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Compensators via $\mathcal{H}_2$-based Model Reduction and Proper Orthogonal Decomposition ⋆

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Abstract:
Low order controllers are essential for the design of real time feedback controllers for systems described by partial differential equations (PDEs). We consider a MinMax control design that does not require full state information by using a state estimate in the feedback law. In this study we compare two reduced order modeling approaches to obtain low order state estimators for a system described by a nonlinear PDE. In the first case we investigate an $\mathcal{H}_2$-model reduction technique for linear systems. In particular, we implement the iterative rational Krylov algorithm (IRKA) to construct a low dimensional linear state estimator. This method maintains stability properties of the original system and is numerically efficient in that it requires only matrix-vector multiplications and sparse linear solvers. In the second case we construct a nonlinear compensator by including nonlinear terms of the state equation in the differential equation for the compensator. Proper orthogonal decomposition (POD) is then used to determine a reduced order model for the resulting nonlinear equation. We apply these approaches to Burgers equation with periodic boundary conditions.

1. INTRODUCTION

The primary goal of this paper is to compare and test two reduced order modeling strategies in practical controller design for systems described by nonlinear partial differential equations (PDEs). Full state information is seldom available for complex physical systems. As an alternative, compensator design uses state measurements for a state estimate and uses this estimate in the feedback control law. For this study we use LQG and MinMax control designs to stabilize the system and attenuate the disturbance to the controlled output map. Distributed parameter control theory is used to design the feedback law after which approximation theory is used to construct finite dimensional controllers for implementation. These controllers are then used to construct low order (linear and nonlinear) state estimators via IRKA (Gugercin et al. [2008]) and POD (Berkooz et al. [1993]).

2. FEEDBACK CONTROL DESIGN

2.1 Abstract Problem Statement

Implementation of distributed parameter control theory requires the abstract form of the PDE:

$$\mathcal{M}\dot{w}(t) = \mathcal{A}w(t) + \mathcal{G}(w(t)) + \mathcal{B}u(t) + \mathcal{D}\eta(t), \qquad (1)$$

with $w(0) = w_0$ and with state measurements

$$y(t) = \mathcal{C}w(t) + \mathcal{E}\eta(t). \qquad (2)$$

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Here w denote the state of the system in a state space X (typically a Hilbert space), $\mathcal{A}$ is a linear operator, $\mathcal{G}$ denotes the nonlinear term, $\mathcal{B}$ is the control input operator, and the disturbance takes the form $\mathcal{D}\eta(t)$.

For this discussion we consider LQG and MinMax approaches. MinMax control design is typically applied to linear systems and results in a linear controller that is determined via the solution of two algebraic Riccati equations. However, this strategy can be applied to nonlinear systems by first linearizing the state equation and designing the standard feedback control law.

We therefore start by linearizing the abstract Cauchy problem (1) about a given a nominal solution, w_{nom}, such as a steady-state solution or a time-averaged solution. For this discussion assume that w_{nom} is the steady state solution,

$$0 = \mathcal{A}w_{\text{nom}} + \mathcal{G}(w_{\text{nom}}) + \mathcal{B}u_{\text{nom}} + \mathcal{D}\eta_{\text{nom}},$$

and $y_{\text{nom}} = \mathcal{C}w_{\text{nom}} + \mathcal{E}\eta_{\text{nom}}$. Define $z(t) = w(t) - w_{\text{nom}}$, $u_z(t) = u(t) - u_{\text{nom}}$, $\eta_z(t) = \eta(t) - \eta_{\text{nom}}$, and $y_z(t) = y(t) - y_{\text{nom}}$. Then z satisfies the abstract Cauchy problem

$$\mathcal{M}\dot{z}(t) = \tilde{\mathcal{A}}z(t) + \tilde{\mathcal{G}}(z(t)) + \mathcal{B}u_z(t) + \mathcal{D}\eta_z(t), \qquad (3)$$

with $z(0) = w_0 - w_{\text{nom}}$, where

$$\tilde{\mathcal{A}}z \equiv \mathcal{A}z + \nabla_w \mathcal{G}(w_{\text{nom}})z$$

and

$$\tilde{\mathcal{G}}(z(t)) \equiv \mathcal{G}(z(t) + w_{\text{nom}}) - \mathcal{G}(w_{\text{nom}}) - \nabla_w \mathcal{G}(w_{\text{nom}})z.$$

Using this form of the equations ensures that the nonlinear term in (3) satisfies

$$\tilde{\mathcal{G}}(z(t)) = z^T(t)\nabla_{ww}\mathcal{G}(\gamma z(t) + (1-\gamma)w_{\text{nom}})z(t)$$

for some $\gamma \in (0,1)$. Thus, $\tilde{\mathcal{G}}$ is properly nonlinear in z.

State measurements of z are produced from (2) by

$$y_z(t) = \mathcal{C}z(t) + \mathcal{E}\eta_z(t). \quad (4)$$

A control that stabilizes (1)–(2) to w_{nom}, equivalently, stabilizes the solution z of (3)–(4) to zero.

Any linear or nonlinear feedback control of the form

$$u_z(t) = -\mathcal{K}z(t) \qquad \text{or} \qquad u_z(t) = -\mathcal{K}(z(t)) \quad (5)$$

requires full state information. In real systems the the full state z is not available due to absence of initial conditions, uncertainties in the model, uncertainties in the parameters/boundary conditions, etc. Therefore, the state z must be estimated using state measurements that may contain their own uncertainties (4). This state estimator must be fast enough to be implemented in real time, and accurate enough to provide a stabilizing feedback control, and hopefully, also maintain some of the optimal performance designed into the control.

2.2 Control and State Estimator

We present only the items essential to the construction of the MinMax controller. For more on the infinite dimensional theory see for example McMillan and Triggiani [1993], McMillan and Triggiani [1994], and Marrekchi [1993]. The finite dimensional theory is presented in Rhee and Speyer [1989] and Başar and Bernard [1991].

Consider the linearized system, ignoring $\tilde{\mathcal{G}}$ in (3),

$$\mathcal{M}\dot{z}(t) = \tilde{\mathcal{A}}z(t) + \mathcal{B}u_z(t) + \mathcal{D}\eta_z(t) \quad (6)$$

$$y_z(t) = \mathcal{C}z(t) + \mathcal{E}\eta_z(t). \quad (7)$$

A MinMax approach requires the design of a state estimator z_c that satisfies the linear system

$$\mathcal{M}\dot{z}_c(t) = \mathcal{A}_c z_c(t) + \mathcal{F}y_z(t), \quad (8)$$

and $z_c(0) = z_{c0}$ is the best available guess to the initial state. The linear feedback law takes the form $u_z(t) = -\mathcal{K}z_c(t)$ where, denoting the operator adjoint with $(^*)$,

$$\begin{aligned}
\mathcal{K} &= R^{-1}\mathcal{B}^*\Pi,\\
\mathcal{F} &= (I - \theta^2 P\Pi)^{-1}P\mathcal{C}^*H^{-1},\\
\mathcal{A}_c &= \tilde{\mathcal{A}} - \mathcal{B}\mathcal{K} - \mathcal{F}\mathcal{C} + \theta^2\mathcal{D}\mathcal{D}^*\Pi
\end{aligned}$$

and Π and P are solutions to algebraic Riccati equations:

$$\tilde{\mathcal{A}}^*\Pi + \Pi\tilde{\mathcal{A}} - \Pi(\mathcal{B}R^{-1}\mathcal{B}^* - \theta^2\mathcal{D}\mathcal{D}^*)\Pi + Q = 0, \quad (9)$$

$$\tilde{\mathcal{A}}P + P\tilde{\mathcal{A}}^* - P(\mathcal{C}^*H^{-1}\mathcal{C} - \theta^2 Q)P + \mathcal{D}\mathcal{D}^* = 0. \quad (10)$$

The parameter $\theta \in [0, \theta_{\max}]$ is chosen to balance performance (for LQG, $\theta = 0$) and robustness ($\theta = \theta_{\max}$). The parameter $\theta_{\max}$ is the largest value of θ where (9) and (10) each have a nonnegative solution and the matrix $I - \theta^2 P\Pi$ is positive definite. Here Q is a nonnegative definite, self-adjoint state weighting operator and R is a positive definite, self-adjoint control weighting operator and $H = \mathcal{E}\mathcal{E}^*$.

Once the solutions to (9) and (10) are known, the stabilizing control for the linear system can also be applied to the nonlinear system, see Burns and King [1998] and Atwell et al. [2001]. The resulting nonlinear observer becomes

$$\mathcal{M}\dot{z}_c(t) = \mathcal{A}_c z_c(t) + \mathcal{F}y_z(t) + \tilde{\mathcal{G}}(z_c(t)), \quad z_c(0) = z_{c_0} \quad (11)$$

and the (nonlinear) feedback law is of the form

$$u_z(t) = -\mathcal{K}z_c(t). \quad (12)$$

The nonlinear closed loop system of the perturbed state z and the estimated state z_c from (11) is defined by

$$\begin{bmatrix} \mathcal{M}\dot{z}(t) \\ \mathcal{M}\dot{z}_c(t) \end{bmatrix} = \begin{bmatrix} \tilde{\mathcal{A}} & -\mathcal{B}\mathcal{K} \\ \mathcal{F}\mathcal{C} & \mathcal{A}_c \end{bmatrix} \begin{bmatrix} z(t) \\ z_c(t) \end{bmatrix} + \begin{bmatrix} \tilde{\mathcal{G}}(z(t)) \\ \tilde{\mathcal{G}}(z_c(t)) \end{bmatrix} + \begin{bmatrix} \mathcal{D} & 0 \\ 0 & \mathcal{F}\mathcal{E} \end{bmatrix} \begin{bmatrix} \eta_z(t) \\ \eta_z(t) \end{bmatrix}$$

$$\begin{bmatrix} z(0) \\ z_c(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ z_{c0} \end{bmatrix}.$$

In the original variables, the system has the form

$$\begin{aligned}
\mathcal{M}\dot{w}(t) &= \mathcal{A}w(t) + \mathcal{G}(w(t)) + \mathcal{B}u(t) + \mathcal{D}\eta(t)\\
y(t) &= \mathcal{C}w(t) + \mathcal{E}\eta(t)\\
u(t) &= -\mathcal{K}\left(w_{\text{nom}} + z_c(t)\right)
\end{aligned}$$

since $u_{\text{nom}} = -\mathcal{K}w_{\text{nom}}$ and z_c satisfies (8) or (11).

2.3 Approximation and Reduced Order Compensator

For the implementation of the control we apply standard finite element techniques. This results in a semi-discrete finite dimensional approximation of (3) and (4)

$$\begin{aligned}
M^N\dot{z}^N(t) = \tilde{A}^N z^N(t) + \tilde{G}^N(z^N(t)) + B^N u_z^N(t)\\
+ D^N\eta_z^N(t),
\end{aligned} \quad (13)$$

with $z^N(0) = z_0^N$ and state measurements

$$y_z^N(t) = C^N z^N(t) + E^N\eta_z^N(t). \quad (14)$$

of order N. In a full order compensator design, these order N approximations are used to compute K^N, F^N and A_c^N and the finite dimensional approximations to the linear compensator equation (8) takes the form

$$M^N\dot{z}_c^N(t) = A_c^N z_c^N(t) + F^N y_z^N(t), \quad (15)$$

and the nonlinear equation (11) the form

$$M^N\dot{z}_c^N(t) = A_c^N z_c^N(t) + F^N y_z^N(t) + \tilde{G}^N(z_c^N(t)). \quad (16)$$

The control law is given by

$$u_z^N(t) = -K^N z_c^N(t). \quad (17)$$

The approximation to the nonlinear closed loop system, which we will refer to as the full order closed loop system, is given by

$$\begin{bmatrix} M^N\dot{z}^N(t) \\ M^N\dot{z}_c^N(t) \end{bmatrix} = \begin{bmatrix} \tilde{A}^N & -B^N K^N \\ F^N\mathcal{C}^N & A_c^N \end{bmatrix} \begin{bmatrix} z^N(t) \\ z_c^N(t) \end{bmatrix} + \begin{bmatrix} \tilde{G}^N(z^N(t)) \\ \tilde{G}^N(z_c^N(t)) \end{bmatrix} + \begin{bmatrix} D^N & 0 \\ 0 & F^N E^N \end{bmatrix} \begin{bmatrix} \eta_z^N(t) \\ \eta_z^N(t) \end{bmatrix}$$

$$\begin{bmatrix} z^N(0) \\ z_c^r(0) \end{bmatrix} = \begin{bmatrix} z_0^N \\ z_{c0}^N \end{bmatrix}. \tag{18}$$

See Marrekchi [1993] for more about the convergence of the finite dimensional approximations to the initial distributed parameter system.

These approximations are used to compute the matrices in the observer equation (16) as well as the control law (17) via solutions to finite dimensional approximations to the algebraic Riccati equations (9) and (10). These finite dimensional discretizations of the PDE system results in very large systems which provides numerous numerical challenges such as solving the approximate algebraic Riccati equations. Furthermore, to enable practical implementation, one must be able to integrate equation (16) in real time.

One approach to address this problem is to implement model reduction techniques. Here we use model reduction techniques to the compensator equation (both linear and nonlinear) where the goal is the design of a robust, low order controller with $r \ll N$. Note that the superscript r refers to the reduced order approximations. The reduced order compensator and control takes the form

$$\begin{aligned} M^r \dot{z}_c^r(t) &= A_c^r z_c^r(t) + F^r y_z^N(t) + \tilde{G}^r(z_c^r(t)), \\ u_z^r(t) &= -K^r z_c^r(t). \end{aligned}$$

For simulation one then considers the coupled system with the full order state and the reduced order state estimate

$$\begin{bmatrix} M^N \dot{z}^N(t) \\ M^r \dot{z}_c^r(t) \end{bmatrix} = \begin{bmatrix} \tilde{A}^N & -B^N K^r \\ F^r \mathcal{C}^N & A_c^r \end{bmatrix} \begin{bmatrix} z^N(t) \\ z_c^r(t) \end{bmatrix} + \begin{bmatrix} \tilde{G}^N(z^N(t)) \\ \tilde{G}^r(z_c^r(t)) \end{bmatrix} + \begin{bmatrix} D^N & 0 \\ 0 & F^r E^N \end{bmatrix} \begin{bmatrix} \eta_z^N(t) \\ \eta_z^N(t) \end{bmatrix}$$

$$\begin{bmatrix} z^N(0) \\ z_c^r(0) \end{bmatrix} = \begin{bmatrix} z_0^N \\ z_{c0}^r \end{bmatrix}.$$

It is important to point out that, when implemented, the control will be coupled with a physical system rather than the discretized differential equation. Thus our goal is not a low order model but rather a low order controller.

Note that $\tilde{G}^N(\cdot)$ and $\tilde{G}^r(\cdot)$ are properly nonlinear. Therefore, if we can reduce the problem such that stability properties of M^r, $\tilde{A}^N - B^N K^r$, and $\tilde{A}^r - F^r C^N$ match their full-order counterparts, the reduced-order compensator should maintain a stable system.

3. MODEL REDUCTION OF THE COMPENSATOR

In this section we present the IRKA algorithm as applied to the linear compensator (8) and the POD algorithm as applied to the nonlinear compensator (11).

3.1 Iterative Rational Krylov Algorithm (IRKA)

We only present the essential background for this $\mathcal{H}_2$-based model reduction technique. For details, see Gugercin et al. [2008]. Consider a linear system

$$\mathscr{E}\dot{x}(t) = \mathscr{A}x(t) + \mathscr{B}u(t), \quad y(t) = \mathscr{C}x(t)$$

with transfer function $H(s) = \mathscr{C}(s\mathscr{E} - \mathscr{A})^{-1}\mathscr{B}$, where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, and $y(t) \in \mathbb{R}^p$ is the output vector.

The objective is to design a high-fidelity reduced system with the same number of inputs and outputs as in the original linear system but with the state dimension $r \ll n$.

The iterative rational Krylov algorithm (IRKA) proposed in Gugercin et al. [2008] constructs a reduced system that satisfies the interpolation-based first order necessary conditions for the optimal $\mathcal{H}_2$ approximation:

$$\|H - H_r\|_{\mathcal{H}_2} = \min_{\dim \tilde{H}_r = r} \|H - \tilde{H}_r\|_{\mathcal{H}_2}, \quad \tilde{H}_r \ \text{stable},$$

where H_r denotes the reduced order transfer function. IRKA is iterative in nature and computationally effective in that it requires only matrix-vector multiplications and sparse linear solvers. The reduced model in IRKA is obtained by a Petrov-Galerkin projection, taking the form

$$\mathscr{E}_r \dot{x}_r(t) = \mathscr{A}_r x_r(t) + \mathscr{B}_r u(t), \quad y_r(t) = \mathscr{C}_r x_r(t)$$

with

$$\mathscr{E}_r = W_r^T \mathscr{E} V_r, \ \mathscr{A}_r = W_r^T \mathscr{A} V_r, \ \mathscr{B}_r = W_r^T \mathscr{B}, \ \mathscr{C}_r = \mathscr{C} V_r$$

where $W_r \in \mathbb{R}^{n \times r}$ and $V_r \in \mathbb{R}^{n \times r}$ are chosen to enforce the interpolation-based necessary conditions for $\mathcal{H}_2$ optimality. For details, see Gugercin et al. [2008]. Also, see Wilson [1970], Meier III and Luenberger [1967], Dooren et al. [2008] and the references therein for other works on $\mathcal{H}_2$ model reduction.

We apply this approach to the linear state estimate equation (15)

$$M^N \dot{z}_c^N(t) = A_c^N z_c^N(t) + F^N \underbrace{y_z^N(t)}_{input}, \ z_c^N(0) = z_{c0}^N, \tag{19}$$

$$\underbrace{u_{out}(t)}_{output} = \mathscr{C} z_c^N(t). \tag{20}$$

In this application the input vector is $y_z^N(t)$ and there are a number of options for the output vector u_{out}. If we choose $\mathscr{C} = -K^N$, the output is $u_{out}(t) = -K^N z_c^N(t)$ which is the control. If we choose $\mathscr{C} = -B^N K^N$, the output is $u_{out}(t) = B^N u^N(t)$ and if we choose $\mathscr{C} = I$, the output is $u_{out}(t) = z_c^N(t)$.

Even though the two approximate algebraic Riccati equations have to be solved, no full order simulations are needed to construct this linear reduced order model. Only the state measurements $y_z^N(t)$ and finite element matrices A_c^N, F^N and the matrices needed for $\mathscr{C}$ are required. These calculations are completed off-line after which the reduced order model for the state estimate can be implemented for real time control.

3.2 Proper Orthogonal Decomposition

The proper orthogonal decomposition (POD) is an empirical method for model reduction of nonlinear systems. It is based on finding one or a set of representative simulations of the nonlinear system, then computing a low dimensional

M^N-orthogonal basis that is optimal for representing the simulation data. This simulation data is referred to as the snapshot set in the literature Sirovich [1987]. The foundation of this process is the singular value decomposition (SVD) or Hilbert-Schmidt decomposition in the abstract setting.

One natural strategy is to perform simulations of the coupled state/compensator equations (18) to build the solution data. See for example Atwell et al. [2001]. Thus, given a set of solutions $\mathcal{W} \equiv \{z_c^N(\,\cdot\,; z_0^N, \eta^N)\}$ parametrized by different initial conditions and different realizations of the disturbance, we compute a low dimensional basis $\{\phi_1, \ldots, \phi_r\}$ that is M^N-orthogonal and is optimal for representing $\mathcal{W}$. Note that we may naturally include other problem parameters that arise, and perhaps modify this basis as other parameters change, see Hay et al. [2008, 2009, 2010].

To determine the POD reduced order model, let $\bar{z}_c$ be the mean, or other centering trajectory of z_c^N and write the POD model of the compensator, z_c^r, as

$$z_c^r(x,t) = \bar{z}_c(x,t) + \sum_{i=1}^{r} \phi_i(x) a_i(t)$$

where $\{\phi_i\}$ are the orthonormal basis functions that "best" represent the data: $z_c^N - \bar{z}_c$.

Substituting z_c^r into the weak form of the compensator equations leads to the reduced order system

$$\dot{a}_i = b_i + [A_c^r a]_i + \underbrace{a^T B_i^r a}_{G^r(a)} + \left[F^r C^N z^N\right]_i,$$

$$a_i(0) = \int_\Omega \phi_i(x)(z_{c0}^N(x) - \bar{z}_c(x))dx = \phi_i^T M^N (z_{c0}^N - \bar{z}_c)$$

where

$$A_c^r = \Phi^T A_c^N \Phi, \qquad F^r = \Phi^T F^N$$

and

$$[B_i^r]_{jk} = \int_\Omega \phi_j \phi_{k,x} \phi_i dx.$$

$$\begin{bmatrix} \dot{z}^N(t) \\ \dot{a}^r(t) \end{bmatrix} = \begin{bmatrix} \tilde{A}^N & -B^N K^M \\ F^r C^N & A^r \end{bmatrix} \begin{bmatrix} z^N(t) \\ a^r(t) \end{bmatrix} + \begin{bmatrix} G^N(z^N(t)) \\ G^r(a^r(t)) \end{bmatrix} + \begin{bmatrix} D^N & 0 \\ 0 & F^r E^N \end{bmatrix} \begin{bmatrix} \eta_z^N(t) \\ \eta_z^N(t) \end{bmatrix}$$

$$\begin{bmatrix} z^N(0) \\ a^r(0) \end{bmatrix} = \begin{bmatrix} z_0^N \\ \Phi^T M^N (z_{c0}^N - \bar{z}_c) \end{bmatrix}.$$

4. NUMERICAL RESULTS

4.1 Burgers Equation

Note that the form of the abstract Cauchy problem in (1) would apply to control problems of several relevant distributed parameters systems such as the Navier-Stokes equations and the Boussinesq equations. However, in this paper, we will restrict our attention to Burgers equation, which has a similar nonlinear term, but avoids complications with high index descriptor systems.

Therefore, we consider

$$\dot{w} = -w\mathbf{a} \cdot \nabla w + \nabla \cdot (\mu \nabla w) + b(x)u(t) + d(x)\eta(t)$$

from which we have the form (1) with the identifications

$$\mathcal{A}w = \nabla \cdot (\mu \nabla w) \quad \text{and} \quad \mathcal{G}w = -w\mathbf{a} \cdot \nabla w$$

along with $\mathcal{B}u(t) = b(\cdot)u(t)$ and $\mathcal{D}\eta(t) = d(\cdot)\eta(t)$. In the one dimensional case, $\mathbf{a} = 1$ and in higher spatial dimensions, $\|\mathbf{a}\| = 1$ is the advection direction. We linearize about $w_{\text{nom}} = 0$, so we have $w = z$, $\mathcal{A} = \tilde{\mathcal{A}}$, and $\mathcal{G} = \tilde{\mathcal{G}}$.

We consider the 1D Burgers equation with parameters and boundary conditions similar to those used by Atwell et al. [2001]. Specifically, we consider periodic boundary conditions:

$$w(t,0) = w(t,1), \quad w_x(t,0) = w_x(t,1),$$

initial conditon

$$w(0,x) = \begin{cases} 0.5\sin(2\pi x), & 0 < x \le 0.5 \\ 0, & 0.5 < x < 1 \end{cases}$$

and the viscosity parameter $\mu = 5 \times 10^{-4}$.

The state measurement y consists of five measurements averaging w over the intervals $[0, 0.1]$, $[0.2, 0.3]$, $[0.4, 0.5]$, $[0.6, 0.7]$, and $[0.8, 0.9]$. The disturbance takes the form $D\eta(t) = 0.75\cos(10t)1(x)$. The design parameters are chosen as follows: $R = 0.1I$ and $H = 1 \times 10^{-5}I$ and $Q = q(x)I$ where $q(x) = \begin{cases} 10, & 0.7 \le x \le 0.9 \\ 1, & \text{elsewhere.} \end{cases}$

We present a collection of results that are representative of the cases that we considered. The MinMax controller with $\theta = \theta_{\max}$ differed very little from the LQG controller where $\theta = 0$ and we only present the LQG case here. We use linear finite elements with $N = 80$ for all high dimensional approximations.

We consider two control input operators and present the two cases in the next sections.

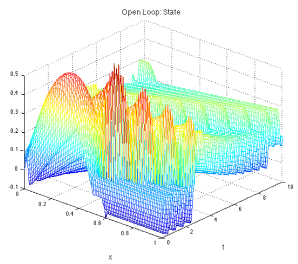
4.2 Low Rank Control Input

Let $Bu(t) = \begin{bmatrix} \chi_{[0.3,0.5]} & \chi_{[0.8,1]} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$.

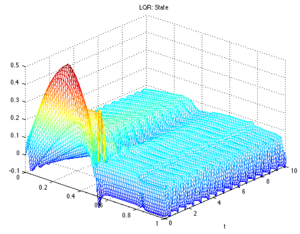
The open loop and closed loop, full state, LQR, simulations are presented in Figures (a) and (b). The effectiveness of the control is clear and we will compare the results using state estimators and reduced order models to this case.

In Figures (c) and (d), we present the resulting state of the coupled full order, closed loop system in (18) for the linear and nonlinear state estimators described in (8) and (11). In this case the results from the linear state estimator compares well to the results from nonlinear compensator. A measure for comparison of the two approaches is the LQG cost and in this case the cost for the linear case is 0.2982 and for the nonlinear case 0.2976.

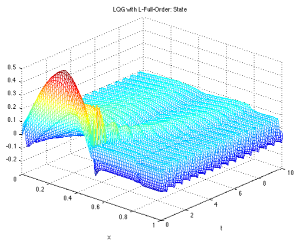
Since the linear compensator performed well in this case, we consider reduced order models of the linear state



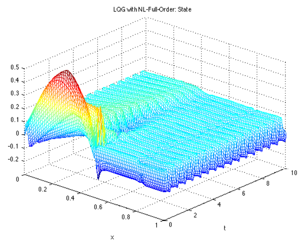
(a) Open Loop



(b) Closed loop, LQR

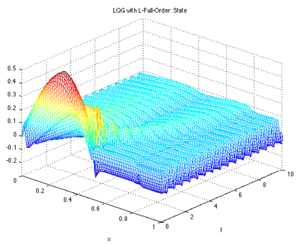


(c) LQG with linear, full order, compensator.

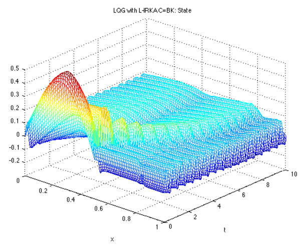


(d) LQG with nonlinear, full order, compensator.

estimator using IRKA. We present the results for the case where $r = 4$ and $\mathscr{C} = -B^N K^N$. Similar results are obtained for the other choices of $\mathscr{C}$.



(e) LQG with linear, full order, compensator.

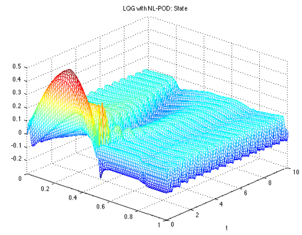


(f) Reduced order linear compensator, IRKA with $r = 4$.

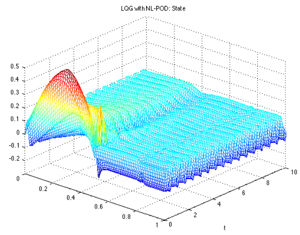
The reduced order model with only $r = 4$ compares well to the full order case as can be seen in Figures (e) and (f). The LQG costs are respectively 0.3253 and 0.3173 for the reduced order compensator using IRKA with $r = 4$ and $r = 8$ respectively. Compare that to the full order LQG cost which is 0.2982.

We also compare the full order, nonlinear compensator LQG model to the POD-based nonlinear reduced order model. As is expected, Figures (g) and (h) compare well to the full order case presented in Figure (d). The LQG cost for the full order nonlinear compensator case is 0.2976, while the cost using IRKA with $r = 8$ is 0.3173 and using POD with $r = 8$ is 0.2974.

It is important to note that the linear reduced order model performed very well in this case. The computational cost implementing IRKA is lower than the cost associated with POD since one does not need full order simulations of the coupled system as required for POD. This example suggests that IRKA could be successfully implemented in the construction of reduced order linear compensators.



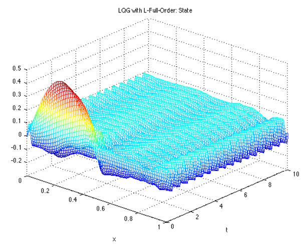
(g) POD with nonlinear compensator, $r = 4$.



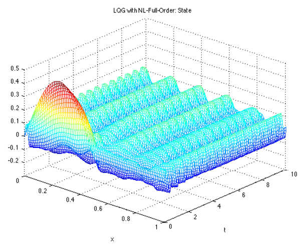
(h) POD with nonlinear compensator, $r = 8$.

4.3 High Rank Control Input

In this example we consider a high rank control input namely $Bu(t) = u(x, t)$. We first of all compare the performance of the linear and nonlinear full order compensators. Following this, the effectiveness of a reduced order linear compensator via IRKA and the effectiveness of a reduced order nonlinear compensator via POD are compared.

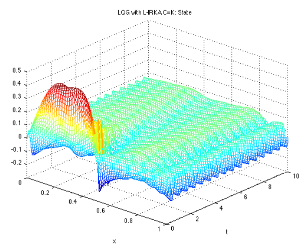


(1) LQG with linear, full order, compensator.

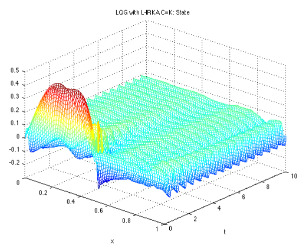


(2) LQG with nonlinear, full order, compensator.

Unlike the previous example, the states associated with a linear and nonlinear compensator differ significantly. We compare the linear reduced order state estimator using IRKA to the full state estimators in Figures (1) and (2). For this example we choose $\mathscr{C} = K$.



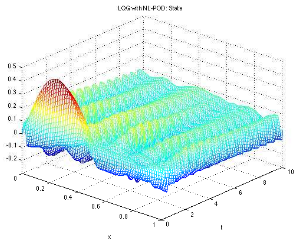
(3) Reduced order linear compensator, IRKA $r = 4$.



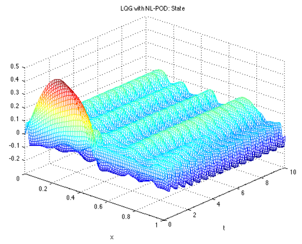
(4) Reduced order linear compensator, IRKA $r = 8$.

As we can see from comparing Figures (1), (3), and (4), the linear reduced model performs reasonably well at approximating the linear full order LQG system but does not capture the dynamics of the nonlinear full order system in Figure (2). The LQG cost for the linear, full order case is 0.1897 while the LQG cost for IRKA with $r = 4$ is 0.2094.

As is expected, the nonlinear reduced order compensator models via POD in Figures (5) and (6), approximate the nonlinear, full order model in Figure (2), better than the IRKA approach. However, there are concerns about the POD model near $x = 1$ as t increases.



(5) POD with nonlinear compensator, $r = 4$.



(6) POD with nonlinear compensator, $r = 8$.

These two examples demonstrate that the full-order nonlinear compensator offers better efficiency for both choices of B. Furthermore, POD of the nonlinear compensator is effective for some systems but there are no guarantees that it works for others, for example, it is important to capture the linear portion of the compensator correctly in the reduced order model.

These observations suggest the need for a hybrid reduced order modeling approach to get the best features of both reduction methods, for example, the computational cost can be significantly reduced by replacing the linear term in the POD model with an IRKA system.

This can be achieved by decomposing the state estimate z_c into two parts, a linear model that is reduced using IRKA and a nonlinear correction that is reduced using POD. For this approach set

$$z_c(t) = \ell_c(t) + n_c(t)$$

where $\ell_c(\cdot)$ is chosen as the solution to the linear problem

$$M\dot{\ell}_c(t) = A_c\ell_c(t) + Fy_z(t) \tag{21}$$

and,

$$M\dot{n}_c(t) = A_c n_c(t) + \tilde{G}\left(\ell_c(t) + n_c(t)\right). \tag{22}$$

The advantage of this approach is that (by adjusting the linear term in n_c) we are guaranteed to produce a combined model that maintains the linear stability properties of the high order model. This does not always happen using POD directly on nonlinear problems, see Aubry et al. [1988], Willcox and Megretski [2005], Borggaard et al. [2008], Wang et al. [2012].

The detail of this approach will appear in a future paper.

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