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Observer-based bang-bang control for a class of nonlinear stochastic systems

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Abstract: This paper is about the bang-bang control strategy for a class of nonlinear stochastic systems. The goal is to give a new approach to stabilize the considered systems which are affected by multiplicative noises by using a state feedback control at first, then an observer-based feedback control in a second time. The so called bang-bang controller permits to ensure the almost sure exponential stability of the closed loop stochastic system.

Keywords: Almost sure exponential stability, stochastic differential equation, bang-bang control, observer-based control.

1. INTRODUCTION

During the last two decades many authors have studied the stability of systems described by a stochastic differential equation (SDE) driven by multiplicative noises. These noises are zero-mean Brownian motions. The SDE are characterized by the presence of random terms in their models in order to take into account some randomness in their evolution. This reflects that the behavior of the solution can not be accurately described using a mathematical model in most physical applications. Thus SDE have generally two parts: the drift one which corresponds to the dominant action of the system and the diffusion one representing the randomness along the dominant behavior. The stochastic modeling permit to obtain pertinent models in various fields such as engineering, finance, biology, population evolution and physics (the reader can see Has'minskii [1980], Mao [1994, 1997], Klebaner [2001], Øksendal [2003], Damm [2004] and references therein).

Notice that there are many types of definition to characterize stability of the equilibrium point. Each definition leads to different designs of control laws and observers for the considered stochastic systems. These definitions allow to express some statistical properties of the solution due to the effect of noises on the solution behavior. The most common stability definitions are the stability in probability, the asymptotic stability, the almost sure exponential stability (ASES) and the p^{th} moment exponential stability Kushner [1967], Kozin [1969], Has'minskii [1980], Mao [1994, 1997], Damm [2004]. In the design of control and observers, the most used stability definition is the mean-square exponential stability (MSES) which corresponds to 2^{nd} moment exponential stability. This stability leads to "good performance" in the design of control laws and observers: for example, the well known $\mathcal{H}_\infty$ control is based on the MSES Damm [2004], Dragan et al. [1997], Hinrichsen and Pritchard [1998], Gershon et al. [2001], Zhang et al. [2005]. The MSES can lead to some conservatism since the equilibrium point of a SDE can be almost surely exponential stable (ASES) but not MSES and, for a wide class of SDE, the MSES of the equilibrium point implies the ASES, but the converse is not true (theorem 4.4.2, Mao [1997]).

This is one of the motivation of this paper; in fact we search here a control law in order to ensure the ASES of the considered systems through a bang-bang control approach. We consider here a class of nonlinear stochastic systems where the nonlinearity is composed by a Lipschitz part and a bilinear in the control input part. And precisely in Longchamp [1980] and in Mohler [1991] it is proposed to use a bang-bang controller to stabilize bilinear deterministic systems when all the states are measured. Then, similarly to them, we decide to use this type of control for the systems we consider in this paper. Notice that at our knowledge, it is the first time that a bang bang approach is used for stochastic systems.

The bang-bang control is a saturated control switching between a minimum value and a maximum value. When some components of the state vector are not available, the bang-bang approach of Longchamp [1980] and Mohler [1991] has been modified in Zhang et al. [2004] and in Gérard et al. [2007] in order to include an observer in the control law. Then a decoupling approach is used in order to ensure the stability of the closed loop system.

This paper is organized as follows. In section 2, some preliminaries results on the ASES of SDE are given. Then, in section 3, the state feedback bang-bang control law ensuring the ASES of the equilibrium point of the considered closed-loop SDE is given. In section 4, it is considered that all the state of the system are not available; so an observer-based bang-bang controller is designed to guarantee the ASES of the closed-loop system. Notice that the proposed designs lead to bilinear matrix inequalities. So an iterative resolution method based on linear matrix inequalities is given in the last part of the appendix. Section 5 concludes the paper by recalling the important results that have been given. More, some useful results on SDE are recalled in the appendix.

Notations. $\mathbb{R}^n$ denotes the n-dimensional euclidean space. $\|A\| = \sqrt{\sum_{i,j} A_{i,j}^2} = \sqrt{\mathrm{tr}(A^T A)}$ is the Euclidean norm of the matrix A, while $\|x\| = \sqrt{x^T x}$ is the Euclidean norm of the vector x. "a.s." means almost surely. For a symmetric matrix A, $A > 0$ means that the matrix A is positive definite. Symbols $<$, $\leqslant$ and $\geqslant$ for matrices are defined similarly.

2. PRELIMINARY RESULTS

We consider the following nonlinear system

$$\mathrm{d}x = (A_{t_0}x + \ell(x) + \sum_{i=1}^{m} u_i A_{t_i} x)\,\mathrm{d}t + \sum_{j=1}^{d} A_{w_j} x\,\mathrm{d}w_{x_j} \quad (1a)$$

$$\mathrm{d}y = Cx\,\mathrm{d}t + Dx\,\mathrm{d}w_y \quad (1b)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input and $y \in \mathbb{R}^p$ is the measured output. $w_x \in \mathbb{R}^d$ and $w_y \in \mathbb{R}$ are independent Brownian motions. The function $\ell(x)$ verifies $\ell(0) = 0$ and the following Lipschitz condition with $\kappa > 0$

$$\|\ell(x) - \ell(\overline{x})\| \leqslant \kappa \|x - \overline{x}\|. \quad (2)$$

We define the ASES as follows.

Definition 1. (Mao [1997], Hu and Mao [2008]) The equilibrium point of the SDE (1a) is ASES if

$$\limsup_{t\to+\infty} \frac{1}{t}\ln(\|x(t)\|) < -\alpha < 0 \qquad \text{a.s.} \quad (3)$$

$\forall\, x_0 \in \mathbb{R}^n$, $\limsup_{t\to+\infty} \frac{1}{t}\ln(\|x(t)\|)$ is the Lyapunov exponent of the solution x.

The ASES of the equilibrum point of the SDE (1a) can be analyzed by using an approach of Lyapunov type. For this we apply the Itô formula (Mao [1997]) to the function $V(x)$ from $\mathbb{R}^n$ to $\mathbb{R}^+$ and we obtain

$$\mathrm{d}V(x) = \mathfrak{L}V(x)\,\mathrm{d}t + \mathfrak{B}V(x)\,\mathrm{d}w_x, \quad (4)$$

with

$$\mathfrak{L}V(x) = \frac{\partial V}{\partial x}(x)f(x) + \frac{1}{2}\,\mathrm{tr}\Big(g^T(x)\frac{\partial^2 V}{\partial x^2}(x)g(x)\Big),$$

$$\mathfrak{B}V(x) = \frac{\partial V}{\partial x}(x)g(x).$$

The ASES of the equilibrum point of the SDE (1a) is given by the following theorem.

Theorem 1. (Mao [1997], Hu and Mao [2008]) Consider that there exist a Lyapunov function positive definite $V(x)$ and constants $c_0 > 0$, $c_1 > 0$, $c_2 \in \mathbb{R}$ and $c_3 \geqslant 0$ such that, if $\forall\, x \neq 0$ and $\forall\, t \geqslant 0$,

$$c_1\|x\|^{c_0} \leqslant V(x), \quad (5)$$

$$\mathfrak{L}V(x) \leqslant c_2 V(x), \quad (6)$$

$$\|\mathfrak{B}V(x)\|^2 \geqslant c_3 V^2(x), \quad (7)$$

then

$$\limsup_{t\to+\infty} \frac{1}{t}\ln(\|x(t)\|) \leqslant \frac{2c_2 - c_3}{2c_0} \quad \text{a.s.} \quad \forall\, x_0 \in \mathbb{R}^n \quad (8)$$

The equilibrium point $x = 0$ is ASES if $c_3 > 2c_2$.

The constant c_2 in the theorem can be positive, negative or zero, then $\mathfrak{L}V(x)$ can not be negative definite in the ASES case, unlike in the MSES one.

3. STATE-FEEDBACK BANG-BANG CONTROL

In this part, we consider that all the components of the state x are available.

Then the following state feedback controller $u(x)$ is considered in order to ensure the ASES of the equilibrium point of the SDE (1a)

$$u(x) = -\begin{bmatrix} |\alpha_1|\,\mathrm{sgn}(x^T A_{t_1}^T P x) \\ \vdots \\ |\alpha_m|\,\mathrm{sgn}(x^T A_{t_m}^T P x) \end{bmatrix} \quad (9)$$

The conditions to be fulfilled are given in the following theorem.

Theorem 2. The control $u(x)$ in (9) is a bang-bang control ensuring the ASES of the equilibrium point of the SDE (1a) if there exist $P = P^T > 0$ and reals $\alpha_1, \ldots, \alpha_m$, $\rho_1 \geqslant 0, \ldots, \rho_d \geqslant 0$ and $\mu > 0$ such that the following inequality

$$\begin{bmatrix} (1,1)_a & A_{w_1}^T P & \cdots & \cdots & A_{w_d}^T P & P \\ PA_{w_1} & -P & 0 & \cdots & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ PA_{w_d} & 0 & \cdots & 0 & -P & 0 \\ P & 0 & \cdots & \cdots & 0 & \dfrac{-I_n}{\mu} \end{bmatrix} < 0 \quad (10)$$

is verified with $\rho = \sum_{j=1}^{d} \rho_j$ and

$$(1,1)_a = (A_{t_0} + \sum_{i=1}^{m} \alpha_i A_{t_i})^T P + P(A_{t_0} + \sum_{i=1}^{m} \alpha_i A_{t_i}) + \kappa^2 \mu^{-1} I_n - \rho P,$$

and if, for every $j = 1, \ldots, d$, one of the two following LMI

$$A_{w_j}^T P + PA_{w_j} - \sqrt{2\rho_j}P > 0, \quad (11)$$

$$A_{w_j}^T P + PA_{w_j} + \sqrt{2\rho_j}P < 0 \quad (12)$$

is verified.

Proof. Let $V(x) = x^T P x$ be a Lyapunov function candidate with $P = P^T > 0$.

The condition (5) holds with $c_0 = 2$ and $c_1 = \lambda_{\min}(P)$.

To prove condition (6), we calculate $\mathfrak{L}V(x)$

$$\mathfrak{L}V(x) = x^T((A_{t_0} + \sum_{i=1}^{m} u_i A_{t_i})^T P + P(A_{t_0} + \sum_{i=1}^{m} u_i A_{t_i}) + \sum_{j=1}^{d} A_{w_j}^T P A_{w_j})x - 2x^T P\ell(x) \quad (13)$$

Since $\ell(x)$ is a Lipschitz function verifying the condition (2) and $\ell(0) = 0$, using lemma 5 (in the Appendix section) with $\mu > 0$ we have

$$\mathfrak{L}V(x) \leqslant x^T((A_{t_0} + \sum_{i=1}^{m} u_i A_{t_i})^T P + P(A_{t_0} + \sum_{i=1}^{m} u_i A_{t_i}) + \sum_{j=1}^{d} A_{w_j}^T P A_{w_j} + \kappa^2 \mu^{-1} I_n + \mu P P)x \quad (14)$$

Applying the theorem of the almost sure exponential stability we must verify that $\mathfrak{L}V(x) - \rho V(x) \leqslant 0$, or equivalently

$$x^T((A_{t_0} + \sum_{i=1}^{m} u_i A_{t_i})^T P + P(A_{t_0} + \sum_{i=1}^{m} u_i A_{t_i}) + \sum_{j=1}^{d} A_{w_j}^T P A_{w_j} + \kappa^2 \mu^{-1} I_n + \mu P P - \rho P)x \leqslant 0 \quad (15)$$

In order to include the control law (9) in the above inequality, the idea is to add and subtract the terms α_i to it, then we obtain

$$x^T((A_{t_0} + \sum_{i=1}^{m} \alpha_i A_{t_i})^T P + P(A_{t_0} + \sum_{i=1}^{m} \alpha_i A_{t_i}) + \sum_{j=1}^{d} A_{w_j}^T P A_{w_j} + \kappa^2 \mu^{-1} I_n + \mu P P - \rho P)x + \underbrace{x^T(\sum_{i=1}^{m}(u_i - \alpha_i) A_{t_i}{}^T P + P\sum_{i=1}^{m}(u_i - \alpha_i) A_{t_i})x} \quad (16)$$

Concerning the underbrace term we have

$$x^T(\sum_{i=1}^{m}(u_i - \alpha_i) A_{t_i}{}^T P + P\sum_{i=1}^{m}(u_i - \alpha_i) A_{t_i})x = x^T(\sum_{i=1}^{m}(-|\alpha_i| \operatorname{sgn}(x^T A_{t_i}^T P x) - \alpha_i) A_{t_i}{}^T P + P\sum_{i=1}^{m}(-|\alpha_i| \operatorname{sgn}(x^T A_{t_i}^T P x) - \alpha_i) A_{t_i})x \quad (17)$$

The sign of $(-|\alpha_i| \operatorname{sgn}(x^T A_{t_i}^T P x) - \alpha_i) x^T A_{t_i}^T P x$ depends on the sign of $x^T A_{t_i}^T P x$ as follows.

(i) If $x^T A_{t_i} P x > 0$ and if

$$|\alpha_i| = \begin{cases} \alpha_i & \text{if} \quad \alpha_i > 0 \\ -\alpha_i & \text{if} \quad \alpha_i < 0 \end{cases} \quad (18)$$

then

$$\begin{cases} \text{if} \quad \alpha_i > 0 \Longleftrightarrow -|\alpha_i| - \alpha_i = -2\alpha_i < 0 \\ \text{if} \quad \alpha_i < 0 \Longleftrightarrow -|\alpha_i| - \alpha_i = 0 \end{cases} \quad (19)$$

(ii) If $x^T A_{t_i} P x < 0$ and if

$$\begin{cases} \text{if} \quad \alpha_i > 0 \Longleftrightarrow |\alpha_i| - \alpha_i = 0 \\ \text{if} \quad \alpha_i < 0 \Longleftrightarrow |\alpha_i| - \alpha_i = -2\alpha_i > 0 \end{cases} \quad (20)$$

then the underbrace part is $\leqslant 0$ and consequently

$$\mathfrak{L}V(x) - \rho V(x) \leqslant 0 \Longleftrightarrow (A_{t_0} + \sum_{i=1}^{m} \alpha_i A_{t_i})^T P + P(A_{t_0} + \sum_{i=1}^{m} \alpha_i A_{t_i}) + \sum_{i=1}^{m} A_{w_j}^T P A_{w_j} + \kappa^2 \mu^{-1} I_n + \mu P P - \rho P \leqslant 0 \quad (21)$$

Applying Schur lemma to this inequality permits to obtain the LMI (10).

Finally, to verify relation (7), we apply $\mathfrak{B}V(x)$ to the equation (1a) and we obtain

$$\mathfrak{B}V(x) = \big[x^T(A_{w_1}^T P + P A_{w_1})x \quad \ldots \quad \ldots \quad x^T(A_{w_d}^T P + P A_{w_d})x\big]. \quad (22)$$

If there exists a real $c_3 = 2\rho$ with $\rho = \sum_{j=1}^{d} \rho_j$ and $\rho_j \geqslant 0$ such that $\|\mathfrak{B}V(x)\|^2 \geqslant c_3 V^2(x)$, then the relation (7) is satisfied. Using the relation (22), we obtain

$$\|\mathfrak{B}V(x)\|^2 - c_3 V^2(x) = \sum_{j=1}^{d} x^T(A_{w_j}^T P + P A_{w_j} - \sqrt{2\rho_j} P)x \times x^T(A_{w_j}^T P + P A_{w_j} + \sqrt{2\rho_j} P)x. \quad (23)$$

The condition $\|\mathfrak{B}V(x)\|^2 \geqslant c_3 V^2(x)$ is verified if, for all j with $j = 1, \ldots, d$, one of the two LMI (11) and (12) is verified.

Since $c_2 < \rho$ and $c_3 = 2\rho \geqslant 0$, we have $c_3 > 2c_2$, and the inequality (6) is verified and the equilibrium point of the SDE (1a) where u is given by (9) is ASES. $\square$

4. OBSERVER-BASED BANG-BANG CONTROL

In this section, we do not have access to all the components of the state x; so, we consider all the system (1) with measurement equation (1b).

The goal of this section is to design the following observer based bang-bang controller for the nonlinear stochastic system (1)

$$\mathrm{d}\widehat{x} = (A_{t_0}\widehat{x} + \sum_{i=1}^{m} u_i A_{t_i}\widehat{x} + \ell(\widehat{x}))\,\mathrm{d}t + (K_0 + \sum_{i=1}^{m} K_i u_i)(\mathrm{d}y - C\widehat{x}\,\mathrm{d}t) \quad (24)$$

The matrices K_i are the gains of the observer to determine such that the SDE (1a) and the observation error $e = x - \widehat{x}$, which is described by the following stochastic differential equation (SDE)

$$\mathrm{d}e = ((A_{t_0} - K_0 C + \sum_{i=1}^{m}(A_{t_i} - K_i C)u_i))e + \ell(x) - \ell(x-e))\,\mathrm{d}t + \sum_{j=1}^{d} A_{w_j} x\,\mathrm{d}w_{x_j} - (K_0 + \sum_{i=1}^{m} K_i u_i) D x\,\mathrm{d}\,w_y, \quad (25)$$

are ASES.

Recall that the bang-bang control input u in SDE (24) and (25) is reconstructed using the estimated state $\widehat{x}$ given by the observer (24) like

$$u(\widehat{x}) = -\begin{bmatrix} |\alpha_1| \operatorname{sgn}(\widehat{x}^T A_{t_1}^T \mathcal{F}\widehat{x}) \\ \vdots \\ |\alpha_m| \operatorname{sgn}(\widehat{x}^T A_{t_m}^T \mathcal{F}\widehat{x}) \end{bmatrix} \tag{26}$$

As $-|\alpha_i| \leqslant u_i \leqslant |\alpha_i|$, for $i = 1, \dots, m$, we can define the following convex polytope

$$\mathcal{P} = \{[-|\alpha_1|, |\alpha_1|] \times \dots \times [-|\alpha_m|, |\alpha_m|]\} \tag{27}$$

In this case the bang-bang (switching) controller is chosen as a parameter varying in the polytope $\mathcal{P}$ where the vertices are noted by $\mathcal{S}$

$$\mathcal{S} = \{[\beta_1, \dots, \beta_m]^T\}, \forall i = 1, \dots, m, \beta_i \in \{-|\alpha_i|, |\alpha_i|\} \tag{28}$$

For all $v = 1, \dots, 2^m$, we define the following vertices $\Gamma_v = [\Gamma_1^v, \dots, \Gamma_m^v] \in \mathcal{S}$.

Then, inserting the bang-bang control (26), the SDE (1a) becomes

$$\mathrm{d}x = (A_{t_0}x + \ell(x) + \sum_{i=1}^{m} \Gamma_i^v A_{t_i} x)\,\mathrm{d}t + \sum_{j=1}^{d} A_{w_j} x \,\mathrm{d}w_{x_j} \tag{29}$$

and the error equation (25) becomes

$$\begin{aligned} \mathrm{d}e = ((A_{t_0} - K_0 C + \sum_{i=1}^{m} (A_{t_i} - K_i C)\Gamma_i^v))e + \ell(x) - \ell(x-e))\,\mathrm{d}t \\ + \sum_{j=1}^{d} A_{w_j} x\,\mathrm{d}w_{x_j} - (K_0 + \sum_{i=1}^{m} K_i \Gamma_i^v) Dx\,\mathrm{d}w_y \end{aligned} \tag{30}$$

Remark 3. Notice that the vertices Γ_i^v (with $i = 1, \dots, m$ and $v = 1, \dots, 2^m$) in (29) and (30) depend on $\widehat{x}$ (see the control $u(\widehat{x})$ in equation (26)). To simplify the notations, we use Γ_i^v instead of $\Gamma_i^v(\widehat{x})$ since the proof of the almost sure exponential stability in theorem 4 must be checked with the values of all vertices in the set $\mathcal{S}$ for the convex polytope $\mathcal{P}$ (see LMI (33) and (34)). □

If we put

$$\begin{aligned} f_1^v(x) &= \left(A_{t_0} + \sum_{i=1}^{m} \Gamma_i^v A_{t_i}\right) x + \ell(x), \\ g_1(x) &= [A_{w_1}x \ \cdots \ A_{w_d}x \ 0], \\ f_2^v(x, e) &= \left(A_{t_0} - K_0 C + \sum_{i=1}^{m} (A_{t_i} - K_i C)\Gamma_i^v\right) e \\ &\qquad + \ell(x) - \ell(x - e), \\ g_2^v(x) &= \left[A_{w_1}x \ \cdots \ A_{w_d}x \ K_0 x + \sum_{i=1}^{m} K_i \Gamma_i^v x\right], \\ w &= \begin{bmatrix} w_x \\ w_y \end{bmatrix}, \end{aligned}$$

for all $v = 1, \dots, 2^m$, the SDE (29) and (30) become

$$\mathrm{d}x = f_1^v(x)\,\mathrm{d}t + g_1(x)\,\mathrm{d}w \tag{31a}$$
$$\mathrm{d}e = f_2^v(x, e)\,\mathrm{d}t + g_2^v(x)\,\mathrm{d}w \tag{31b}$$

The stochastic system (31) have a triangular form since e does not appear in the SDE (31a).

In association with (31), the following "decoupled" SDE are defined

$$\mathrm{d}\overline{x} = f_1^v(\overline{x})\,\mathrm{d}t + g_1(\overline{x})\,\mathrm{d}w \tag{32a}$$
$$\mathrm{d}\overline{e} = f_2^v(0, \overline{e})\,\mathrm{d}t \tag{32b}$$

The condition for ASES with the bang-bang observer-based control given by (24) and (26) are presented in the following theorem.

Theorem 4. The equation (24), in assocation with the control given by (26) is an observer-based bang-bang control ensuring the ASES of the equilibrium point of the SDE (29) and (30) if there exist matrices $\mathcal{F} = \mathcal{F}^T > 0$, $Q = Q^T > 0$ and Y_i $(i = 0, \dots, m)$ and reals $\alpha_1, \dots, \alpha_m$, $\rho_1 \geqslant 0, \dots, \rho_d \geqslant 0$ and $\mu_0 > 0$, $\mu_1 > 0$ such that the following LMI

$$\begin{bmatrix} (1,1) & A_{w_0}^T \mathcal{F} & \dots & A_{w_d}^T \mathcal{F} & \mathcal{F} \\ \mathcal{F}A_{w_0} & -\mathcal{F} & 0 & 0 & 0 \\ \vdots & 0 & 0 & \ddots & 0 \\ \mathcal{F}A_{w_d} & 0 & 0 & 0 & -P \\ \mathcal{F} & 0 & 0 & 0 & \dfrac{-I_n}{\mu_0} \end{bmatrix} < 0 \tag{33}$$

$$\begin{bmatrix} (1,1)_b & \mathcal{Q} \\ \mathcal{Q} & \dfrac{-I_n}{\mu_1} \end{bmatrix} < 0 \tag{34}$$

are verified for all $v = 1, \dots, 2^m$ with $\rho = \sum_{j=1}^{d} \rho_j$ and

$$\begin{aligned} (1,1) = (A_{t_0} + \sum_{i=1}^{m} \Gamma_i^v A_{t_i})\mathcal{F} + \mathcal{F}(A_{t_0} + \sum_{i=1}^{m} \Gamma_i^v A_{t_i}) \\ + \sum_{j=1}^{d} A_{w_j}^T \mathcal{F} A_{w_j} + \kappa^2 \mu_0^{-1} I_n - \rho\mathcal{F} \end{aligned}$$

$$\begin{aligned} (1,1)_b = \sum_{i=1}^{m} ((\mathcal{Q}A_{t_i} - Y_i C)^T \Gamma_i^v + (\mathcal{Q}A_{t_i} - Y_i C)\Gamma_i^v) \\ + (\mathcal{Q}A_{t_0} - Y_0 C)^T + (\mathcal{Q}A_{t_0} - Y_0 C) + \mu_1^{-1}\kappa^2 I_n. \end{aligned}$$

and if, for any $j = 1, \dots, d$, one of the two LMI

$$A_{w_j}^T \mathcal{F} + \mathcal{F}A_{w_j} - \sqrt{2\rho_j}\mathcal{F} > 0, \tag{35}$$
$$A_{w_j}^T \mathcal{F} + \mathcal{F}A_{w_j} + \sqrt{2\rho_j}\mathcal{F} < 0 \tag{36}$$

is verified.

The gain matrices K_i are given by $K_i = \mathcal{Q}^{-1} Y_i$ for $i = 0, \dots, m$.

Proof. To verify the ASES, the theorem 6 proven in Barbata et al. [2012] is used. This theorem allows to decompose the problem into two subproblems, i.e. we study the ASES of the equilibrium point of the SDE (32a) and the ASES of the equilibrium point of the SDE (32b) separately.

<u>ASES of the equilibrium point of the SDE (32a):</u>

Let $V_x(\overline{x}) = \overline{x}^T \mathcal{F}\,\overline{x}$ be a Lyapunov function candidate with $\mathcal{F} = \mathcal{F}^T > 0$.

The condition (5) holds with $c_0 = 2$ and $c_1 = \lambda_{\min}(\mathcal{F})$.

By using Itô calculus on SDE (29) and assumption 1, applying the theorem of the almost sure exponential stability requires to verify that $\mathfrak{L}V_x(x)-\rho V_x(x) \leqslant 0$, or equivalently that

$$(A_{t_0} + \sum_{i=1}^{m} \Gamma_i^v A_{t_i})\mathcal{F} + \mathcal{F}(A_{t_0} + \sum_{i=1}^{m} \Gamma_i^v A_{t_i}) + \sum_{j=1}^{d} A_{w_j}^T \mathcal{F} A_{w_j} + \kappa^2 \mu_0^{-1} I_n + \mu_0 \mathcal{F}\mathcal{F} - \rho\mathcal{F} \leqslant 0 \quad (37)$$

Applying the Schur lemma to (37) leads to inequality (33) (for all $v = 1, \ldots, 2^m$).

The application of $\mathfrak{B}V_x(\overline{x})$ to the SDE (29) gives

$$\mathfrak{B}V_x(\overline{x}) = \big[\overline{x}^T(A_{w_1}^T P + PA_{w_1})\overline{x} \quad \ldots \quad \ldots \quad \overline{x}^T(A_{w_d}^T P + PA_{w_d})\overline{x}\big]. \quad (38)$$

The condition $\|\mathfrak{B}V_x(\overline{x})\|^2 \geqslant c_3 V_x^2(\overline{x})$ is verified with $c_3 = 2\rho$ and $\rho = \sum_{j=1}^{d} \rho_j$ if, for all j with $j = 1, \ldots, d$, one of the two LMI (35) and (36) is verified.

Since $c_2 < \rho$, $c_3 = 2\rho \geqslant 0$, we have $c_3 > 2c_2$ and, using the convexity of the polytope $\mathcal{P}$ (see (27)), the inequality (6) is verified and the equilibrium point of the SDE (32a) is ASES.

ASES of the equilibrium point of the SDE (32b):

Let $V_e(\overline{e}) = \overline{e}^T \mathcal{Q}\overline{e}$ be a Lyapunov function candidate with $\mathcal{Q} = \mathcal{Q}^T > 0$.

The SDE (32b) is given by

$$\dot{\overline{e}} = \left(A_{t_0} - K_0 C + \sum_{i=1}^{m} (A_{t_i} - K_i C)\Gamma_i^v\right)\overline{e} - \ell(-\overline{e}) \quad (39)$$

and is deterministic, i.e. it is an ordinary differential equation (ODE).

Using assumption 1, the time-derivative of $V_e(\overline{e})$ along the trajectory of the ODE (39) yields

$$\dot{V}_e(\overline{e}) \leqslant \overline{e}^T \left(\sum_{i=1}^{m} ((A_{t_i} - K_i C)^T \Gamma_i^v \mathcal{Q} + Q(A_{t_i} - K_i C)\Gamma_i^v) + (A_{t_0} - K_0 C)^T \mathcal{Q} + \mathcal{Q}(A_{t_0} - K_0 C) + \mu_1^{-1}\kappa^2 I_n + \mu_1 \mathcal{Q}\mathcal{Q}\right)\overline{e} \quad (40)$$

Using the convexity of the polytope $\mathcal{P}$, the equilibrium point of the SDE (32b) is exponentially stable if the inequality

$$\sum_{i=1}^{m} ((A_{t_i} - K_i C)^T \Gamma_i^v \mathcal{Q} + \mathcal{Q}(A_{t_i} - K_i C)\Gamma_i^v) + (A_{t_0} - K_0 C)^T \mathcal{Q} + \mathcal{Q}(A_{t_0} - K_0 C) + \mu_1^{-1}\kappa^2 I_n + \mu_1 \mathcal{Q}\mathcal{Q} < 0 \quad (41)$$

is satisfied for all $v = 1, \ldots, 2^m$. Indeed, if inequality holds, then there exists a real $\xi > 0$ such that

$$\dot{V}_e(\overline{e}) \leqslant -\xi\, \overline{e}^T \overline{e} \leqslant \frac{-\xi}{\lambda_{\max}(\mathcal{Q})} V_e(\overline{e})$$

and we obtain

$$\|\overline{e}\| \leqslant \sqrt{\frac{\lambda_{\max}(\mathcal{Q})}{\lambda_{\min}(\mathcal{Q})}}\, \|\overline{e}_0\|^{\frac{-\xi}{\lambda_{\max}(\mathcal{Q})} t}$$

where $\overline{e}_0 = \overline{e}(t_0)$ with the initial time $t_0 = 0$.

If we put $Y_i = \mathcal{Q}K_i$ for $i = 0, \ldots, m$, applying the Schur lemma to inequality (41) leads to the inequality (34). The proof is ended since the ASES and the exponential stability are equivalent for the ODE (32b). □

5. CONCLUSION

In this article we present a new method to stabilize a nonlinear stochastic system with a Lipschitz nonlinearity using a bang-bang control strategy. The stabilization is done for the state feedback case and for the observer-based feedback case. This kind of control realizes the almost sure exponential stability of the equilibrium point of the closed-loop stochastic nonlinear system.

APPENDIX

First Result

Lemma 5. (Petersen [1987]) We consider three matrices $A \in \mathbb{R}^{n\times q}$, $B \in \mathbb{R}^{p\times n}$ and $C \in \mathbb{R}^{q\times p}$ with $C^T C \leqslant I_p$, then, for all real $\mu > 0$, then

$$2x^T ACBx \leqslant \mu x^T AA^T x + \frac{1}{\mu} x^T B^T Bx. \quad (42)$$

Second Result

Consider the two following SDE

$$\mathrm{d}\, x_1 = f_1(x_1)\,\mathrm{d}\, t + g_1(x_1)\,\mathrm{d}w \quad (43a)$$
$$\mathrm{d}x_2 = f_2(x_1, x_2)\,\mathrm{d}\, t + g_2(x_1)\,\mathrm{d}w \quad (43b)$$

and

$$\mathrm{d}\overline{x}_1 = f_1(\overline{x}_1)\,\mathrm{d}\, t + g_1(\overline{x}_1)\,\mathrm{d}w \quad (44a)$$
$$\mathrm{d}\overline{x}_2 = f_2(0, \overline{x}_2)\,\mathrm{d}t \quad (44b)$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $\overline{x}_1 \in \mathbb{R}^{n_1}$, $\overline{x}_2 \in \mathbb{R}^{n_2}$ and w is an independent multidimensional Brownian motion.

Assumption 1. For all $t \geqslant 0$, there exists a real $k > 0$ such that

$$\|f_2(x_1, x_2) - f_2(0, \overline{x}_2)\| \leqslant k\left(\|x_1\| + \|x_2 - \overline{x}_2\|\right), \quad (45)$$
$$\mathrm{tr}\big((g_i(x_1) - g_i(\overline{x}_1))(g_i(x_1) - g_i(\overline{x}_1))^T\big) \leqslant k\,\|x_1 - \overline{x}_1\|^2 \quad i = 1, 2. \quad (46)$$

Theorem 6. (Barbata et al. [2012]) Under assumption 1, the equilibrium point of the SDE (43) is almost surely exponentially stable if and only if the equilibrium point of the SDE (44) is almost surely exponentially stable.

On the determination of the coefficients α_i

In theorem 2, the inequality (10) is bilinear since the coefficients α_i and the Lyapunov matrix P should be determined.

This bilinear inequality can be "transformed" in to linear matrix inequalities (LMI) in order to use standard LMI solver for convex optimization problem.

To do that, we consider two inequalities in the sequel:

- the inequlity I_1 which corresponds to the inequality (10) by replacing the term $(1.1)_a$ by

$$(1,1)_a = (A_{t_0} + \sum_{i=1}^{m} \alpha_i A_{t_i})^T P + P(A_{t_0} + \sum_{i=1}^{m} \alpha_i A_{t_i}) + \kappa^2 \mu^{-1} I_n - \rho P - \beta I_n$$

with $\beta \in \mathbb{R}$.
- the inequality I_2 which corresponds either to inequality (11) or to inequality (12).

The idea is to minimize a real β. If we find $\beta < 0$, then the ASES is obtained. But, since β can be positive, the inequality $I_1 < 0$ can be solved even if the inequality (10) is not verified, which allows to continue to try to find a feasible solution to the theorem 2.

We present below the sketch of an iterative algorithm in order to obtain the solutions α_i and P.

- **Step 0.**
Set $k = 0$ and choose $\alpha_1 \in \mathbb{R}, \ldots, \alpha_m \in \mathbb{R}$ and $\rho > 0$.
Find $P = P^T > 0$, $\mu > 0$ and $\beta \in \mathbb{R}$ that solve the LMI problem
$$\min_{\beta} \text{ subject to } I_1 < 0, I_2 < 0$$
Set $\beta_0 = \beta$, $P_0 = P$, $\alpha_{1_0} = \alpha_1$, $\ldots$, $\alpha_{m_0} = \alpha_m$, $\mu_0 = \mu$ and $\rho_0 = \rho$.
  - If $\beta_0 < 0$, go to step 3.
  - If $\beta_0 \geqslant 0$, go to step 1.
- **Step 1.**
$k \longleftarrow k + 1$.
Set $P = P_{k-1}$.
Find $\alpha_1 \in \mathbb{R}$, $\ldots$, $\alpha_m \in \mathbb{R}$, $\mu > 0$, $\rho > 0$ and $\beta \in \mathbb{R}$ that solve the LMI problem
$$\min_{\beta} \text{ subject to } I_1 < 0, I_2 < 0$$
Set $\beta_k = \beta$, $P_k = P$, $\alpha_{1_k} = \alpha_1$, $\ldots$, $\alpha_{m_k} = \alpha_m$, $\mu_k = \mu$ and $\rho_k = \rho$.
  - If $\beta_k < 0$, go to the step 3.
  - If $\beta_k \geqslant 0$, go to the step 2.
- **Step 2.**
$k \longleftarrow k + 1$.
Set $\alpha = \alpha_{1_{k-1}}$, $\ldots$, $\alpha = \alpha_{m_{k-1}}$ and $\rho = \rho_{k-1}$.
Find $P = P^T > 0$, $\mu > 0$ and $\beta \in \mathbb{R}$ that solve the LMI problem
$$\min_{\beta} \text{ subject to } I_1 < 0, I_2 < 0$$
Set $\beta_k = \beta$, $P_k = P$, $\alpha_{1_k} = \alpha_1$, $\ldots$, $\alpha_{m_k} = \alpha_m$, $\mu_k = \mu$ and $\rho_k = \rho$.
  - If $\beta_k < 0$, go to the step 3.
  - If $\beta_k \geqslant 0$, go to the step 1.
- **Step 3.**
End.

Several refinements can be made to this algorithm: stopping criterion, limiting the number of iterations, ...

For the theorem 4, we propose to use the scalars α_i obtained in theorem 2.

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