Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Robust Nonlinear Regulation: Continuous Internal Models and Hybrid Identifiers ⋆

Francesco Forte * **Lorenzo Marconi** *

* *Department of Electrical, Electronic and Information Engineering "Guglielmo Marconi", University of Bologna, Bologna, Italy, (e-mail: f.forte@unibo.it, lorenzo.marconi@unibo.it)*

Abstract: We consider the problem of output regulation for the class of minimum-phase nonlinear systems described in normal form. We assume that the ideal steady state control input fulfills a nonlinear regression law that is linearly parametrized in the uncertain parameters and we propose an internal model-based design that combines high-gain and identification tools. The identification tool by which the internal model is updated, is discrete-time by thus obtaining a hybrid internal model. The present paper is part of a wider research activity of the authors in which the attempt is to combine high-gain tools typically used in the context of nonlinear output regulation with identification tools that are here used to estimate the optimal regression law by best fitting the friend and its time derivatives.

1. INTRODUCTION

1.1 Background

Nonlinear output regulation is a well established research field in the nonlinear control theory that received increasing attention by the community in the last years. The problem of designing internal model-based regulators is particularly challenging in presence of uncertainties in the regulated plant, and in particular in the ideal steady state control law that secures zero regulation error. In the literature relevant results addressing the design of *robust* regulators, such as Huang [1995] and Serrani et al. [2001], can be found. In Ding [2003], Delli Priscoli et al. [2006] and Marino et al. [2008] efficient approaches to the problem with the use of the adaptive observers theory can be found, both in the linear case and in the nonlinear case and in global and semiglobal case. Hybrid tools have been also investigated in the attempt of designing robust regulators able to offset parametric uncertainties and exogenous disturbances. For instance, Serrani [2006] studies the interconnection of a hybrid adaptive law with a feedforward model of the disturbance. Recently (Isidori et al. [2012]), a design solution that does not rely upon conventional adaptation schemes has been proposed. The solution relies upon high-gain methods originally proposed in Byrnes et al. [2004] by using regression-like arguments to derive a nonlinear internal model able to offset the presence of uncertainties in the steady state control law (see also Marino et al. [2011]).

1.2 Contribution

The paper deals with the problem of adaptive output regulation in the particular case in which the steady state control action satisfies a nonlinear regression law that is linearly parametrized in the uncertainties. The design solution proposed by the authors in this context builds on the high-gain methods of Byrnes et al. [2004] for the design of the internal model. The novelty of the paper is to augment the internal model with an hybrid system aiming to identify on-line the regression law involving the ideal steady state control input (the so-called "friend") and its time derivatives. The main idea is to employ the internal model structure presented in Byrnes et al. [2004] also as a "dirty derivative observer" of the friend and its time derivatives that are then used in the dynamic hybrid identifier for online identification of the internal model structure.

1.3 Organization

The paper is organized as follows. In Section 2 some useful preliminaries about high-gain nonlinear output regulation and the main idea developed in the article. are presented. Section 3 presents the details of the proposed approach while Section 4 describes a simple example in order to test the proposed methodology.

2. PRELIMINARIES

2.1 Nonlinear Output Regulation

In this section we briefly recall some basic concepts regarding the nonlinear output regulation with high gain methods (Byrnes et al. [2003], Byrnes et al. [2004]) that are instrumental for the main result of the paper. We consider the following class of nonlinear systems in *normal form* and with *relative degree* equals to one [1]

$$\dot{w} = s(w) \tag{1}$$
$$\dot{z} = f(z, w, e) \tag{2}$$
$$\dot{e} = q(z, w, e) + b(z, w, e)u\,. \tag{3}$$

⋆ This research has been conducted in part under the collaborative project SHERPA (ICT 600958) supported by the European Community under the 7th Framework Programme.

[1] All the forthcoming results can be easily extended to systems with higher relative degree by means of standard tools.

In the previous system one can recognize two main subsystems: the first, described by (1), is the so-called *exosystem* with state $w \in W \subset \mathbb{R}^s$ generating possible *references* signals to be tracked and/or possible *disturbances* that must be rejected. The set W is a compact set that is assumed to be invariant for the exosystem dynamics (1). The second subsystem is the *controlled plant* given in (2)-(3) in which $(z, e) \in \mathbb{R}^n \times \mathbb{R}$ is the state, $u \in \mathbb{R}$ is the control input, and e is the regulation error. All functions in the overall system, i.e. $s(\cdot)$, $f(\cdot,\cdot,\cdot)$, $q(\cdot,\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot)$ are smooth in their arguments, with the function $b(\cdot,\cdot,\cdot)$, the so-called high-frequency gain of the system, that is assumed to be bounded from below by a positive number $\underline{b}$, i.e

$$b(z, w, e) \geq \underline{b} \qquad \forall (z, w, e) \in \mathbb{R}^n \times W \times \mathbb{R}\,.$$

In this framework, the control objective can be formulated as follows: given the sets $W \subset \mathbb{R}^s$, $Z \subset \mathbb{R}^n$ and $E \subset \mathbb{R}$ of initial conditions for the system (1)–(3), design a controller processing the error e, namely

$$\begin{aligned} \dot{\xi} &= \alpha(\xi, e), \qquad \xi \in \mathbb{R}^d \\ u &= \beta(\xi, e) \end{aligned}$$

with initial conditions in $\Xi \subset \mathbb{R}^d$ such that all trajectories of the closed loop system starting from $W \times Z \times E \times \Xi$ are bounded and $\lim_{t\to\infty} e(t) = 0$ uniformly in the initial conditions.

We shall approach the previous problem under assumptions that are customary in the literature of output regulation. In particular we assume the existence of a smooth function $\pi : \mathbb{R}^s \to \mathbb{R}^n$ that solves the regulator equation

$$L_{s(w)}\pi(w) = f(\pi(w), w, 0), \tag{4}$$

for all $w \in W$. This implies the existence of a compact set

$$\mathcal{A} := \{(w, z) \in W \times \mathbb{R}^n \;:\; z = \pi(w)\}$$

that is invariant for the dynamics

$$\dot{w} = s(w), \qquad \dot{z} = f(z, w, 0)\,. \tag{5}$$

The previous system is easily recognized to be the zero dynamics of system (1)-(3) relative to the input u and to the output e. As in most of the literature about output regulation, we make a minimum-phase assumption on system (5) that is formalized as follows.

Assumption. The set $\mathcal{A}$ is locally asymptotically stable for (5) with a domain of attraction of the form $W \times \mathcal{D}$ with $\mathcal{D}$ an open set of $\mathbb{R}^n$ such that $\mathcal{D} \supset Z$. $\triangleleft$

For notational convenience, in the following we let $\mathbf{z} := \mathrm{col}(w, z)$ so that system (5) can be compactly rewritten as

$$\dot{\mathbf{z}} = F(\mathbf{z}) \quad \text{with} \quad F(\mathbf{z}) := \mathrm{col}(s(w),\, f(z, w, 0))\,. \tag{6}$$

Furthermore, with a mild abuse of notation, we let $q(\mathbf{z}, e) = q(w, z, e)$ and $b(\mathbf{z}, e) = b(w, z, e)$.

In the design of the regulator a crucial role is played by the function $c(\mathbf{z})$ defined as

$$c(\mathbf{z}) = -q(\mathbf{z}, 0)/b(\mathbf{z}, 0)\,. \tag{7}$$

This function is readily seen to be the "friend" associated to the zero dynamics of system (1)–(3) (see Isidori [1995]), namely the control input that makes the set $\mathcal{A} \times \{0\}$ invariant for the system (1)–(3). In the context of output regulation, the output signals generated by system (6) with output (7) with initial conditions ranging in $\mathcal{A}$ are the steady state control inputs that must be generated by the controller in order to keep the regulation error identically to zero. It is thus apparent that system (6) with output (7) with initial conditions ranging in $\mathcal{A}$ plays a crucial role in the design of the regulator.

As a matter of fact it is a well-known fact (Marconi et al. [2007]) that the output regulation problem is solved if one is able to design smooth functions $M : \mathbb{R}^d \to \mathbb{R}^d$, $G : \mathbb{R}^d \to \mathbb{R}^{d\times 1}$, and $\gamma : \mathbb{R}^d \to \mathbb{R}$, such that, for some smooth function $\tau : \mathbb{R}^s \times \mathbb{R}^n \to \mathbb{R}^d$, the set

$$\mathrm{graph}(\tau(\mathbf{z})) := \{(\mathbf{z}, \xi) \in \mathcal{A} \times \mathbb{R}^d \;:\; \xi = \tau(\mathbf{z})\}$$

is locally asymptotically stable for the system

$$\dot{\mathbf{z}} = F(\mathbf{z}), \qquad \dot{\xi} = M(\xi) + G(\xi)c(\mathbf{z}) \tag{8}$$

with a domain of attraction $W \times \mathcal{D} \times \mathcal{C}$ with $\mathcal{C}$ an open set of $\mathbb{R}^d$ satisfying $\mathcal{C} \supset \Xi$, and, in addition,

$$\gamma(\tau(\mathbf{z})) = c(\mathbf{z}) \qquad \forall\, \mathbf{z} \in \mathcal{A}\,. \tag{9}$$

In this context, in fact, the controller that solves the problem at hand is a system of the form

$$\begin{aligned} \dot{\xi} &= M(\xi) + G(\xi)(\gamma(\xi) + v) \\ u &= \gamma(\xi) + v \\ v &= -\kappa(e) \end{aligned} \tag{10}$$

where $\kappa(\cdot)$ is a properly defined class-$\mathcal{K}$ function. As a matter of fact, the closed loop system given by (1)–(3) and (10) is a system that has relative degree one relative to the input v and output e and has a zero dynamics precisely given by (8). Furthermore, due to (9), the set $\mathrm{graph}(\tau(\mathbf{z})) \times \{0\}$ is an invariant set for the closed loop system with $v = 0$. Under this circumstances, standard high-gain arguments can be used to show that an "high-gain" function [2] $\kappa(\cdot)$ succeeds in making the set $\mathrm{graph}(\tau(\mathbf{z})) \times \{0\}$ locally asymptotically stable with a domain of attraction containing the compact set of initial conditions.

As shown in Marconi et al. [2008], functions $M(\cdot)$, $G(\cdot)$ and $\gamma(\cdot)$ with the desired properties can be always constructed by following a design procedure that, however, is not, in general, constructive. A relevant context where a *constructive* design procedure can be given is the one presented in Byrnes et al. [2004] asking that the friend $c(\mathbf{z})$ fulfills a regression formula of the form

$$L^d_{F(\mathbf{z})}c(\mathbf{z}) = \varphi(c(\mathbf{z}), L_{F(\mathbf{z})}c(\mathbf{z}), \ldots, L^{d-1}_{F(\mathbf{z})}c(\mathbf{z})), \quad \forall\, \mathbf{z} \in \mathcal{A} \tag{11}$$

for some d and some locally Lipschitz function $\varphi : \mathbb{R}^d \to \mathbb{R}$. In this case, in fact, the theory of high-gain observers (Gauthier et al. [2001]) can be used to show that the above properties are fulfilled with

$$G(\xi) = G := \mathrm{col}(\lambda_1 g,\, \lambda_2 g^2,\, \ldots,\, \lambda_d g^d)\,, \tag{12}$$

where g is a design parameter and the λ_i's that are coefficients of an Hurwitz polynomial,

$$M(\xi) := \mathrm{col}(\xi_2,\, \ldots,\, ,\xi_d,\, ,\varphi_s(\xi)) - G\xi_1\,, \tag{13}$$

where $\varphi_s(\cdot)$ is a uniformly bounded and globally Lipschitz function such that $\varphi(c(\mathbf{z}), L_{F(\mathbf{z})}c(\mathbf{z}), \ldots, L^{d-1}_{F(\mathbf{z})}c(\mathbf{z})) = \varphi_s(c(\mathbf{z}), L_{F(\mathbf{z})}c(\mathbf{z}), \ldots, L^{d-1}_{F(\mathbf{z})}c(\mathbf{z}))$ for all $\mathbf{z} \in \mathcal{A}$, and $\gamma(\xi) = \xi_1$. By choosing $M(\cdot)$, G, and $\gamma(\cdot)$ in this way, it turns out that there exists a $g^\star > 1$ (depended on the Lipschitz

[2] The $\kappa(e)$ can be indeed taken as a linear function ke with k a sufficiently large gain if the set $\mathrm{graph}(\tau(\mathbf{z}))$ is also locally *exponentially* stable for (8).

constant and on the bound of $\varphi_s(\cdot)$) such that, having defined

$$\begin{aligned}\tau(\mathbf{z}) &= \mathrm{col}(\tau_1(\mathbf{z}),\ldots,\tau_d(\mathbf{z}))\\ &:= \mathrm{col}(c(\mathbf{z}),\ldots,L_{F(\mathbf{z})}^{d-1}c(\mathbf{z}))\,,\end{aligned} \tag{14}$$

the set graph($\tau(\mathbf{z})$) is locally asymptotically stable for (8) and (9) is fulfilled.

2.2 Main Idea

In this paper we propose a design procedure that builds on the framework of Byrnes et al. [2004], by thus assuming a relation of the form (11) involving the friend $c(\mathbf{z})$ and its time derivatives. Our goal is to present a design procedure that accounts for possible uncertainties in the function $\varphi(\cdot)$. For sake of simplicity we assume that the function $\varphi(\cdot)$ is linearly parametrized in the uncertainties, namely we assume that there exist a $p > 0$ and a known smooth function $\phi : \mathbb{R}^d \to \mathbb{R}^p$ such that

$$\varphi(\xi_1,\ldots,\xi_d) = \theta^T\phi(\xi_1,\ldots,\xi_d) \tag{15}$$

where $\theta \in \mathbb{R}^p$ is a vector of uncertainties. We assume that $\theta \in P$, with P a known compact set of $\mathbb{R}^p$. By bearing in mind that what really matter in the design of the regulator is the value of $\varphi(\cdot)$ evaluated at $\tau(\mathbf{z})$, it is apparent that a crucial objective in the regulator design is to estimate the vector θ by computing the best "fitting" between the d-th derivative of the friend, $L_F^d c(\mathbf{z})$ and the regressor $\phi(c(\mathbf{z}),\ldots,L_F^{d-1}c(\mathbf{z}))$, for all possible $\mathbf{z}\in\mathcal{A}$. The problem at hand can be clearly cast as a identification problem. If the friend $c(\mathbf{z})$ and its derivative up to the order $d+1$ were known, the problem could be addressed by running identification algorithms, such as least-square methods, to compute the parameter that best fits the data. Since $c(\mathbf{z}),\ldots,L_F^d c(\mathbf{z})$ are not measurable in the output regulation context, the idea that is pursued in the paper is to estimate their value by employing the "dirty derivative" (using the terminology in Teel et al. [1995]) features of the internal model of the form indicated at the end of the previous section. Namely, the ability of the ξ-system in (8), with $M(\cdot)$ and G given in (13) and (12) to roughly estimate the friend $c(\mathbf{z})$ and its time derivative up to the order $d-1$, with an estimation error that can be arbitrarily decreased by increasing g, *regardless* the specific expression $\varphi_s(\cdot)$ in (13) (provided that a bound on the Lipschitz constant is fixed). Since the identification problem potentially requires the knowledge also of the $L_F^d c(\mathbf{z})$, the regulator that is presented later has dimension $d+1$, namely one more with respect to the one presented above. The extra state variable ξ_{d+1}, that is redundant as far as the internal model property is concerned, has precisely the role of providing a "dirty estimate" of $L_F^d c(\mathbf{z})$ that turns out to be crucial in estimating the actual value of θ.

In order to precisely present the regulator structure, let $\phi' : \mathbb{R}^{d+1}\to\mathbb{R}$ be the locally Lipschitz function defined as

$$\phi'(\zeta_1,\ldots,\zeta_{d+1}) = \sum_{i=1}^{d}\frac{\partial\phi(\zeta_1,\ldots,\zeta_d)}{\partial\zeta_i}\zeta_{i+1}$$

and let $\phi'_s : \mathbb{R}^{d+1}\to\mathbb{R}$ a globally Lipschitz and uniformly bounded function such that

$$\phi'_s(c(\mathbf{z}),\ldots,L_{F(\mathbf{z})}^d c(\mathbf{z})) = \phi'(c(\mathbf{z}),\ldots,L_{F(\mathbf{z})}^d c(\mathbf{z}))\ \forall\,\mathbf{z}\in\mathcal{A}\,.$$

Then, along the lines of Byrnes et al. [2004], our controller takes the following hybrid form whose flow and jump dynamics are governed by a clock variable, denoted by τ_c, that resets every $\tau_{c,\max}$ instances, in details it flows according to

$$\begin{aligned}&\dot\tau_c = 1, \quad \tau_c\in[0,\tau_{c,\max}]\\ &\begin{bmatrix}\dot\xi_{[1,d]}\\ \dot\xi_{d+1}\end{bmatrix} = \begin{bmatrix}\xi_{[2,d+1]}\\ \hat\theta^T\phi'_s(\xi)\end{bmatrix} + [\lambda_1 g,\ldots,\lambda_{d+1}g^{d+1}]^T u\end{aligned} \tag{16}$$

and jumps according to

$$\begin{aligned}&\tau_c^+ = 0, \quad \tau_c\in\{\tau_{c,\max}\}\\ &\begin{bmatrix}\xi_{[1,d]}^+\\ \xi_{d+1}^+\end{bmatrix} = \begin{bmatrix}\xi_{[1,d]}\\ \hat\theta^T\phi_s(\xi_{[1,d]})\end{bmatrix}\end{aligned} \tag{17}$$

with $u = \xi_1 + v$ and where, for compactness, $\xi_{[a,b]}$ denotes the sub-vector of $\xi = (\xi_1,\ldots,\xi_{d+1})^T$ containing only the components from a to b, v is a residual input and $\hat\theta$ is an estimate of the uncertainty θ. The estimation of θ, as said, takes advantage of the fact that system (16)-(17) can be tuned to have ξ approximating $c(\mathbf{z}),\ldots,L_F^d c(\mathbf{z})$. In our design the *dynamic identifier* is a hybrid system that flows according to

$$\left.\begin{aligned}\dot\tau_c &= 1\\ \dot\eta &= F_\eta(\eta,\xi) \quad \eta\in\mathbb{R}^m\end{aligned}\right\}\quad \tau_c\in[0,\tau_{c,\max}] \tag{18}$$

and jumps according to

$$\left.\begin{aligned}\tau_c^+ &= 0\\ \eta^+ &= J_\eta(\eta,\xi)\end{aligned}\right\}\quad \tau_c\in\{\tau_{c,\max}\} \tag{19}$$

with the estimate $\hat\theta$ of the form

$$\hat\theta = \Gamma_\eta(\eta) \tag{20}$$

where the functions $F_\eta(\cdot)$, $J_\eta(\cdot)$ and $\Gamma_\eta(\cdot)$ are smooth functions. Intuitively, the previous system must be designed so that if ξ is replaced by $c(\mathbf{z}),\ldots,L_F^d c(\mathbf{z})$, then $\hat\theta$ asymptotically converge to θ. The fact that ξ is not coincident with the friend $c(\mathbf{z})$ and its time derivative, will require additional robustness properties that will be detailed in Section 3.1.

3. MAIN RESULT

3.1 Identifier Design Requirement

We start by making explicit the requirements to be fulfilled by the dynamic identifier (18), (19), (20). With a mild abuse of notation with respect to the previous section we let

$$\tau_e(\mathbf{z}) = \mathrm{col}(c(\mathbf{z})\,,\, L_F c(\mathbf{z})\,,\ldots,\, L_F^d c(\mathbf{z}))$$

Identifier Design Requirement. System (18), (19), (20) is said to satisfy a "Identifier Design Requirement" if there exists a smooth function $\sigma : \mathbb{R}\times\mathbb{R}^s\times\mathbb{R}^n\to\mathbb{R}^m$ such that the hybrid system flowing according to

$$\left.\begin{aligned}\dot\tau_c &= 1, \quad \dot{\mathbf{z}} = F(\mathbf{z})\\ \dot\eta &= F_\eta(\eta,\tau_e(\mathbf{z})+d) \quad \eta\in\mathbb{R}^m\end{aligned}\right\}\quad \tau_c\in[0,\tau_{c,\max}] \tag{21}$$

and jumping according to

$$\left.\begin{aligned}\tau_c^+ &= 0, \quad \mathbf{z}^+ = 0\\ \eta^+ &= J_\eta(\eta,\tau_e(\mathbf{z})+d)\end{aligned}\right\}\quad \tau_c\in\{\tau_{c,\max}\} \tag{22}$$

is ISS with restrictions relative to the set

$$\begin{aligned}\mathrm{gr}(\sigma(\tau_c,\mathbf{z})) = \{(\tau_c,\mathbf{z},\eta)\in[0,\tau_{c,\max}]\times\mathcal{A}\times\mathbb{R}^m :\\ \eta = \sigma(\tau_c,\mathbf{z})\}\end{aligned}$$

with respect to the input d. That is (see Liberzon et al. [2012]), there exists $r > 0$ such that, letting $\zeta = \mathrm{col}(\tau_c, \mathbf{z}, \eta)$, for all $\zeta(0,0) \in [0, \tau_{c,\max}] \times (W \times Z) \times X$, and for all bounded d satisfying $\|d(\cdot,\cdot)\|_\infty \le r$, the trajectory of system (21)-(22) is bounded and

$$\|\zeta(t,j)\|_{\mathrm{gr}(\sigma(\tau_c,\mathbf{z}))} \le \max\{\beta(\|\zeta(0,0)\|_{\mathrm{gr}(\sigma(\tau_c,\mathbf{z}))}, t+j)\,, \gamma(\|d(t,j)\|)\}$$

where $\beta(\cdot,\cdot)$ and $\gamma(\cdot)$ are respectively a class-$\mathcal{KL}$ and a class-$\mathcal{K}$ function. Furthermore, there exists $\bar{\theta} > 0$ such that

$$\|\Gamma_\eta(\eta)\| \le \bar{\theta} \qquad \forall\, \eta \in \mathbb{R}^m \tag{23}$$

and the following holds

$$L_F^d c(\mathbf{z}) = \Gamma_\eta(\sigma(\tau_c, \mathbf{z}))^T\, \phi(c(\mathbf{z})\,,\, L_F c(\mathbf{z})\,, \ldots,\, L_F^{d-1} c(\mathbf{z})) \tag{24}$$

for all $\mathbf{z} \in \mathcal{A}$ and for all $\tau_c \in [0, \tau_{c,\max}]$. ◁

We will show later in Section 3.3 a possible design of a system of the form (18), (19), (20) fulfilling the previous requirement.

3.2 Closed loop analysis

Again with a mild abuse of notation with respect to the previous section, we let

$$G(\xi) = G := \mathrm{col}(\lambda_1 g,\, \lambda_2 g^2,\, \ldots,\, \lambda_d g^d,\, \lambda_{d+1} g^{d+1})\,, \tag{25}$$

where g is a design parameter and the λ_i's that are coefficients of an Hurwitz polynomial, and

$$M(\xi) := \mathrm{col}(\xi_2,\, \ldots,\, ,\xi_{d+1},\, ,\hat{\theta}^T \phi_s'(\xi)) - G\xi_1\,, \tag{26}$$

The closed loop system is a hybrid system flowing when $\tau_c \in [0, \tau_{c,\max}]$ according to

$$\begin{aligned} \dot{\tau}_c &= 1 \\ \dot{\mathbf{z}} &= F(\mathbf{z}) + \Upsilon(\mathbf{z}, e)e \\ \dot{\xi} &= M(\xi) + G(\xi_1 + v) + B(\Gamma_\eta(\eta) - \theta)^T \phi_s'(\xi) \\ \dot{\eta} &= F_\eta(\eta, \xi) \\ \dot{e} &= q(\mathbf{z}, e) + b(\mathbf{z}, e)(\xi_1 + v) \end{aligned} \tag{27}$$

and jumping whenever $\tau_c \in \{\tau_{c,\max}\}$ according to

$$\begin{aligned} \tau_c^+ &= 0 \\ \mathbf{z}^+ &= \mathbf{z} \\ \xi^+ &= \mathrm{col}(\xi_{[1,d]}, \Gamma_\eta^T(J_\eta(\eta, \xi))\phi_s(\xi_{[1,d]})) \\ \eta^+ &= J_\eta(\eta, \xi) \\ e^+ &= e \end{aligned} \tag{28}$$

where $B \in \mathbb{R}^{d+1 \times 1}$ is the column vector with all the entries that are zero except the last one that is 1, and $\Upsilon(\cdot,\cdot)$ is an appropriately defined function. This is a system that has relative degree one with respect to the input v and output e and a zero dynamics that flows when $\tau_c \in \{\tau_{c,\max}\}$ according to

$$\begin{aligned} \dot{\tau}_c &= 1 \\ \dot{\mathbf{z}} &= F(\mathbf{z}) \\ \dot{\xi} &= M(\xi) + Gc(\mathbf{z}) + B(\Gamma_\eta(\eta) - \theta)^T \phi_s'(\xi) \\ \dot{\eta} &= F_\eta(\eta, \xi) \end{aligned} \tag{29}$$

and jumping according to

$$\begin{aligned} \tau_c^+ &= 0 \\ \mathbf{z}^+ &= \mathbf{z} \\ \xi^+ &= \mathrm{col}(\xi_{[1,d]}, \Gamma_\eta^T(J_\eta(\eta, \xi))\phi_s(\xi_{[1,d]})) \\ \eta^+ &= J_\eta(\eta, \xi) \end{aligned} \tag{30}$$

for all $\tau_c \in \{\tau_{c,\max}\}$. In the following we analyze such a system, by showing that if the dynamic identifier (18), (19), (20) fulfills the previous design requirement then, for a sufficiently large value of g, the zero dynamics have a well-defined attractor that is locally asymptotically stable. To this purpose, we regard the zero dynamics as interconnection of two systems. The first is a hybrid system described by

$$\begin{aligned} \dot{\tau}_c &= 1, \quad \tau_c \in [0, \tau_{c,\max}] \\ \dot{\mathbf{z}} &= F(\mathbf{z}) \\ \dot{\xi} &= M(\xi) + Gc(\mathbf{z}) + B\Lambda(\eta, \xi, \mathbf{z}) + Bu_1 \end{aligned} \tag{31}$$

and

$$\begin{aligned} \tau_c^+ &= 0, \quad \tau_c \in [0, \tau_{c,\max}] \\ \mathbf{z}^+ &= \mathbf{z} \\ \xi^+ &= \mathrm{col}(\xi_{[1,d]}, \Gamma_\eta^T(J_\eta(\eta, \xi))\phi_s(\xi_{[1,d]})) \end{aligned} \tag{32}$$

with $\Lambda(\eta, \xi, \mathbf{z}) = (\Gamma_\eta(\eta) - \theta)^T(\phi_s'(\xi) - \phi_s'(\tau_e(\mathbf{z})))$, regarded as a system with state $(\mathbf{z}, \xi) \in W \times \mathbb{R}^n \times \mathbb{R}^{d+1}$, input $u_1 \in \mathbb{R}$ and output $y_1 \in \mathbb{R}^{d+1}$ defined as

$$y_1 = \xi - \tau_e(\mathbf{z})\,.$$

The second is an hybrid system flowing according to

$$\begin{aligned} \dot{\tau}_c &= 1, \quad \tau_c \in [0, \tau_{c,\max}] \\ \dot{\mathbf{z}} &= F(\mathbf{z}) \\ \dot{\eta} &= F_\eta(\eta, \tau_e(\mathbf{z}) + u_2) \end{aligned} \tag{33}$$

and jumping according to

$$\begin{aligned} \tau_c^+ &= 0, \quad \tau_c \in \{\tau_{c,\max}\} \\ \mathbf{z}^+ &= \mathbf{z} \\ \eta^+ &= J_\eta(\eta, \tau_e(\mathbf{z}) + u_2) \end{aligned} \tag{34}$$

The latter is regarded as a system with state $(\tau_c, \mathbf{z}, \eta) \in [0, \tau_{c,\max}] \times W \times \mathbb{R}^n \times \mathbb{R}^m$, input $u_2 \in \mathbb{R}^{d+1}$ and output $y_2 \in \mathbb{R}$ defined as

$$y_2 = (\Gamma_\eta(\eta) - \theta)^T \phi_s'(\tau_e(\mathbf{z}))\,.$$

It easy to see that the zero dynamics are obtained by the interconnection

$$u_1 = y_2 \qquad u_2 = y_1\,.$$

Proposition 1. Consider system (31)-(32) with input u_1 and output u_2, and let $\Gamma_\eta(\cdot)$ appearing in $\Lambda(\cdot)$ be fulfilling (23) for some $\bar{\theta}$. Then there exists a $g^\star \ge 1$ such that for all $g \ge g^\star$ such a system is pre-ISS relative to the set

$$\mathcal{E} := \{(\tau_c, \mathbf{z}, \xi) \in \mathbb{R} \times \mathcal{A} \times \mathbb{R}^{d+1} \;:\; \xi = \tau_e(\tau_c, \mathbf{z})\}$$

with respect to the input u_1. In particular, there exists a c such that for all $\mathbf{z}(0) \in W \times Z$, $\xi(0) \in \Xi$, and for all bounded $u_1(t)$, the trajectories of the system are bounded and the following asymptotic bound holds true

$$\limsup_{t\to\infty} \|y_1(t)\| \le \frac{c}{g} \limsup_{t\to\infty} \|u_1(t)\|\,.$$

In other words there exists a locally Lipschitz Lyapunov function $V : \mathbb{R} \times \mathcal{A} \times \mathbb{R}^{d+1} \to \mathbb{R}_{\ge 0}$ such that the following hold

- there exist positive $\underline{\alpha}_\xi$, $\bar{\alpha}_\xi$ such that
$$\underline{\alpha}_\xi ||(\tau_c, \mathbf{z}, \xi)||_{\mathcal{E}} \le V(\tau_c, \mathbf{z}, \xi) \le \bar{\alpha}_\xi ||(\tau_c, \mathbf{z}, \xi)||_{\mathcal{E}};$$
- there exist $\chi_\xi > 0$ and $c_\xi > 0$ such that
$$V(\tau_c, \mathbf{z}, \xi) \ge \frac{\chi_\xi}{g} ||u_1|| \Rightarrow \dot{V}(\tau_c, \mathbf{z}, \xi) \le -c_\xi V(\tau_c, \mathbf{z}, \xi);$$
- there exists $0 < \lambda_\xi < 1$ such that we have
$$V(\tau_c, \mathbf{z}, \xi) \le \max\{\lambda_\xi V(\tau_c, \mathbf{z}, \xi), \chi_\xi ||u_1||\}.$$

Furthermore, let $\|u_1\|_\infty \le \bar{u}$. Then for all $\epsilon > 0$ and for all $t^\star > 0$ there exists a $g_2^\star > 0$ such that for all $g \ge g_2^\star$ the following holds

$$\|y_1(t)\| \le \epsilon \qquad \forall\, t \ge t^\star\,.$$

Now we move the attention to system (33)-(34) with input u_2 and output y_2. By the "Identifier Design Requirement" such a system is pre-ISS relative to the set $\mathrm{gr}(\sigma(\tau_c, \mathbf{z}))$ *with non zero restrictions* r *on the input* u_2. In the following it will be shown that the restriction on the input is fulfilled in (arbitrarily small) finite time. To this end, we observe that, by the requirement (23), by the fact that $\theta \in P$, P a compact set, and since $\phi_s'(\cdot)$ is a bounded function, there exists a positive $\bar{\delta}$ such that

$$\|y_2\| \le \bar{\delta} \qquad \forall\, (\eta, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^{s+n}\,.$$

From this, using the claim in the second part of Proposition 1, it is immediately concluded that for all $t^\star > 0$, there exists a $g^\star$ such that for all $g \ge g^\star$ $\|u_2(t)\| \le r$ for all $t \ge t^\star$.

Theorem 1. There exists a $g^\star > 0$ such that for all $g \ge g^\star$ the set

$$\mathcal{B} = \{(\tau_c, \mathbf{z}, \eta, \xi) \in [0, \tau_{c,\max}] \times \mathcal{A} \times \mathbb{R}^m \times \mathbb{R}^{d+1} \;:\; \eta = \sigma(\tau_c, \mathbf{z})\,, \xi = \tau_e(\tau_c, \mathbf{z})\}$$

is locally asymptotically stable with a domain of attraction containing the set of initial conditions $[0, \tau_{c,\max}] \times W \times Z \times X \times \Xi$. Furthermore, $q(\mathbf{z}, 0) + b(\mathbf{z}, 0)\xi_1 = 0$ for all $(\tau_c, \mathbf{z}, \eta, \xi) \in \mathcal{B}$.

Proof: (Sketch) Using $L_F c(\mathbf{z}) = \theta^T \phi_s'(\tau_e(\mathbf{z}))$ for all $\mathbf{z} \in \mathcal{A}$ and using (24), it turns out that

$$\|y_2\| \le c_1 \|(\tau_c, \mathbf{z}, \eta)\|_{\mathrm{gr}(\sigma(\tau_c, \mathbf{z}))}$$

for some positive c_1. From this the asymptotic stability of the set $\mathcal{B}$ follows by the small gain theorem in Liberzon et al. [2012] by using the fact that the restriction on the input u_2 can be fulfilled in an arbitrarily small time (so that finite escape time can be prevented) and using the first part of Proposition 1. The second part follows from the definition of $\tau_e(\mathbf{z})$ and of $c(\mathbf{z})$. ◁

The result of the previous theorem is instrumental to the analysis of the overall closed-loop system (27)-(28). The fact that such a system has relative degree one (from the input v to the error e), that the zero dynamics has an asymptotically stable attractor $\mathcal{B}$, and that $\mathcal{B} \times \{0\}$ is an invariant set for the system with $v = 0$ (as it immediately follows from the second part of Theorem 1), allows one to claim the following final theorem.

Theorem 2. Let g be fixed according to Theorem 1. Then there exists a class-$\mathcal{K}$ function $\kappa(\cdot)$ such that the set $\mathcal{B} \times \{0\}$ is locally asymptotically stable for (27)-(28) with $v = \kappa(e)$ with a domain of attraction that contains the set of initial conditions $[0, \tau_{c,\max}] \times W \times Z \times X \times \Xi \times E$.

3.3 A Possible Design of the Identifier

In this part we briefly sketch a possible design of the hybrid identifier fulfilling the requirements specified in section 3.1. As typical in adaptive control we make a persistence of excitation assumption detailed in the following.

Assumption. (Persistence of excitation) There exist positive $T^\star$ and δ such that

$$\det \int_t^{t+T} \phi(\tau(\mathbf{z}(\mathbf{z}_0, s)))\, \phi(\tau(\mathbf{z}(\mathbf{z}_0, s)))^T ds \ge \delta$$

for all $t \ge 0$, $T \ge T^\star$ and $\mathbf{z}_0 \in \mathcal{A}$, having defined $\mathbf{z}(\mathbf{z}_0, s)$ the trajectory of $\dot{\mathbf{z}} = F(\mathbf{z})$ at time s with initial condition $\mathbf{z}_0$. ◁

Note that, by continuity arguments, there exist positive r and δ' such that

$$\det \int_t^{t+T} \phi(\tau(\mathbf{z}(\mathbf{z}_0, s)) + d)\, \phi(\tau(\mathbf{z}(\mathbf{z}_0, s)) + d)^T ds \ge \delta'$$

for all $t \ge 0$, $T \ge T^\star$ and $\mathbf{z}_0 \in \mathcal{A}$, and for all d such that $\|d\| \le r$. Now, starting from $L_F^d c(\mathbf{z}) = \theta^T \phi(c(\mathbf{z}), \ldots, L_F^{d-1} c(\mathbf{z}))$, post multiplying both sides for $\phi(c(\mathbf{z}), \ldots, L_F^{d-1} c(\mathbf{z}))^T$, and taking the integral from 0 to $\tau_{c,\max}$, with $\tau_{c,\max} > T^\star$, it turns out that

$$\theta^T = \int_0^{\tau_{c,\max}} L_F^d c(\mathbf{z}(\mathbf{z}_0, s))\, \phi(\tau(\mathbf{z}(\mathbf{z}_0, s)))^T \cdot \left[\int_0^{\tau_{c,\max}} \phi(\tau(\mathbf{z}(\mathbf{z}_0, s)) \phi(\tau(\mathbf{z}(\mathbf{z}_0, s)))^T\right]^{-1} \tag{35}$$

This expression motivates a possible identifier flowing according to

$$\dot{\tau}_c = 1\,,\ \dot{\eta}_1 = \xi_{d+1} \phi(\xi)^T\,,\ \dot{\eta}_2 = \phi(\xi)\phi(\xi)^T\,,\ \dot{\eta}_3 = 0$$

when $\tau_c \in [0, \tau_{c,\max}]$, and jumping according to

$$\tau_c^+ = 0\,,\ \eta_1^+ = 0\,,\ \eta_2 = 0\,,\ \eta_3 = \eta_1 \eta_2^\dagger$$

when $\tau_c \in \{\tau_{c,\max}\}$, where $\eta_2^\dagger$ is the pseudo inverse of η_2, and taking $\hat{\theta} = \eta_3^T$. The estimate parameter $\hat{\theta}$ is kept constant in the clock interval and it is updated at each clock by properly elaborating the value of η_1 and η_2. The latter integrate the quantities $\xi_{d+1}\phi(\xi)^T$ and $\phi(\xi)\phi(\xi)^T$ during the clock interval in order to obtain a parameter estimation in the spirit of (35). It turns out that the previous system has the ISS property with restrictions required in Section 3.1.

4. SIMULATION RESULTS

In this section we show a simple, yet relevant, example of application of the theory proposed above. As regulated plant we consider a controlled Van der Pol oscillator described by

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -x_1(t) + (1 - x_1^2(t))x_2(t) - w_1(t) + u(t)$$

with initial conditions $(x_1(0), x_2(0)) = (x_{10}, x_{20})$, where $u(t)$ is the control input, while $w_1(t)$ is an exogenous input generated by an autonomous Duffing oscillator (playing the role of exosystem)

$$\dot{w}_1(t) = w_2(t) \qquad\qquad w_1(0) = w_{10}$$
$$\dot{w}_2(t) = -\alpha^\star w_1^3(t) - \beta^\star w_1(t) \qquad w_2(0) = w_{20}$$

with $\theta := \mathrm{col}(\alpha^\star, \beta^\star)$ that is the constant vector uncertainties. Let the regulation error be defined as $e(t) := x_1(t) + x_2(t)$ and let $z(t) := x_1(t)$. Simple calculations show that the system in the new coordinates is in the form (2)-(3), namely

$$\dot{z}(t) = -z(t) + e(t) \tag{36}$$

$$\dot{e}(t) = (2 - z^2(t))e(t) + (z^2(t) - 3)z - w_1(t) + u(t) \tag{37}$$

with initial conditions $(z(0), e(0)) = (z_0, e_0) := (x_{10}, x_{10} + x_{20})$. It is readily seen that in this case $\mathbf{z} = (w_1, w_2, z)$ and

$c(\mathbf{z}) = w_1$. For the design of the internal model part it is possible to observe that the regression law (11) is directly fulfilled by the steady state input with $d = 2$, leading to a third order internal model. On the other hand, the hybrid identifier has dimension equals to 9, because of the presence of a clock $\tau_c(t) \in \mathbb{R}$, a vector $\eta_1(t) \in \mathbb{R}^2$, a matrix $\eta_2(t) \in \mathbb{R}^{2\times 2}$ and the vector of estimated parameters $\hat{\theta}(t) := \mathrm{col}(\hat{\alpha}(t), \hat{\beta}(t))$. In Table 1 it is possible to find the list of the parameters used in this example, with

$(\alpha^\star, \beta^\star) = (1, -0.5)$	$(w_{10}, w_{20}) = (2.4, 1.3)$
$(x_{10}, x_{20}) = (1.2, 2.3)$	$(g, k, \tau_{c,\max}) = (10, 100, 1)$
$(\xi_{10}, \xi_{20}, \xi_{30}) = (0, 0, 0)$	$(\tau_{c,0}, \eta_{10}, \eta_{20}, \hat{\theta}_0) = (0, 0, 0, 0)$

Table 1. List of parameters used in the simulation.

$(\xi_{10}, \xi_{20}, \xi_{30})$ and with $(\tau_{c,0}, \eta_{10}, \eta_{20}, \hat{\theta}_0)$ that are the initial conditions of the internal model and the hybrid identifier respectively. The outcome of the simulation is represented in Figure 1, where the error variable $e(t)$ and the error between the real and estimated parameters are plotted. At

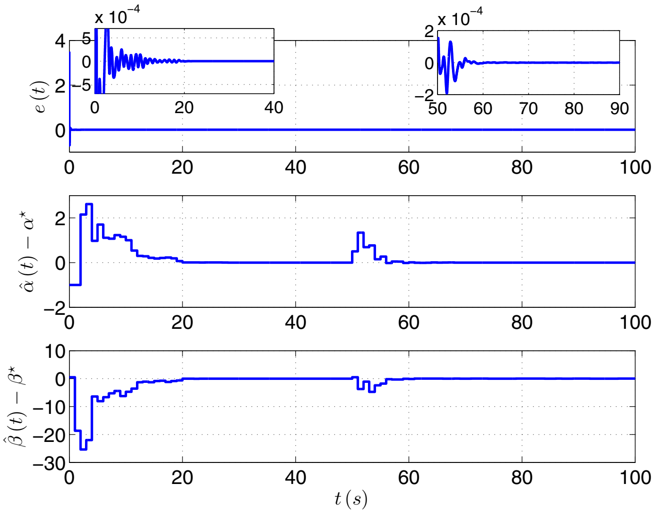

Fig. 1. The behaviors of the three errors, namely, the regulation error $e(t) = x_1(t)+x_2(t)$, the error between $\alpha^\star$ and $\hat{\alpha}(t)$ and finally the error between $\beta^\star$ and $\hat{\beta}(t)$.

time $t = 50\,s$ we simulated a change of value in the vector of real parameters from $(1, -0.5)$ to $(0.5, -1)$ just to check the efficiency of the hybrid identifier. Fig. 1 confirms good performances of the proposed controller. In particular, observing the two small figures, it is possible to see how the error $e(t)$ reaches exactly the zero value only when the estimation error converges to zero.

5. CONCLUSIONS

We have considered the problem of designing an internal model based controller by putting particular attention on the class of controlled plants described by equations (1)-(2)-(3) and on the particular framework in which the steady state control input is generated by a nonlinear regression law, linear in all the uncertain parameters. We have shown how it is possible to construct an overall regulator composed by two main systems: the first is the internal model designed according to Byrnes et al. [2004], which plays the role of the steady state input generator, and the hybrid identifier that provides an estimate of the regression law underlying the friend and its time derivatives, by thus guaranteeing an asymptotically zero regulation error. The proposed method have been also validated by simulations. Future work in this direction are on the investigation of other identification techniques, such as neural network, wavelets, and others, and to address the output regulation problem in a stochastic environment.

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