P) \cdot |C^T C|)^{1/2}$ (where C satisfies $y = Cx$), and it also guarantees the finite $\mathcal{L}_2$ gain with $\gamma_2 = \gamma_L$. This completes the proof. □

Remark 3. The proof of Theorem 3 shows that parameters (κ, ρ) are required to be positive in the reset controller (9), but they may not be small. ○

The main result of this paper shows the $\mathcal{L}_2$ gain performance improvement.

Theorem 3. ($\mathcal{L}_2$ gain estimate) Consider the reset control system (9). Assume that there exist $P > 0$ and (A_c, B_c, C_c) such that BMI (13) is satisfied for the base linear system. Then, with the proposed reset control laws (8) and (17), the following asymptotic estimate holds with $\kappa > 0, \rho > 0$ for the $\mathcal{L}_2$ gain γ of the closed loop from d to y.

$$
\gamma \leq \gamma_L \tag{27}
$$

where γ_L is the gain from d to y of the system without resets.

Proof. From Theorem 2, it is known that the base linear system satisfies $2x^T P(Ax + Bd) + |y|^2 - \gamma_L^2 |d|^2 \leq 0$, for all x and d. The flow condition follows trivially from the fact that the reset control system coincides with the base linear system when flows, and the $V(x)$ is decreasing when jump occurs according to (20). According to the proposed reset condition and the proof of Theorem 2 (see (20)), it is required that $\kappa > 0$. As the base linear system is stable (from Theorem 1), so the ρ does not need to be small. Hence, for any $\kappa > 0, \rho > 0$, the $\mathcal{L}_2$ gain of the reset system must be less than or equal to the prescribed $\mathcal{L}_2$ gain γ of its base linear system. □

Remark 4. Note that with the proposed reset condition, it is not guaranteed that the reset action would occur. In addition, if the temporal regularization variable satisfies $\rho \to \infty$, the reset action does not occur. Under both situations, the system (8) will always flow and we can recover the $\mathcal{L}_2$ gain γ_L from the base linear system, that is $\gamma = \gamma_L$. ○

Remark 5. The results obtained in Nešić et al. [2011] are a special case of this paper. More specifically, when a scalar plant and the first order controller is considered as in Nešić et al. [2011] with a symmetric positive definite matrix $P \in \mathbb{R}^{2\times 2}$, the flow set with the proposed reset condition will be $\mathcal{F} = \{x \in \mathbb{R}^2 | \kappa v^2 + 2P_{12} v x_r \geq 0\}$ ($v = -x_p, x = [x_p, x_r]^T$). Compared with the reset condition in Nešić et al. [2011] (where $\mathcal{F} = \{x \in \mathbb{R}^2 | \varepsilon v^2 + 2 v x_r \geq 0\}$), if $P_{12} > 0$, we can select $\kappa = \varepsilon P_{12}$ such that our reset condition is exactly the same as the one proposed in Nešić et al. [2011]. ○

Remark 6. In this work, the reset controller (8) needs to have its dynamics (see equation (8a)). Thus the proposed method is not applicable to proportional only controller. ○

Remark 7. The proposed reset controller uses full state feedback. If not all states are measurable, appropriate observers are needed. With some modification, it is possible to extend the obtained results to observer-based reset controller design. ○

5. SIMULATION

In order to demonstrate the effectiveness of the proposed reset controllers, a simulation example is presented.

Example: Consider the following plant

$$
\dot{x}_p = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} x_p + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d. \tag{28}
$$

The disturbance signal takes the following form

$$
d(t) = 4 \exp(-0.2t) \sin(20t). \tag{29}
$$

Given a prescribed $\mathcal{L}_2$ gain $\gamma_L = 3$, as discussed in Remark 1, we select P randomly as follows

$$
P = \begin{bmatrix} 9.5503 & -3.4475 & 3.2077 & 4.1028 \\ -3.4475 & 13.9004 & -0.5048 & -9.7408 \\ 3.2077 & -0.5048 & 5.9801 & -1.4455 \\ 4.1028 & -9.7408 & -1.4455 & 9.1851 \end{bmatrix} \tag{30}
$$

Then, solving the condition (13) in Theorem 1, we get the following linear controller such that the closed loop base linear system is stable

$$
\begin{aligned}
\dot{x}_c &= \underbrace{\begin{bmatrix} -6.5711 & 7.8467 \\ -8.8239 & 1.4911 \end{bmatrix}}_{A_c} x_c + \underbrace{\begin{bmatrix} -5.3377 & -8.8704 \\ -3.9644 & 1.4919 \end{bmatrix}}_{B_c} x_p \\
u &= \underbrace{\begin{bmatrix} 7.3423 & -1.1431 \end{bmatrix}}_{C_c} x_c
\end{aligned} \tag{31}
$$

Select $\kappa = 0.1, \rho = 0.1$, and initial conditions $x = [5 \ -13 \ \ 0 \ \ 0]^T$. The comparison control results between linear controller and reset controller are shown in Figure 2, where $y_L(t)$ denotes output with linear controller, and $y_R(t)$ denotes output with reset controller. The disturbance signal is shown in Figure 3. We can see that the disturbance attenuation ability is enforced by reset control.

Remark 8. In this simulation, the only criteria to select P is to ensure that the BMI condition (13) is satisfied. As the reset controller is highly dependent on the choice of P, how to select such a P to get the best $\mathcal{L}_2$ gain improvement is still an open problem.

6. CONCLUSION

By constructing new Lyapunov-based reset rules, we show the sufficient stability results for general SISO LTI systems, and provide a theoretical proof that the $\mathcal{L}_2$ gain performance with the proposed reset controller can be improved. Note that the proposed reset rules cannot guarantee the occurrence of resets, and if no reset happens, the base linear system is recovered. Simulation example demonstrates the effectiveness of the proposed methods.

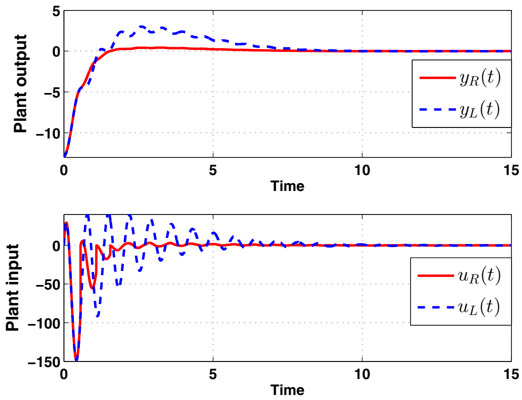

Fig. 2. Comparison results between using reset controller (8) and linear controller (2)

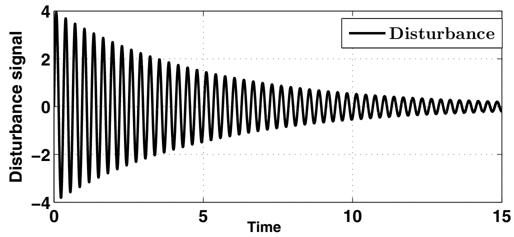

Fig. 3. Disturbance signal

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