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On componentwise ultimate bound minimisation for switched linear systems via closed-loop Lie-algebraic solvability

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Abstract: We present a novel state feedback design method for perturbed discrete-time switched linear systems. The method aims at achieving (a) closed-loop stability under arbitrary switching and (b) minimisation of ultimate bounds for specific state components. Objective (a) is achieved by computing state feedback matrices so that the closed-loop A matrices generate a solvable Lie algebra (i.e. admit simultaneous triangularisation). Previous results derived an iterative algorithm that computes the required feedback matrices, and established conditions under which this procedure is possible. Based on these conditions, objective (b) is achieved by exploiting available degrees of freedom in the iterative algorithm.

Keywords: Switched systems, Eigenstructure assignment, Ultimate bounds.

1. INTRODUCTION

In the last decade there has been increasing research activities in the areas of stability and stabilisability of switched systems; see, for example, Liberzon [2003], Shorten et al. [2007], Lin and Antsaklis [2009]. A problem of interest is that of stability under arbitrary switching between subsystems, which consists in obtaining conditions that guarantee stability of the switched system for every switching signal. Finding these conditions in general is difficult except for special cases where the subsystems are pairwise commutative, symmetric or normal [Liberzon, 2003]. An equivalent condition to stability under arbitrary switching is the existence of a common Lyapunov function for all subsystems. As a special case, one can study the existence of a common quadratic Lyapunov function (CQLF).

While most efforts on stability and stabilisation of switched systems deal with asymptotic stability of the origin (as equilibrium point of the system), it might not be possible to achieve asymptotic stability in some situations, such as, for example, when the switched system is subject to non-vanishing disturbances. In these cases, the concern is practical stability of the system in the sense that its trajectories ultimately lie in a bounded region sufficiently close to the origin. In order to have an acceptable system performance, it is desirable for the system *ultimate bounds*, characterising these regions, to be sufficiently small. However, for a discrete-time system no feedback law can produce arbitrarily small closed-loop ultimate bounds since the latter are bounded from below by the effect of the disturbance on the state equations. For switched systems, this limitation could be more severe as each subsystem may have different disturbance characteristics.

In this paper, we address closed-loop stability under arbitrary switching and ultimate bound minimisation simultaneously, for discrete-time switched systems. To this purpose, our first contribution is to derive conditions in terms of eigenstructure of the perturbed switched system in order for the trajectories of one component of the state to lie within the minimum possible ultimate bound. Then, we exploit an algorithm from Haimovich and Braslavsky [2013], which iteratively seeks feedback matrices for each subsystem, and a common transformation matrix so that the closed-loop A matrices are stable and can be simultaneously transformed into upper-triangular by means of the computed transformation. Simultaneous triangularisation of the closed-loop A matrices (generation of a solvable Lie algebra) in addition to stability of each subsystem guarantees the existence of a CQLF [Liberzon, 2003]. Haimovich and Braslavsky [2013] give specific conditions on the number of states, inputs and subsystems under which the Lie-algebraic state feedback design is possible for almost every set of system parameters. The main contribution of the present paper is then to show how the degrees of freedom that remain available when the aforementioned conditions are satisfied can be exploited in order to minimise the ultimate bound for one arbitrary state component.

Notation. The index set $\{1, 2, \ldots, N\}$ is denoted $\underline{N}$. The nullspace of matrix A is denoted $\ker A$ and its image, img A. For $x \in \mathbb{C}^{n\times m}$, its j-th row is denoted $x_{(j,:)}$, its transpose x', its conjugate transpose x^* and its Moore-Penrose pseudoinverse $x^\dagger$. $\mathrm{d}(\mathcal{S})$ denotes the dimension of the vector space $\mathcal{S}$. The j-th component of $x_k \in \mathbb{C}^n$ is denoted $x_{j,k}$. An eigenvalue $\lambda \in \mathbb{C}$ is *stable* if $|\lambda| < 1$.

2. TIGHTEST ULTIMATE BOUNDS

Consider a perturbed discrete-time switched linear system

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u_{\sigma(k)}(k) + H_{\sigma(k)}d(k) \quad (1)$$

where the switching function $\sigma(\cdot)$ takes values in $\underline{N}$, $x \in \mathbb{R}^n$, $u_i \in \mathbb{R}^{m_i}$, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m_i}$ and have full column rank, and $H_i \in \mathbb{R}^{n\times z}$, for every $i \in \underline{N}$. The disturbance variable $d \in \mathbb{R}^z$ is componentwise bounded by $|d(\cdot)| \leq \mathbf{d}$, where $\mathbf{d} \in \mathbb{R}^z$ is a nonnegative vector. We are interested in state-feedback control design of the form

$$u_{\sigma(k)}(k) = K_{\sigma(k)}x(k) \quad (2)$$

so that the resulting perturbed closed-loop system

$$x(k+1) = A^{cl}_{\sigma(k)}x(k) + H_{\sigma(k)}d(k), \quad (3)$$

with $A_i^{cl} = A_i + B_iK_i$, simultaneously admits a CQLF and exhibits the minimum possible ultimate bound for one state component.

Componentwise ultimate bound minimisation for non-switched discrete-time systems has recently been studied in Heidari et al. [2013], where conditions were derived so that the ultimate bound on one (or more) state components is minimised to its least possible value via eigenvalue-eigenvector assignment. For discrete-time switched systems, there are also limitations on the lowest achievable ultimate bound for any state component. Indeed, the ultimate bound associated to a specific state component of (3) can never be smaller than the effect of the perturbation on that component. This is formalised in the following result.

Lemma 1. An ultimate bound on the j-th state component of the switched system (3) can never be smaller than

$$\mathbf{b}_j^{\min} \doteq \max_{i\in\underline{N}} \left[\max_{|d|\leq\mathbf{d}} |[H_i]_{(j,:)}d| \right]. \quad (4)$$

Proof. An ultimate bound on the j-th state component can never be smaller than that corresponding to the case when the j-th row of A_i^{cl} is zero for every $i \in \underline{N}$. Then, the result follows from direct analysis of (3). □

We remark that the expression (4) is independent of the closed-loop matrices A_i^{cl}, $i \in \underline{N}$, because it corresponds to the case when the j-th row of every $A_i^{cl} = A_i + B_iK_i$ is zero. Lemma 2 below gives conditions on a common transformation V and resulting transformed matrices M_i, such that

$$A_i^{cl} = A_i + B_iK_i = VM_iV^{-1} \quad (5)$$

has its j-th row equal to zero $\forall i \in \underline{N}$ and M_i is upper triangular.

Lemma 2. The j-th ultimate bound of the discrete-time switched system (1) can be minimised to its minimum possible value (4) if there exist feedback matrices K_i for all $i \in \underline{N}$ and an invertible V such that $M_i = V^{-1}(A_i + B_iK_i)V$ are stable, upper triangular, and have the form

$$M_i = \begin{bmatrix} \Delta_i & \delta_i \\ 0 & 0 \end{bmatrix}, \quad (6)$$

where Δ_i is the $(n-1)$th leading principal of the upper-triangular matrix M_i, δ_i is an arbitrary vector and the transformation matrix V is such that its j-th row has a nonzero element at the last column and is zero everywhere else, that is,

$$V_{(j,:)} = [0_{1\times n-1} \ V_{j,n}], \quad V_{j,n} \neq 0. \quad (7)$$

Proof. Using (6) and (7), the j-th row of the closed-loop matrix of each subsystems is

$$\begin{aligned} [A_i^{cl}]_{(j,:)} &= [A_i + B_iK_i]_{(j,:)} \\ &= [VM_iV^{-1}]_{(j,:)} = V_{(j,:)}M_iV^{-1} \\ &= [0_{1\times n-1} \ V_{j,n}] \begin{bmatrix} \Delta_i & \delta_i \\ 0 & 0 \end{bmatrix} V^{-1} = 0_{1\times n} \end{aligned} \quad (8)$$

and hence, the ultimate bound on the j-th state component is minimum as in (4). □

Haimovich and Braslavsky [2013] developed an algorithm that iteratively seeks feedback matrices K_i and the transformation V so that $M_i = V^{-1}(A_i + B_iK_i)V$ are stable and upper triangular. In the next section, we modify this algorithm in order to achieve the additional conditions of Lemma 2 and hence yield closed-loop matrices $A_i^{cl} = A_i + B_iK_i$ with zero j-th row.

3. STABILISATION AND ULTIMATE BOUND MINIMISATION BY FEEDBACK

Haimovich and Braslavsky [2013] give conditions on the number of states n, the number of subsystems N, and the number of inputs of each subsystem m_i, $i \in \underline{N}$, so that the stabilising feedback matrices K_i and the simultaneous triangularisation transformation V will exist for almost every set of system parameters, i.e. for almost all possible entries of the matrices A_i and B_i, for all $i \in \underline{N}$. When these conditions are satisfied, Haimovich and Braslavsky [2013] also show that, in addition, the closed-loop eigenvalues for every subsystem can be arbitrarily selected. In this section, we modify the feedback design algorithm of Haimovich and Braslavsky [2013] so that all available degrees of freedom are exploited to achieve minimum ultimate bounds. These degrees of freedom consist in the selection of some closed-loop eigenvalues and the construction of a unitary matrix with specific properties.

Consider the discrete-time switched linear system (1) with state-feedback law (2), yielding the closed-loop system (3). The proposed modified algorithm is shown below as Algorithm ITBF in Figure 1. This algorithm is an extension of the algorithm in Haimovich and Braslavsky [2013], where the main modifications are: (a) the state component to be minimised, namely j, has to be supplied as input data, (b) the common eigenvector assignment (CEA) procedure of Haimovich and Braslavsky [2013] is replaced by the common *shifted* eigenvector assignment (CSEA) procedure in (10), and (c) the unitary matrix construction (14) has to satisfy the additional constraints (15). As in Haimovich and Braslavsky [2013], the proposed algorithm seeks feedback matrices K_i so that the closed-loop matrices A_i^{cl} in (3) are stable and simultaneously triangularisable, but with the additional property that the condition in Lemma 2 is fulfilled.

A brief description of the algorithm is as follows. After initialisation, the algorithm iterates the following steps: common eigenvector computation for the internal subsystems identified by A_i^ℓ, B_i^ℓ [performed by procedure CSEA

Algorithm ITBF: Iterative triangularisation and ultimate bound minimisation by feedback

Data: $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m_i}$ for $i \in \underline{N}$, and j
Output: K_i for $i \in \underline{N}$
Initialisation: $A_i^1 \doteq A_i$, $B_i^1 \doteq B_i$, $K_i^0 \doteq 0$, $U_1 \doteq \mathrm{I}$, $\ell \leftarrow 0$, $k^1 \leftarrow j$;
repeat

$$\ell \leftarrow \ell+1, \quad n_\ell \leftarrow n-\ell+1, \tag{9}$$

$$[v_1^\ell, \{F_i^\ell\}_{i=1}^N] \leftarrow \mathrm{CSEA}(\{A_i^\ell\}_{i=1}^N, \{B_i^\ell\}_{i=1}^N, k^\ell), \tag{10}$$

$$A_i^{\ell,\mathrm{CL}} \doteq A_i^\ell + B_i^\ell F_i^\ell, \tag{11}$$

$$K_i^\ell \leftarrow K_i^{\ell-1} + F_i^\ell\left(\prod_{r=1}^{\ell} U_r^*\right), \tag{12}$$

$$V_{(:,\ell)} \leftarrow \left(\prod_{r=1}^{\ell} U_r\right) v_1^\ell. \tag{13}$$

if $\ell < n$ **then**
Construct a unitary matrix (14) satisfying (15), with arbitrary $\hat{k}^\ell \neq 1$, and assign (16)–(19):

$$\left[v_1^\ell | v_2^\ell | \cdots | v_{n_\ell}^\ell\right] \in \mathbb{C}^{n_\ell \times n_\ell}, \quad \text{where} \tag{14}$$

$$v_{k^\ell,i}^\ell = 0 : i = 1,\ldots,n_\ell,\ i \neq \hat{k}^\ell,\ v_{k^\ell,\hat{k}^\ell}^\ell \neq 0 \tag{15}$$

$$k^{\ell+1} \leftarrow \hat{k}^\ell - 1 \tag{16}$$

$$U_{\ell+1} \leftarrow [v_2^\ell | \cdots | v_{n_\ell}^\ell], \tag{17}$$

$$A_i^{\ell+1} \leftarrow U_{\ell+1}^* A_i^{\ell,\mathrm{CL}} U_{\ell+1}, \tag{18}$$

$$B_i^{\ell+1} \leftarrow U_{\ell+1}^* B_i^\ell. \tag{19}$$

end if
until $\ell = n$;
$K_i \leftarrow K_i^n$;

Fig. 1. ITBF algorithm.

in (10)], state feedback and transformation update [performed at (12) and (13)], and internal matrices' update for the next iteration [at (14) to (19)]. The internal subsystem matrices change dimensions during the execution of the algorithm because one state dimension is eliminated at each iteration. During initialisation, the internal matrices for iteration $\ell = 1$ are set to coincide with the subsystem matrices: $A_i^1 = A_i$, $B_i^1 = B_i$. The CSEA procedure in (10) seeks a vector v_1^ℓ having specific structure (which will be explained later), and corresponding (internal) feedback matrices F_i^ℓ, so that v_1^ℓ is a feedback-assignable eigenvector common to all internal subsystems, with corresponding stable eigenvalues. That is, if Procedure CSEA is successful, then v_1^ℓ satisfies $\|v_1^\ell\| = 1$ and $(A_i^\ell + B_i^\ell F_i^\ell)v_1^\ell = \lambda_i^\ell v_1^\ell$ for some scalars λ_i^ℓ satisfying $|\lambda_i^\ell| < 1$ for all $i \in \underline{N}$.

Existence of such v_1^ℓ is ensured by the structural condition of Haimovich and Braslavsky [2013], as we next explain. Define $m_i^\ell \doteq \mathrm{rank}(B_i^\ell) = \mathrm{d}(\mathrm{img}\, B_i^\ell)$, and factor $B_i^\ell = b_i^\ell r_i^\ell$, where $r_i^\ell : \mathbb{R}^{m_i} \to \mathbb{R}^{m_i^\ell}$ has full row rank and $b_i^\ell : \mathbb{R}^{m_i^\ell} \to \mathbb{R}^{n_\ell}$ has full column rank. Note that $\mathrm{img}\, B_i^\ell = \mathrm{img}\, b_i^\ell$. Let Λ^ℓ be a vector with components λ_i^ℓ, $i \in \underline{N}$, i.e.

$$\Lambda^\ell \doteq [\lambda_1^\ell, \lambda_2^\ell, \ldots, \lambda_N^\ell]', \tag{20}$$

and build the matrices

$$R_\ell(\Lambda^\ell) \doteq \begin{bmatrix} \lambda_1^\ell I - A_1^\ell \\ \vdots \\ \lambda_N^\ell I - A_N^\ell \end{bmatrix}, \; B_\ell \doteq \mathrm{blkdiag}\left[b_1^\ell, \ldots, b_N^\ell\right],$$

$$Q_\ell(\Lambda^\ell) \doteq [R_\ell(\Lambda^\ell), -B_\ell], \tag{21}$$

where blkdiag denotes block diagonal concatenation.

Lemma 3. (Structural condition) (Haimovich and Braslavsky [2013]) Let

$$p_\ell \doteq n_\ell + \sum_{i=1}^N m_i^\ell - N n_\ell. \tag{22}$$

Then,

(a) $\mathrm{d}(\ker Q_\ell(\Lambda^\ell)) \geq p_\ell$ for every choice of Λ^ℓ as in (20).
(b) A vector that can be assigned by feedback as a common eigenvector with corresponding eigenvalues λ_i^ℓ for $i \in \underline{N}$ exists if and only if $\mathrm{d}(\ker Q_\ell(\Lambda^\ell)) > 0$. Consequently, if

$$p_\ell > 0, \tag{23}$$

then a feedback-assignable common eigenvector exists for every choice of corresponding eigenvalues.
(c) If $Q_\ell(\Lambda^\ell) w^\ell = 0$ with $w^\ell \neq 0$ partitioned as

$$w^\ell \doteq [v', u_1', \ldots, u_N']', \qquad \text{then } v \neq 0, \text{ and} \tag{24}$$

$$(A_i^\ell + B_i^\ell F_i^\ell)v = \lambda_i^\ell v, \quad \text{for } i \in \underline{N}, \tag{25}$$

for every F_i^ℓ satisfying $r_i^\ell F_i^\ell v = u_i$. For each $i \in \underline{N}$ one such F_i^ℓ is $F_i^\ell = (r_i^\ell)^\dagger u_i v^\dagger$. ◦

If the structural condition (23) holds, the nullspace of $Q_\ell(\Lambda^\ell)$ is not empty and, thus, we can find $w^\ell \in \ker Q_\ell(\Lambda^\ell)$. Let $\mathrm{d}(\ker Q_\ell(\Lambda^\ell)) = \xi_\ell \geq p_\ell$, define $d_\ell \doteq n_\ell + \sum_{i=1}^N m_i^\ell$ and let $W^\ell \in \mathbb{C}^{d_\ell \times \xi_\ell}$ be a basis for $\ker Q_\ell(\Lambda^\ell)$. Then, from Lemma 3(c), the vector $w^\ell \neq 0$ has the form

$$w^\ell = W^\ell \alpha^\ell \tag{26}$$

where $\alpha^\ell \in \mathbb{C}^{\xi_\ell}$ is an arbitrary vector. Once (26) is obtained, a feedback-assignable common eigenvector v_1^ℓ provided by procedure CSEA at iteration ℓ of algorithm ITBF can be computed by selecting the first n_ℓ components of w^ℓ to construct v [cf. (24)] and then letting

$$v_1^\ell = v/\|v\|. \tag{27}$$

In summary, the Algorithm ITBF (shown in Figure 1) seeks for feedback matrices K_i so that

(1) the closed-loop matrices $A_i^{cl} = A_i + B_i K_i$ are stable and simultaneously triangularisable, and
(2) the j-th ultimate bound, with j an arbitrary state index, is minimised to its smallest value.

Algorithm ITBF runs Procedure CSEA on its internal system matrices A_i^ℓ and B_i^ℓ. This procedure selects the common eigenvector such that the resulting triangularising transformation matrix V fulfils the condition of Lemma 2.

3.1 Solvability of Algorithm ITBF

In this section we first revisit some results from Haimovich and Braslavsky [2013] that analyse the structural condition (23). Then we present a new result on how to exploit the available degrees of freedom in the ITBF algorithm to guarantee satisfaction of (23) at each iteration, which ensures the successful generation of the desired common triangularising transformation.

The structural condition (23) depends on m_i^ℓ, the rank of B_i^ℓ. At the first iteration $\ell = 1$, $n_\ell = n$, $m_i^\ell = m_i$ and thus,

Procedure CSEA (Common Shifted Eigenvector Assignment)

Input: $A_i^\ell \in \mathbb{R}^{n_\ell \times n_\ell}$, $B_i^\ell \in \mathbb{R}^{n_\ell \times m_i}$, for $i \in \underline{N}$ and k
Output: v_1^ℓ with $v_{k,1}^\ell = 0$, F_i^ℓ for $i \in \underline{N}$
Factor $B_i^\ell = b_i^\ell r_i^\ell$ with $b_i^\ell \in \mathbb{R}^{n_\ell \times m_i^\ell}$ and $m_i^\ell = \text{rank}(B_i^\ell)$;
if $p_\ell = n_\ell + \sum_{i=1}^N m_i^\ell - Nn_\ell > 1$ **then**
 if $\ell < n$ **then**
  Select $\lambda_i^\ell \in \mathbb{R}$ stable and construct Λ^ℓ as in (20);
  Construct $Q_\ell(\Lambda^\ell)$ as in (21) and compute W^ℓ as in (26);
  Select α^ℓ as in (40), Lemma 8(b) satisfying Theorem 6(b);
  Compute w^ℓ as in (26) and partition it as in (24);
  $v_1^\ell = v/\|v\|$;
 else
  if $\ell = n$ **then**
   Select $\lambda_i^n = 0$ and construct Λ^n as in (20);
   Select $v_1^n = 1$;
  end if
 end if
end if

Fig. 2. CSEA when the structural condition is satisfied.

$p_1 = (1-N)n + \sum_{i=1}^N m_i$. At subsequent iterations, $m_i^{\ell+1}$ depends on the vector v_1^ℓ given by Procedure CSEA as

$$m_i^{\ell+1} = \begin{cases} m_i^\ell & \text{if } v_1^\ell \notin \text{img}\, B_i^\ell, \\ m_i^\ell - 1 & \text{if } v_1^\ell \in \text{img}\, B_i^\ell. \end{cases} \tag{28}$$

From (28), $m_i^{\ell+1} = m_i^\ell - 1$ when $m_i^\ell = n_\ell$, because $v_1^\ell \in \mathbb{R}^{n_\ell} = \text{img}\, B_i^\ell$. The next lemma follows from (22) and (28).

Lemma 4. (Haimovich and Braslavsky [2013]). Consider Algorithm ITBF at iteration ℓ and p_ℓ as in (22), with $m_i^\ell = \text{rank}(B_i^\ell)$. Then, $p_{\ell+1} \geq p_\ell - 1$, with equality if and only if

$$v_1^\ell \in \mathcal{B}^\ell, \quad \text{with } \mathcal{B}^\ell \doteq \bigcap_{i\in\underline{N}} \mathcal{B}_i^\ell \quad \text{and } \mathcal{B}_i^\ell \doteq \text{img}\, B_i^\ell. \tag{29}$$

$\circ$

From Lemma 4, if at iteration ℓ, $p_\ell = 1$ and $v_1^\ell \in \bigcap_{i\in\underline{N}} \mathcal{B}_i^\ell$, then $p_{\ell+1} = p_\ell - 1 = 0$ and no common eigenvector can be found; hence, the ITBF algorithm terminates unsuccessfully. We thus want to avoid this situation.

Let $\mathcal{S}_i^\ell$ denote the set of vectors $v \in \mathcal{B}_i^\ell = \text{img}\, B_i^\ell$ for which there exist a matrix F_i^ℓ and a stable scalar λ so that

$$(A_i^\ell + B_i^\ell F_i^\ell)v = \lambda v. \tag{30}$$

That is, $\mathcal{S}_i^\ell$ is the set of feedback-assignable eigenvectors for (A_i^ℓ, B_i^ℓ), with associated stable eigenvalue, which are contained in $\mathcal{B}_i^\ell$.

Suppose $\{\mathcal{S}_i^\ell : i \in \underline{N}\}$ are transverse subspaces (a generic property of randomly selected subspaces, see, Haimovich and Braslavsky [2013, Lemma 3]). Define the quantities

$$\rho_i^\ell \doteq \text{d}(\mathcal{S}_i^\ell), \qquad q_\ell \doteq n_\ell + \sum_{i\in\underline{N}} \rho_i^\ell - Nn_\ell, \tag{31}$$

$$\mathcal{S}^\ell \doteq \textstyle\bigcap_{i\in\underline{N}} \mathcal{S}_i^\ell, \qquad \rho^\ell \doteq \text{d}(\mathcal{S}^\ell). \tag{32}$$

The next lemma relates the quantities p_ℓ, q_ℓ and ρ^ℓ, defined in (22) and (31)–(32), and is central for the solvability of the proposed algorithm at all iterations. The proof follows from arguments in Haimovich and Braslavsky [2013].

Lemma 5. Let $p_\ell > 0$, $\{\mathcal{S}_i^\ell : i \in \underline{N}\}$ be transverse and (A_i^ℓ, B_i^ℓ) be controllable. Then, $p_\ell \geq \rho^\ell = \max\{0, q_\ell\}$, with $p_\ell = \rho^\ell$ if and only if $m_i^\ell = n_\ell$ for all $i \in \underline{N}$.

If the common eigenvector v_1^ℓ lies within $\mathcal{B}^\ell$, then $v_1^\ell \in \mathcal{B}^\ell$ as in (29), and the structural condition $p_{\ell+1} = p_\ell - 1 > 0$ may not be satisfied. To avoid this situation and guarantee that (23) continues to be satisfied, we provide in part (b) of Theorem 6 below a new result on exploiting the degrees of freedom to choose the common eigenvector such that $v_1^\ell \notin \mathcal{B}^\ell$ and hence, $p_{\ell+1} \geq p_\ell$.

Theorem 6. Let $\{\mathcal{S}_i^1 : i \in \underline{N}\}$ be transverse, $q_1 \geq 0$ and (A_i, B_i) be controllable for all $i \in \underline{N}$. Then,

(a) $p_\ell > 0$ for $\ell = 1, \ldots, n$.
(b) It is always possible to select α^ℓ in (26) such that p_ℓ is non-decreasing for $\ell = 1, \ldots, n$ or, if for some $k < n$ we have $p_k < p_{k-1}$ then $p_\ell = n_\ell$ for $\ell = k, \ldots, n$.
(c) There exist feedback gains K_i such that the set $\mathcal{Z} = \{A_i + B_iK_i : i \in \underline{N}\}$ consists of stable matrices and generates a solvable Lie-algebra. Hence, such that the closed-loop system admits a CQLF.

Proof. For the proof of (a) and (c) see Haimovich and Braslavsky [2013], Theorem 2. Here we show that by proper selection of α^ℓ in (26), p_ℓ is non-decreasing for all iterations until $p_\ell = n_\ell$ and remains equal to n_ℓ afterwards.

From Lemma 3(a), we have $\text{d}(\ker Q_\ell(\Lambda^\ell)) = \xi_\ell \geq p_\ell$. Thus, a basis for the nullspace of $Q_\ell(\Lambda^\ell)$ has the form (see (26))

$$W^\ell = \begin{bmatrix} w_1^\ell \ldots w_{\xi_\ell}^\ell \end{bmatrix} = \begin{bmatrix} v_1 & \ldots & v_{\xi_\ell} \\ u_{11} & \ldots & u_{1\xi_\ell} \\ \vdots & & \vdots \\ u_{N1} & \ldots & u_{N\xi_\ell} \end{bmatrix}, \ \text{rank}(W^\ell) = \xi_\ell \geq p_\ell, \tag{33}$$

where the partition of each vector follows from (24). From (24) and (26), the common eigenvector is determined as

$$v = \begin{bmatrix} v_1 & \ldots & v_{\xi_\ell} \end{bmatrix} \alpha^\ell. \tag{34}$$

First, we show that the subspace generated by the v_r vectors is also of dimension ξ_ℓ, i.e.

$$\text{rank}(\begin{bmatrix} v_1 & \ldots & v_k & \ldots & v_{\xi_\ell} \end{bmatrix}) = \xi_\ell. \tag{35}$$

Suppose, for a proof by contradiction, that v_k, for some $k \in \{1, \ldots, \xi_\ell\}$, is a linear combination of v_r, $r \neq k$, i.e.

$$v_k = \sum_{r=1, r\neq k}^{\xi_\ell} v_r \gamma_r. \tag{36}$$

where at least one coefficient γ_r is nonzero. Then, since W^ℓ in (33) is in the nullspace of $Q_\ell(\Lambda^\ell)$ we have

$$Q_\ell(\Lambda^\ell) w_r^\ell = 0, \quad r = 1, \ldots, \xi_\ell \tag{37}$$

and by replacing (21) and (33) in (37), for $i \in \underline{N}$ we obtain

$$(\lambda_i^\ell I - A_i^\ell)v_r - b_i^\ell u_{ir} = 0, \quad r = 1, \ldots, \xi_\ell. \tag{38}$$

For $r = k$, using (36) in (38) we obtain

$$(\lambda_i^\ell I - A_i^\ell)v_k - b_i^\ell u_{ik} = \sum_{r=1, r\neq k}^{\xi_\ell} (\lambda_i^\ell I - A_i^\ell) v_r \gamma_r - b_i^\ell u_{ik}$$

$$= b_i^\ell \sum_{r=1, r\neq k}^{\xi_\ell} u_{ir}\gamma_r - b_i^\ell u_{ik} = b_i^\ell \Bigg(\sum_{r=1, r\neq k}^{\xi_\ell} u_{ir}\gamma_r - u_{ik} \Bigg) = 0.$$

Since the b_i^ℓ matrices have full column rank, the above implies

$$\sum_{r=1, r\neq k}^{\xi_\ell} u_{ir}\gamma_r - u_{ik} = 0. \tag{39}$$

This means that $u_{ik} = \sum_{r=1,r\neq k}^{\xi_\ell} u_{ir}\gamma_r$, for $i \in \underline{N}$, which together with (36) yields $w_k^\ell = \sum_{r=1,r\neq k}^{\xi_\ell} w_r^\ell \gamma_r$, i.e. $\text{rank}(W^\ell) < \xi_\ell$, which contradicts our assumption in (33). Hence, (35) holds and the common eigenvector v in (24) can be chosen in a space of dimension ξ_ℓ.

In Haimovich and Braslavsky [2013, Theorem 2] it is proved by induction that when $\{\mathcal{S}_i^1 : i \in \underline{N}\}$ is transverse, $q_1 \geq 0$ and (A_i, B_i) is controllable, then for $\ell = 1, \ldots, n$, $\{\mathcal{S}_i^\ell : i \in \underline{N}\}$ is transverse and (A_i^ℓ, B_i^ℓ) is controllable. Hence, from part (a) in the current theorem and Lemma 5, we know that $p_\ell \geq \rho^\ell$ with equality if and only if $m_i^\ell = n_\ell$ for $i \in \underline{N}$. We consider two cases, $p_\ell = \rho^\ell$ and $p_\ell > \rho^\ell$.

When $p_\ell = \rho^\ell$, then $m_i^\ell = n_\ell$ for $i \in \underline{N}$ and all (internal) input matrices are invertible. We then have $p_\ell = \rho^\ell = n_\ell$ and hence, the dimension of $\mathcal{S}^\ell$ is n_ℓ (see (32)) and any common eigenvector $v_1^\ell \in \mathbb{C}^{n_\ell}$ is also in $\mathcal{S}^\ell$. Then, from Lemma 4, $p_{\ell+1} = p_\ell - 1 = n_\ell - 1$. On the other hand, the reduction of subsystems dimension in the next step (see (9)) yields $n_{\ell+1} = n_\ell - 1$ which results in $p_{\ell+1} = n_{\ell+1}$. When $p_\ell > \rho^\ell$, from (35) and $\xi_\ell \geq p_\ell$, it is always possible to select α^ℓ in (34) such that the resulting v is not in $\mathcal{S}^\ell$. From (27) we then have $v_1^\ell \notin \mathcal{S}^\ell$, hence $v_1^\ell \notin \mathcal{B}^\ell$ and (29) does not hold. Thus, from Lemma 4 we have $p_{\ell+1} > p_\ell - 1$ which results in $p_{\ell+1} \geq p_\ell$. Part (b) is then proved. □

Remark 7. Note that Theorem 6(b) shows that, under the same solvability conditions given in Haimovich and Braslavsky [2013], it is always possible to maximise the available degrees of freedom, characterised by $\xi_\ell(\geq p_\ell)$. ○

3.2 Iterative ultimate bound minimisation

At each iteration of the ITBF algorithm, $\xi_\ell(\geq p_\ell)$ represents the degrees of freedom to choose the common eigenvector such that a desirable property is satisfied. Indeed, as seen from (26), the vector α^ℓ can be selected to make proper use of the available degrees of freedom to shape the common eigenvector v_1^ℓ given in (27) in a specific way. This is shown in the following result, whose proof is straightforward and thus omitted.

Lemma 8. At iteration ℓ of the ITBF Algorithm, consider w^ℓ defined in (24), (26), where $W^\ell \triangleq W^\ell(\Lambda^\ell)$, with the eigenvalue vector Λ^ℓ of the form (20). If the structural condition (23) holds, then the j-th element of the common eigenvector v_1^ℓ given in (27) can be made zero, that is $v_{j,1}^\ell = W_{(j,:)}^\ell \alpha^\ell / \|v\| = 0$, in the following cases:

(a) If $p_\ell = 1$ and $\xi_\ell = 1$ for all stable eigenvalue vector Λ^ℓ, i.e. $W^\ell \triangleq W^\ell(\Lambda^\ell)$ is a column vector, and the solution Λ^ℓ of the equation $W_j^\ell(\Lambda^\ell) = 0$ is stable, then place the eigenvalues of the subsystems at the elements of Λ^ℓ.
(b) If $p_\ell > 1$, then $\xi_\ell > 1$ and select the vector $\alpha^\ell \neq 0$ such that

$$\alpha^\ell \in \ker W_{(j,:)}^\ell. \tag{40}$$

The above lemma states that if $p_\ell > 1$ at each iteration of the ITBF algorithm, then by proper selection of α^ℓ as in (40) it is possible to shift the common eigenvector so that the desired element of the matrix V in (5) is zero. Otherwise, if $\xi_\ell = p_\ell = 1$, then we should check if the solution of the equation $W_j^\ell(\Lambda^\ell) = 0$ has elements with magnitude smaller than one. This will generically not hold so we concentrate on the case $p_\ell > 1$.

Theorem 9. Consider the perturbed switched discrete-time system (1), and let j be an arbitrary number in $\{1, 2, \ldots, n\}$. Let $\{\mathcal{S}_i^1 : i \in \underline{N}\}$ be transverse, $q_1 \geq 0$, $p_1 > 1$ and (A_i, B_i) be controllable for all $i \in \underline{N}$. Then, the j-th ultimate bound of system (1) can be minimised to its minimum possible value (4) by executing Algorithm ITBF in Figure 1, if at each iteration, α^ℓ that satisfies Theorem 6(b) can be chosen to also satisfy (40).

Proof. The assumption $q_1 \geq 0$ together with $\{\mathcal{S}_i^1 : i \in \underline{N}\}$ being transverse and (A_i, B_i) being controllable for all $i \in \underline{N}$, satisfy the conditions in Theorem 6. From the assumption $p_1 > 1$ and Theorem 6(b), since p_ℓ is non-decreasing at all iterations while $p_\ell < n_\ell$, we have $p_\ell > 1$ for all those iterations. (When $p_\ell = n_\ell$, whether $n_\ell = 1$ or $n_\ell > 1$, no more iterations are necessary since the algorithm can be terminated in one step, see Remark 10). Thus, we can select α^ℓ as in Lemma 8(b) and Theorem 6(b). In the remainder of the proof, the iterative ultimate bound minimisation is explained.

The aim is to iteratively construct the columns of the matrix V through (13)-(17) to achieve the final form (7). Since the columns of V are the result of a product of matrices (c.f. (13)), the idea is to propagate the location of zero and nonzero elements in relevant rows of these matrices so that the end result is the j-th row of V having all zero elements except at the last column. At the first iteration, since $U_1 = I_n$, to have $V_{j,1} = 0$, Procedure CSEA needs to select the common eigenvector such that $v_{j,1}^1 = 0$. Then, to construct a unitary matrix with v_1^1 as its first column, the j-th row of the unitary matrix has $n-1$ zero elements and one nonzero element at its $\hat{k}^1$-th place, $\hat{k}^1 \neq 1$. Thus, the j-th row of U_2 in (17) has $n_2 - 1 = n-2$ zeros and a nonzero entry at $k^2 = \hat{k}^1 - 1$ [cf. (16)]. Hence,

$$U_1 U_2 = I_n \begin{bmatrix} * \ldots & * & \ldots * \\ \vdots & \vdots & \vdots \\ 0 \ldots & v_{j,\hat{k}^1-1}^1 & \ldots 0 \\ \vdots & \vdots & \vdots \\ * \ldots & * & \ldots * \end{bmatrix} \doteq \begin{bmatrix} * \ldots & * & \ldots * \\ \vdots & \vdots & \vdots \\ 0 \ldots & v_{j,k^2}^1 & \ldots 0 \\ \vdots & \vdots & \vdots \\ * \ldots & * & \ldots * \end{bmatrix}_{n\times n_2}$$

where $*$ is a non-specified entry. It can be seen that the matrix U_1U_2 has its j-th row with one nonzero element at the k^2-th place ($k^2 = \hat{k}^1 - 1$) and otherwise zero. Hence, to achieve $V_{j,2} = 0$, from (13) we need to have $v_{k^2,1}^2 = 0$ for the common eigenvector v_1^2. Accordingly, the unitary matrix (14) constructed using this v_1^2 will have its j-th row with $n_3 - 1$ zeros and one nonzero element at its $\hat{k}^2$-th place, $\hat{k}^2 \neq 1$. Continuing with the same procedure, at iteration ℓ the matrix $\prod_{r=1}^{\ell} U_r$ is of size $n \times n_\ell$ and has its j-th row equal to zero except for its k^ℓ-th component. Thus, to have $V_{j,\ell} = 0$, the common eigenvector assignment should satisfy $v_{k^\ell,1}^\ell = 0$. At the last iteration, the matrix $\prod_{r=1}^{n} U_r$ is of size $n \times 1$ with its j-th element being nonzero:

$$\left(\prod_{r=1}^{n-1} U_r\right) U^n = \begin{bmatrix} * & * \\ \vdots & \vdots \\ 0 & v_{j,2}^{n-2} \\ \vdots & \vdots \\ * & * \end{bmatrix}_{n\times 2} \begin{bmatrix} * \\ v_{2,2}^{n-1} \end{bmatrix}_{2\times 1} = \begin{bmatrix} * \\ \vdots \\ v_{j,1}^{n-1} \\ \vdots \\ * \end{bmatrix}_{n\times 1}$$

Thus, in order to have $V_{j,n} \neq 0$, the scalar v_1^n needs to be nonzero. Since we have a scalar system at this iteration, the common eigenvector can be taken to be $v_1^n = 1$ for arbitrary eigenvalues. From (13), multiplying v_1^n by the vector $\prod_{r=1}^{n-1} U_r$ displayed above results in $V_{j,n} \neq 0$. By executing the above procedure, the j-th row of V takes the form (7). Setting the last eigenvalue of all subsystems to 0, all M_i matrices take the form (6), and the j-th ultimate bound will be minimised to its lowest value (4). □

Remark 10. If at any iteration ℓ, B_i^ℓ for $i \in \underline{N}$ have rank n_ℓ, then the control input matrices are invertible and we can assign arbitrary eigenvalues for all subsystems with common eigenvector matrix I_{n_ℓ}, that is, all remaining iterations from ℓ to n can be subsumed in one step by replacing v_1^ℓ in (13) with the matrix $V_1^\ell \triangleq I_{n_\ell}$. Since the matrix $\prod_{r=1}^{\ell} U_r$ is of size $n \times n_\ell$ with its j-th row having a nonzero k^ℓ-th component and otherwise zero:

$$\prod_{r=1}^{\ell} U_r = \begin{bmatrix} * \dots & * & \dots * \\ \vdots & \vdots & \vdots \\ 0 \dots & v_{j,k^\ell}^\ell & \dots 0 \\ \vdots & \vdots & \vdots \\ * \dots & * & \dots * \end{bmatrix}_{n\times n_\ell},$$

by multiplying this matrix with the common eigenvector matrix I_{n_ℓ}, the last $n-\ell+1$ columns of the matrix V in (13) are $V_{(:,\ell:n)} = \left(\prod_{r=1}^{\ell} U_r\right)$, and thus, the j-th row of V has the form $V_{(j,:)} = \begin{bmatrix} 0_{1\times(\ell+k^\ell-1)} & v_{j,k^\ell}^\ell & 0_{1\times(n-\ell-k^\ell)} \end{bmatrix}$. Then, a property similar to that of Lemma 2 holds by assigning to zero the eigenvalue associated with v_{j,k^ℓ}^ℓ:

$$M = \begin{bmatrix} \Delta^1_{(\ell+k^\ell-1)} & \delta^1_{(\ell+k^\ell-1)\times 1} & \Delta^2_{(\ell+k^\ell-1)\times(n-\ell-k^\ell)} \\ 0_{1\times(\ell+k^\ell-1)} & 0 & 0_{1\times(n-\ell-k^\ell)} \\ 0_{(n-\ell-k^\ell)\times(\ell+k^\ell-1)} & 0_{(n-\ell-k^\ell)\times 1} & \Delta^3_{(n-\ell-k^\ell)} \end{bmatrix}.$$

The eigenvalues of the upper triangular matrices $\Delta^1_{(\ell+k^\ell-1)}$ and $\Delta^3_{(n-\ell-k^\ell)}$ are stable and can be arbitrarily chosen. ∘

4. NUMERICAL EXAMPLE

Consider a switched system with $N = 2$ and matrices

$$A_1 = \begin{bmatrix} 1.8039 & -4.5216 & 2.4237 & -2.4003 \\ -1.2862 & 2.3826 & 4.3745 & 2.5897 \\ -4.2177 & -4.6200 & 0.1336 & 4.9334 \\ -0.4365 & 4.5424 & -2.5910 & -1.4329 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 3.2700 & 2.0058 & -2.2256 & -1.9570 \\ -1.9192 & -2.5813 & -4.9389 & -2.0914 \\ -0.9764 & 2.5983 & -1.2529 & -2.5748 \\ 3.8423 & -2.0907 & -0.6307 & 4.3668 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 2.5286 & 2.3072 & 1.4007 \\ -3.8995 & -2.3882 & -3.6796 \\ 0.9705 & -4.0519 & -0.4718 \\ -0.6940 & -0.4904 & 1.5220 \end{bmatrix}, H_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 3.6019 & -0.1038 & 3.6166 \\ -1.0277 & -2.3019 & -4.6737 \\ -0.2058 & 4.8974 & -1.6804 \\ 0.6500 & -3.1632 & 2.4875 \end{bmatrix}, H_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{d} = 1.$$

We aim to minimise the 4-th ultimate bound. At the first iteration, $q_1 = 0$, $p_1 = 2$ and, for arbitrary eigenvalues $\Lambda^1 = [\lambda_1^1\ \lambda_2^1]' = [0.0398\ -0.1141]'$, Procedure CSEA yields

$$V_{(:,1)} = v_1^1 = \begin{bmatrix} 0.5355 \\ 0.7953 \\ 0.2840 \\ 0 \end{bmatrix} \quad \text{with} \quad \alpha^1 = \begin{bmatrix} -0.8968 \\ 0.4424 \end{bmatrix}. \qquad (41)$$

The matrix (17) with $\hat{k}^1 = 3$ is $U_2 = \begin{bmatrix} -0.8445 & 0 & 0 \\ 0.5043 & 0 & -0.3363 \\ 0.1801 & 0 & 0.9418 \\ 0 & 1 & 0 \end{bmatrix}$. At the next iteration, $n_2 = 3$ and $m_1^2 = m_2^2 = 3$ and hence, the input matrices are invertible. Since $\hat{k}^2 = \hat{k}^1 - 1 = 2$, we need $\lambda_i^3 = 0$. Thus, assigning the remaining eigenvalues at $\Lambda^2 = [\lambda_1^2\ \lambda_2^2]' = [-0.8924\ 0.0960]'$, $\Lambda^3 = [\lambda_1^3\ \lambda_2^3]' = [0\ 0]'$, $\Lambda^4 = [\lambda_1^4\ \lambda_2^4]' = [0.7244\ 0.1337]'$, and computing the eigenvector matrix as in Remark 10, $V_1^2 = I_3$, yields

$$V_{(:,2:4)} = I_4\ U_2\ V_1^2 = \begin{bmatrix} -0.8445 & 0 & 0 \\ 0.5043 & 0 & -0.3363 \\ 0.1801 & 0 & 0.9418 \\ 0 & 1 & 0 \end{bmatrix}. \qquad (42)$$

The resulting feedback gains and upper triangular closed-loop matrices are

$$K_1 = \begin{bmatrix} 0.1300 & 2.7027 & -0.3796 & -0.6779 \\ -1.0793 & -0.1029 & -0.6018 & 1.0040 \\ -0.0017 & -1.7852 & 1.3354 & 0.9558 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 2.4581 & -1.4435 & -0.9600 & 4.1290 \\ -0.0229 & -0.5822 & -0.0379 & 0.1981 \\ -2.2160 & 0.4773 & 0.4561 & -2.5826 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 0.0398 & -0.5862 & -0.8573 & 3.6896 \\ 0 & -0.8924 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.7244 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} -0.1141 & -9.0128 & 6.6379 & -6.2035 \\ 0 & 0.0960 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1337 \end{bmatrix}.$$

The matrices V (see (41) and (42)) and M_i satisfy the conditions of Lemma 1 and Remark 10. For random disturbance and switching signals, the trajectories of the switched system depicted in Figure 3 show that the 4-th state is kept within the smallest possible bound.

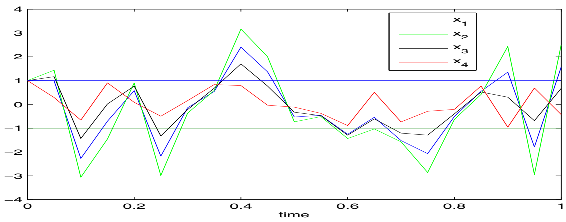

Fig. 3. Switched system state trajectories.

5. CONCLUSION

This paper has derived conditions to achieve a minimum ultimate bound for one component of the state of a discrete-time switched linear system under arbitrary switching and non-vanishing perturbations. A procedure to satisfy the derived conditions has been presented via an iterative algorithm which simultaneously triangularises all subsystem matrices and gives the minimum achievable value for one of the ultimate bounds.

REFERENCES

H. Haimovich and J. Braslavsky. Sufficient conditions for generic feedback stabilisability of switching systems via lie-algebraic solvability. *IEEE Trans. on Aut. Control*, 58(3):814–820, March 2013.

R. Heidari, M. Seron, and J. Braslavsky. On eigenvalue-eigenvector assignment for componentwise ultimate bound minimisation in MIMO LTI discrete-time systems. In *Australian Control Conf. Perth, Australia*, 2013.

D. Liberzon. *Switching in systems and control.* Birkhäuser, 2003.

H. Lin and P. J. Antsaklis. Stability and stabilisability of switched linear systems: a survey of recent results. *IEEE Trans. on Aut. Control*, 54(2):308–322, 2009.

R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King. Stability criteria for switched and hybrid systems. *SIAM Review*, 49(4): 545–592, 2007.