Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

On Maximally Stabilizing Traffic Signal Control with Unknown Turn Ratios ⋆

K. Savla * **E. Lovisari** ** **G. Como** **

* *Sonny Astani Department of Civil and Environmental Engineering at the University of Southern California, Los Angeles, CA* ksavla@usc.edu
** *Department of Automatic Control, Lund University, Sweden.* {enrico.lovisari, giacomo.como}@control.lth.se

Abstract
This paper designs distributed dynamic traffic signal control policies for urban traffic networks. Vehicles at the end of an approach to an intersection queue up in separate lanes corresponding to different possible turn maneuvers at the upcoming intersection, according to fixed turn ratios. The departure rate of vehicles from the queue is governed by traffic signal control at the intersections. We consider traffic signal control architectures under which, at every intersection, only a subset of non-conflicting approaches get green light simultaneously. We propose a class of *cooperative* green light policies, under which traffic signal control at an intersection requires information only about the occupancy levels on the lanes incoming at that intersection. In particular, such a policy does not require information about turn ratios, external arrival rates or departure rates. We show that such minimalist policies are maximally stabilizing for acyclic network topologies under some scenarios, and, when the network is stabilizable, the network admits a globally asymptotically stable equilibrium under these policies. Simulations to illustrate the applicability of our results to cyclic network topologies are also presented.

Keywords: Distributed Traffic Signal Control, Monotone Systems, Nonlinear Control, Switching Networks.

1. INTRODUCTION

Traffic congestion is a major societal issue faced by many cities. Rapid advancements in traffic sensing technology has made it possible to use real-time traffic information to regulate traffic flow. This has opened up the possibility of replacing traditional fixed-timing traffic signal controllers with dynamic controllers. Motivated by such possibilities, this paper addresses the problem of designing dynamic signal control policies for urban traffic networks.

An overview of the problem and practices of urban traffic signal control can be found in Papageorgiou et al. (2007); Stevanovic (2010). Classical strategies consist of using extensive surveys to obtain network parameters, which are then used to design traffic light plans, which are either fixed, e.g., see Robertson (1997), or constantly re-tuned as in SCOOT, e.g., see Bretherton et al. (1998). Classical control techniques have also been used for traffic signal control, e.g., see Cai et al. (2009); Aboudolas et al. (2010). However, these works do not provide any guarantees with respect to performance metrics of interest such as throughput, delay, and robustness to disruptions. Recently, well-known algorithms for routing in data networks, such as the back-pressure algorithm and its throughput analysis, have been adapted to the traffic signal control setting, e.g., see Wongpiromsarn et al. (2012); Varaiya (2013). However, these algorithms require the traffic signal controllers to have explicit knowledge about the turn ratios representing the route choice behavior of drivers. Such a requirement may be impractical, especially in dynamic scenarios such as disruptions. Recently, distributed adaptive signal control algorithms that rely on the estimation of turn ratios at short time scales have also been proposed, e.g., see Le et al. (2013). In this paper, we propose distributed dynamic traffic signal control policies that require no information about the turn ratios, external arrival rates or departure rates, nor do they estimate these parameters, and are still provably maximally stabilizing under some scenarios.

We study traffic networks in which every intersection is signalized. Vehicles at the end of an approach to an intersection queue up in separate lanes corresponding to different turn maneuvers, according to fixed turn ratios. Every lane is assumed to have finite saturated flow capacity and unbounded occupancy capacity. The departure rate of the vehicles from the queue is governed by traffic signal control at the intersections. A turn maneuver from a lane incoming to an intersection to the corresponding downstream road is called a *movement* associated with that intersection. A *phase* at an intersection is a collection of non-conflicting movements at that intersection. We assume that the set of phases is pre-determined for every intersection, and consider traffic signal control architectures under which, at every intersection, only one phase is given green light at any given time. In a typical practical setting, traffic signal controller at an intersection operates periodically with a

⋆ E. Lovisari and G. Como were partially supported by the Swedish Research Council through the Junior Research Grant Information Dynamics in Large Scale Networks and the Linnaeus Center LCCC.

fixed *cycle time*, sequentially activating the set of phases within each cycle. The challenge is then to optimally split the cycle time among the phases every cycle. In this paper, we consider a continuous time fluid approximation of this setting.

The contributions of this paper are as follows. First, we extend the continuous time traffic signal control formulation from our previous work in Savla et al. (2013) to a setting where a phase consists of multiple non-conflicting movements. The analysis of this general formulation is made difficult because of the resulting switched dynamics. Our second contribution is to identify a critical low dimensional sub-system, for which the switching vanishes after a finite time. We describe the class of *cooperative* signal control policies whose actions are monotonic with respect to the state values of the low dimensional system. A distinct feature of these policies is that they require no information about the turn ratios. Third, we show that cooperative green light policies are maximally stabilizable for acyclic network topologies under specific scenarios. To this aim, we make use of an analysis tool from our recent work in Como et al. (2013). Due to space constraints, we refer to Savla et al. (2014) for missing proofs of the technical results.

We conclude this section by defining important notations to be used throughout the paper. Let $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$ be the sets of real, nonnegative real, and positive real numbers, respectively. For finite sets $\mathcal{A}$ and $\mathcal{B}$, and a set $\mathcal{C}$, $|\mathcal{A}|$ denotes the cardinality of $\mathcal{A}$, $\mathcal{C}^{\mathcal{A}}$ the space of $\mathcal{C}$-valued vectors whose components are indexed by elements of $\mathcal{A}$, and $\mathcal{C}^{\mathcal{A}\times\mathcal{B}}$ the space of matrices whose entries from $\mathcal{C}$ are indexed by pairs in $\mathcal{A}\times\mathcal{B}$. A directed multi-graph is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ and $\mathcal{E}$ stand for the node set and the link set, respectively, and are both finite. They are endowed with two vectors: $\sigma, \tau \in \mathcal{V}^{\mathcal{E}}$. For every $i \in \mathcal{E}$, σ_i and τ_i stand for the tail and head nodes respectively of link i. We assume that there are no self-loops, i.e., $\tau_i \neq \sigma_i$, for all $i \in \mathcal{E}$. On the other hand, we allow for parallel links. For a node $v \in \mathcal{V}$, let $\mathcal{E}_v^+ := \{i : \sigma_i = v\}$ and $\mathcal{E}_v^- := \{i : \tau_i = v\}$ be the sets of incoming and outgoing links, respectively, at node v. For a finite set $\mathcal{A}$, let $S_{\mathcal{A}} := \{x \in \mathbb{R}_+^{\mathcal{A}} : \sum_{i\in\mathcal{A}} x_i = 1\}$ denote the simplex over the elements of $\mathcal{A}$, and

$$\bar{S}_{\mathcal{A}} := \{x \in \mathbb{R}_+^{\mathcal{A}} : \sum_{i\in\mathcal{A}} x_i \leq 1\}. \tag{1}$$

2. PROBLEM FORMULATION

We represent the graph topology of the traffic network by a directed multigraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, whose specification is as follows. Every traffic intersection is assigned a node, with $\mathcal{N}$ representing the set of nodes corresponding to traffic intersections. Every road approaching an intersection is split into a set of *lanes*, one for each possible turn maneuver at the upcoming intersection (e.g., see Figure 1). For convenience, we introduce a *virtual* intersection at the point where a road splits into different lanes, and assign a node to every virtual intersection. Let $\mathcal{U}$ be the set of nodes corresponding to virtual intersections. The node set of $\mathcal{G}$ is $\mathcal{V} := \mathcal{N} \cup \mathcal{U}$. For clarity in terminology, henceforth, we reserve the term *road* only for the portion upstream of the virtual intersection. We assign one directed link to every road, and one directed link to every lane. Let $\mathcal{R}$ and $\mathcal{L}$ be the sets of links corresponding to the roads and lanes, respectively. Therefore, the link set of $\mathcal{G}$ is $\mathcal{E} := \mathcal{R} \cup \mathcal{L}$.

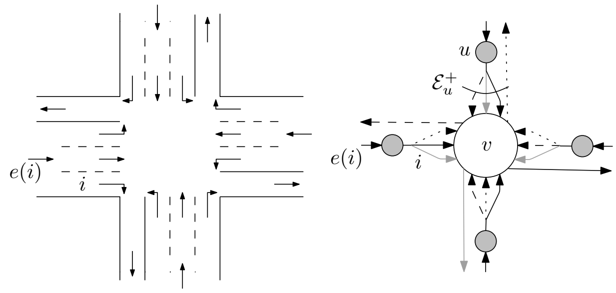

Figure 1. The left hand side figure corresponds to an intersection with 12 incoming lanes, 4 incoming roads, and 4 outgoing roads, and the right hand side figure corresponds to its representation in terms of a directed multigraph. Each incoming road is split into three lanes, each associated with a possible movement. Shaded disks (e.g., u) correspond to virtual intersections, and the white disk v corresponds to real intersection. $e(i)$ denotes the unique road upstream of lane i.

We model external arrivals to and departures from the network to be at the roads $\mathcal{R}$. Upon entering a road $r \in \mathcal{R}$, the external traffic homogeneously joins other traffic incoming to r from the upstream parts of the network. We denote the corresponding arrival rate vector by $\lambda \in \mathbb{R}_+^{\mathcal{R}}$. For every lane $i \in \mathcal{L}$, let $C_i > 0$ be its saturated flow capacity, i.e., the rate at which cars depart lane i when given green light. Let $C \in \mathbb{R}_{++}^{\mathcal{L}\times\mathcal{L}}$ be a diagonal matrix, whose diagonal entries correspond to the saturated flow capacities of the lanes. With a slight abuse of notation, we use C_i to refer to the i-th diagonal element of C. For every road $r \in \mathcal{R}$, let $\theta_r \in [0, 1]$ be the departure rate from r, i.e., the fraction of flow on r leaving the network. Let $\theta = \{\theta_r : r \in \mathcal{R}\}$ be the vector of departure ratios. For every virtual intersection $u \in \mathcal{U}$, the probability vector β^u denotes the turn ratio at u, i.e., β_i^u, $i \in \mathcal{E}_u^+$, is the fraction of flow through node u that turns to lane i. These turn ratios represent the *static* route choice behavior of drivers. Under the static route choice assumption, it is not necessary to distinguish between arrival rates for different destinations. Let $\beta = \{\beta^u : u \in \mathcal{N}\}$ be the set of turn ratios at all the intersections.

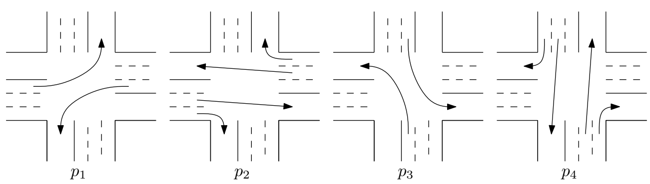

Figure 2. Illustration of a four phase architecture at an intersection.

Throughout this paper, we shall restrict ourselves to network topologies satisfying the following natural property:

Assumption 1. For every lane $i \in \mathcal{L}$, there exists at least one road $r \in \mathcal{R}$ with $\theta_r > 0$ along with at least one directed path from σ_i to τ_r in $\mathcal{G}$.

We denote by $\rho(t) \in \mathbb{R}_+^{\mathcal{L}}$ the vector of occupancy levels on the lanes at time $t \geq 0$. The dynamics in the occupancy levels is governed by the external inflow into the network λ, departure rates θ, the turn ratios β at the virtual intersections, and the traffic signal control policies at the intersections $\mathcal{N}$. We next formalize the traffic signal control architecture.

The choice of lane i according to β^{σ_i} uniquely determines the road $r \in \mathcal{E}_{\tau_i}^+$ that the drivers in i will turn into when i is given a green light by the signal control policy at the intersection τ_i. Such a pair $(i, r) \in \mathcal{E}_{\tau_i}^- \times \mathcal{E}_{\tau_i}^+$ is referred to as a *movement* associated with τ_i, and a phase at τ_i is a set of nonconflicting movements associated with τ_i (see Figure 2). For every intersection $v \in \mathcal{N}$, let Φ_v be the set of phases at v, and $\Phi = \{\Phi_v : v \in \mathcal{N}\}$ be the set of phases at all the intersections. For every intersection $v \in \mathcal{N}$, let $\phi(s) \subseteq \mathcal{L}$ be the set of lanes in $\mathcal{E}_v^-$ which receive green light simultaneously when phase s is activated. Conversely, for each $i \in \mathcal{E}_v^-$, let $\psi(i) \subseteq \Phi_v$ be the set of phases which contain i, i.e., $\psi(i) := \{p \in \Phi_v \mid i \in \phi(p)\}$.

We assume that every traffic intersection is signalized. We specify traffic signal control policies in the form of the amount of green light given to every phase in response to occupancy levels. The formal definition of such green light policies is as follows.

Definition 1. A *green light policy* is a set of Lipschitz continuous functions $\{h^v : \mathbb{R}^{\mathcal{L}} \to \bar{S}_{\Phi_v}, v \in \mathcal{N}\}_{v \in \mathcal{N}}$, where $\bar{S}_{\Phi_v}$ is as defined in (1).

Throughout the paper, we shall often drop the superscript in h^v and refer to $h(\rho)$ as the vector of green light control signals. In a typical practical setting, traffic signal controller at an intersection operates periodically with a fixed *cycle time*, sequentially activating the set of phases in every cycle. The challenge then is to specify the green light times for all the phases during every cycle. In this paper, we consider a continuous time setting with fluid approximation of traffic flow, where the output of the green light policies can be interpreted as the short term average of the fraction of cycle time for different phases of the practical setting. However, a rigorous justification of this connection is left as a future work. Under the continuous time fluid approximation, the dynamics in occupancy levels is described as:

$$\dot{\rho}_i(t) = T_i\left(f_i^{\text{in}}(\rho) - C_i \sum_{p \in \psi(i)} h_p(\rho)\right), \quad \forall i \in \mathcal{L}, \qquad (2)$$

where $f_i^{\text{in}}(\rho)$ is the inflow to lane i, and $T_i(x(\rho)) := x(\rho) - \min\{0, x(\rho)\}\mathbb{1}_{\rho_i = 0}$ ensures that the occupancy level on an empty lane has a nonnegative derivative when it is given green light. It follows that

$$T_i(x(\rho)) \geq x(\rho), \qquad (3)$$

for all $\rho \in \mathbb{R}_+^{\mathcal{L}}$. If $e(i) \in \mathcal{R}$ denotes the unique road upstream of lane i, i.e., $\tau_{e(i)} = \sigma_i$, then for all $\rho \in \mathbb{R}_+^{\mathcal{L}}$,

$$f_i^{\text{in}}(\rho) = \left(\left(1 - \theta_{e(i)}\right) f_{e(i)}^{\text{in}}(\rho) + \lambda_{e(i)}\right) \beta_i^{\sigma_i}, \qquad (4)$$

where $f_{e(i)}^{\text{in}}(\rho)$ is the inflow to road $e(i)$ from the upstream part of the network, defined as

$$f_{e(i)}^{\text{in}}(\rho) = \sum_{j:(j,e(i)) \text{ is a movement}} f_j^{\text{out}}(\rho)$$

if $\mathcal{E}_{\sigma_{e(i)}}^- \neq \emptyset$, and $f_{e(i)}^{\text{in}}(\rho) = 0$ otherwise, and where $f_j^{\text{out}}(\rho) = C_j \sum_{s \in \psi(j)} h_s(\rho)$ if $\rho_j > 0$, or if $\rho_j = 0$ and $C_j \sum_{s \in \psi(j)} h_s(\rho) \leq f_j^{\text{in}}(\rho)$; and $f_j^{\text{out}}(\rho) = f_j^{\text{in}}(\rho)$ otherwise.

Remark 1. The discontinuity in the right hand side of (2) due to the composition with T_i is one of the key features that differentiates the setting of this paper from our previous work in Savla et al. (2013).

In order to write the dynamics in (2) in a compact form, we introduce the *routing matrix* $P \in [0,1]^{\mathcal{L} \times \mathcal{L}}$ and the *lane-phase matrix* $A \in \{0,1\}^{\mathcal{L} \times \Phi}$ as follows:

$$P_{ij} = \begin{cases} \beta_i^{\sigma_i}(1 - \theta_{e(i)}), & \text{if } (j, e(i)) \text{ is a movement} \\ 0, & \text{otherwise,} \end{cases}$$

$$A_{is} = \begin{cases} 1, & \text{if } s \in \psi(i) \\ 0, & \text{otherwise.} \end{cases}$$

Let $f^{\text{in}}(\rho)$ and $f^{\text{out}}(\rho)$ be the vectors of inflows and outflows respectively. One can check that

$$f^{\text{in}}(\rho) = P f^{\text{out}}(\rho) + b$$

where $b \in \mathbb{R}_+^{\mathcal{L}}$ is such that $b_i = \lambda_{e(i)} \beta_i^{\sigma_i}$ for all $i \in \mathcal{L}$. Moreover, if A_i is the i-th row of A, then $f_i^{\text{out}}(\rho) = C_i A_i h(\rho)$ if $\rho_i > 0$ or $\rho_i = 0$ and $C_i A_i h(\rho) \leq f_i^{\text{in}}(\rho)$; and $f_i^{\text{out}}(\rho) = f_i^{\text{in}}(\rho)$ otherwise. Using these notations, the dynamics in (2) can be written compactly as:

$$\dot{\rho} = (-I + P) f^{\text{out}}(\rho) + b \qquad (5)$$

For given $\lambda \in \mathbb{R}_+^{\mathcal{R}}$, $\theta \in [0,1]^{\mathcal{R}}$ and $\beta \in \prod_{u \in \mathcal{U}} S_{\mathcal{E}_u^+}$, let $f_i(\lambda, \theta, \beta)$ denote the flow *induced* on lane $i \in \mathcal{L}$:

$$f(\lambda, \theta, \beta) := (I - P)^{-1} b \qquad (6)$$

where $I - P$ is invertible because, under Assumption 1, P is a strictly sub-stochastic matrix.

The idea behind induced flow, as seen from (5), is that the equilibrium outflow from the lanes is indeed $f(\lambda, \theta, \beta)$, so the induced flow is the steady state load induced on different lanes of the network as a result of external arrival rates λ, departure rates θ and turn ratios β. The flow induced by given λ, θ and β is called *feasible* if, for all $i \in \mathcal{L}$, $f_i(\lambda, \theta, \beta) \in [0, C_i]$. Essentially, an induced flow is called feasible, if it is non-negative and satisfies lane-wise capacity constraints. Throughout this paper, we shall implicitly assume that λ, θ and β are such that the induced flow $f(\lambda, \theta, \beta)$ is feasible. Furthermore, the quadruple $(\lambda, \theta, \beta, C)$ is called feasible if it satisfies the following definition.

Definition 2. A given set of arrival rates, departure rates, turn ratios, and lane capacities $(\lambda, \theta, \beta, C)$ is called *feasible* if there exists $z \in \mathbb{R}_+^{\mathcal{L}}$ such that $\sum_{i \in \mathcal{E}_v^-} z_i \leq 1$ for all $v \in \mathcal{N}$ and

$$Az \geq C^{-1} f(\lambda, \theta, \beta). \qquad (7)$$

Let Γ denote the set of feasible $(\lambda, \theta, \beta, C)$.

The motivation behind Definition 2 is that, if the routing policy also has knowledge of $C^{-1} f(\lambda, \theta, \beta)$, then a routing policy of the kind $h(\rho) \equiv z$, where z satisfies (7) is stabilizing as per the following definition.

Definition 3. For given $\lambda \in \mathbb{R}_+^{\mathcal{R}}$, $\theta \in [0,1]^{\mathcal{R}}$ and $\beta \in \prod_{u \in \mathcal{U}} S_{\mathcal{E}_u^+}$, the traffic network is *stabilizable* if there exists a green light policy $\{h^v\}_{v\in\mathcal{N}}$ such that, under the dynamics in (2), from any initial condition $\rho(0) \in \mathbb{R}_+^{\mathcal{L}}$, we have

$$\limsup_{t\to\infty} \rho_i(t) < +\infty \quad \forall i \in \mathcal{L}. \tag{8}$$

The primary objective of this paper is to design distributed green light policies that are maximally stabilizing. A necessary condition for stability is provided in the following proposition.

Proposition 1. If the traffic network is stabilizable under a green light policy, then $(\lambda, \theta, \beta, C) \in \Gamma$.

Proof 1. Consider a green light policy such that (8) holds true for all $\rho(0) \in \mathbb{R}_+^{\mathcal{L}}$. Then,

$$\frac{1}{t}\int_0^t f^{\mathrm{out}}(\rho(s))ds \overset{t\to\infty}{\longrightarrow} (I-P)^{-1}b = f(\lambda,\theta,\beta),$$

where the equality follows from (6). Notice that either $f_j^{\mathrm{out}}(\rho(t)) = C_jA_jh(\rho)$, or $f_j^{\mathrm{out}}(\rho(t)) = f_j^{\mathrm{in}}(\rho(t)) \leq C_jA_jh(\rho)$. This implies that

$$\frac{1}{t}\int_0^t f^{\mathrm{out}}(\rho(s))ds \leq \frac{1}{t}\int_0^t CAh(\rho(s))ds\,.$$

In turn, if $z(t) = \frac{1}{t}\int_0^t h(\rho(s))ds$, then

$$Az(t) \geq C^{-1}\frac{1}{t}\int_0^t f^{\mathrm{out}}(\rho(s))ds \overset{t\to\infty}{\longrightarrow} C^{-1}f(\lambda,\theta,\beta)$$

and thus for large enough t there exists a vector $z(t)$ satisfying the conditions in (7).

In the next section, we prove that the same condition in Proposition 1 is also sufficient under a specific class of green light policies,

3. STABILITY UNDER COOPERATIVE GREEN LIGHT POLICIES

In this section, we show that a class of green light policies, called *cooperative* green light policies, is maximally stabilizing under some scenarios. We restrict our attention to network topologies satisfying the following:

Assumption 2. $\mathcal{G}$ contains no cycles.

We make the following assumption about disjoint property of the phases. We present results for the case when phases can have at most one overlapping lane in the extended version Savla et al. (2014).

Assumption 3. $\phi(p) \cap \phi(q) = \emptyset$ for all $p, q \in \Phi$, $p \neq q$.

Without loss of generality, and to avoid cumbersome technicalities, we also assume that:

Assumption 4. For all $v \in \mathcal{N}$, and for all $p \in \Phi_v$, $\frac{f_i(\lambda,\theta,\beta)}{C_i} \neq \frac{f_j(\lambda,\theta,\beta)}{C_j}$, for all $i, j \in \phi(p)$, $i \neq j$.

Let

$$\omega_p(\rho, C) := \max_{i\in\phi(p)} \frac{\rho_i}{C_i}, \quad \forall p \in \Phi, \tag{9}$$

be the *critical (scaled) phase occupancies*, with $w(\rho, C) \in \mathbb{R}_+^{\Phi}$ being the vector notation. Our analysis relies on studying the low-dimensional subsystem comprising of the critical phase occupancies. Note that such a system is a switched system, where the switching is state-dependent.

Remark 2. (9) implies that the computation of critical phase occupancies for an intersection requires information only about the actual occupancies and the saturated flow capacities of all the lanes incoming to that intersection, and in particular, it does not require information about the external arrival rates, departure rates or the turn ratios for any part of the network.

Henceforth, for brevity in notation, we drop the explicit dependence of ω on ρ and C.

We now specify a class of *cooperative green light policies* that require information only about the critical phase occupancies for their implementation. Moreover, these policies are *distributed* in the sense that the green light computations at an intersection depend only on the critical occupancies on phases associated with that intersection. We impose the following natural constraint on these policies: for all $p \in \Phi$,

$$\omega_p \to 0^+ \implies h_p(\omega) \to 0^+. \tag{10}$$

(10) implies that, if all the lanes in a phase are empty, then that phase gets no green light.

Definition 4. (Cooperative green light policy). A *cooperative green light policy* is a set of Lipschitz continuous functions $\{h^v : \mathbb{R}^{\Phi} \to \bar{S}_{\Phi_v},\, v \in \mathcal{N}\}_{v\in\mathcal{N}}$ that, in addition to (10), satisfies the following for every $\omega \in \mathbb{R}_+^{\Phi}$ and $v \in \mathcal{N}$:

a) for all $p \in \Phi_v$:
$\frac{\partial}{\partial\omega_q}h_p^v(\omega) < 0$ for all $q \in \Phi_v$, $q \neq p$, and
$\frac{\partial}{\partial\omega_q}h_p^v(\omega) = 0$ for all $q \notin \Phi_v$;

b) $\frac{\partial}{\partial\omega_p}\sum_{q\in\Phi_v} h_q^v(\omega) > 0$, for all $p \in \Phi_v$;

c) for any $\mathcal{I} \subseteq \Phi_v$, if $\sum_{p\in\mathcal{I}}\omega_p \to +\infty$ and $\frac{\sum_{p\in\mathcal{I}}\omega_p}{\sum_{q\in\Phi_v}\omega_q} \to 1^-$, then $\sum_{p\in\mathcal{I}} h_p^v(\omega) \to 1^-$.

Remark 3. Note that the domain of the green light functions in Definition 4 is $\mathbb{R}_+^{\Phi}$, whereas it is $\mathbb{R}_+^{\mathcal{L}}$ in Definition 1. However, one can interpret a cooperative green light policy as the composition $h\circ\omega$, whose domain is $\mathbb{R}_+^{\mathcal{L}}$. In this case, the Lipschitz continuity of ω from (9), together with the Lipschitz continuity of h, implies Lipschitz continuity of $h \circ \omega$. Hereafter, we shall implicitly refer to this composition whenever we refer to a cooperative green light policy.

Example 1. An example of a cooperative green light policy is as follows:

$$h_p(\omega) = \frac{\omega_p}{\sum_{q\in\Phi_v}\omega_q + \kappa}, \quad p \in \Phi_v\,, v \in \mathcal{N}, \tag{11}$$

where $\kappa > 0$ is a constant.

Remark 4. It is interesting to note that the expression for green time allocation, as given in (11), is similar to the standard protocol for green time allocation in pre-timed traffic signal timing literature, e.g., see (Papacostas and Prevedouros, 2001, Section 4.6.4). The key difference is that, unlike existing protocols that use critical *induced flows*, we use critical phase occupancies in (11). With such a modification we can guarantee maximal network throughput in some scenarios, without knowledge of the external arrival rates, departure rates or turn ratios. Moreover, we emphasize that the result on maximal throughput is valid under any green time allocation policy satisfying Definition 4, of which (11) is just an example.

In order to state the main results of this section, we need a few additional notations. Let $\tilde{\lambda}_i := \frac{f_i(\lambda,\theta,\beta)}{C_i}$, for all $i \in \mathcal{L}$, be the scaled induced flows on the lanes. For all $p \in \Phi$, let

$$\mathcal{M}_p(\lambda,\theta,\beta,C) := \operatorname{argmax}\{\tilde{\lambda}_i : i \in \phi(p)\} \qquad (12)$$

be the lane with the highest scaled induced flow among all the lanes in $\phi(p)$, and let

$$\mathcal{M}(\lambda,\theta,\beta,C) := \{\mathcal{M}_p(\lambda,\theta,\beta,C) : p \in \Phi\}. \qquad (13)$$

With this last definition, one can see that, under Assumption 3, the feasible set in Definition 2 specializes to:

$$\Gamma = \left\{ (\lambda,\theta,\beta,C) \,\middle|\, \sum_{i \in \mathcal{M} \cap \mathcal{E}_v^-} \tilde{\lambda}_i \le 1, \forall v \in \mathcal{N} \right\}. \qquad (14)$$

Theorem 1. For a traffic network satisfying Assumptions 1–4, consider the dynamics in (5) under a cooperative green light policy. If $(\lambda,\theta,\beta,C) \in \Gamma$, then there exists an equilibrium $\rho^* \in \mathbb{R}_+^{\mathcal{L}}$ such that

$$\lim_{t\to+\infty} \rho(t) = \rho^* \qquad \forall\, \rho(0) \in \mathbb{R}_+^{\mathcal{L}}.$$

Moreover, $\rho_i^* > 0$ if $i \in \mathcal{M}$ and $\rho_i^* = 0$ otherwise, and

$$\lim_{t\to+\infty} f_i^{\text{out}}(\rho(t)) = f_i(\lambda,\theta,\beta) \quad \forall i \in \mathcal{L}.$$

Remark 5. Theorem 1 implies that the traffic network admits a globally asymptotically stable equilibrium under a cooperative green light policy for all $(\lambda,\theta,\beta,C) \in \Gamma$, when the phases are disjoint. This, combined with Proposition 1, implies that cooperative green light policies are maximally stabilizing when the phases are disjoint.

Theorem 1 is proved by induction. The base case corresponds to a local system which consists of an isolated intersection, as illustrated in Figure 1. Our stability analysis for the local system relies on identification of a *critical* low-dimensional sub-system of the underlying networked dynamical system and in establishing global asymptotic stability of the local component of this sub-system using ℓ_1 contraction principle for monotone dynamical systems that we recently developed in Como et al. (2013). We then use a cascade argument from monotone control systems, e.g., see Angeli and Sontag (2003), to extend the stability result to the entire network. We refer to Savla et al. (2014) for details.

4. SIMULATIONS

In this section, we provide simulation results to suggest that Theorem 1 also extends to cyclic network topologies. In addition, we offer a comparison with back-pressure traffic signal control policies, e.g., see Wongpiromsarn et al. (2012); Varaiya (2013).

We considered the grid-like network shown in Figure 3. Each of the four intersections have 12 incoming lanes, 4 incoming roads, and 4 outgoing roads, as shown in Figure 4. We assume the same turn ratio at every virtual intersection by setting (see Figure 4) $\beta_{i_1} = \beta_{i_4} = \beta_{i_7} = \beta_{i_{10}} = 1/2$, $\beta_{i_2} = \beta_{i_5} = \beta_{i_8} = \beta_{i_{11}} = 1/3$, and $\beta_{i_3} = \beta_{i_6} = \beta_{i_9} = \beta_{i_{12}} = 1/6$. The saturated lane flow capacities are also symmetrical across all the four intersections, and for a given intersection, they are specified as follows (see Figure 4): $C_{i_1} = C_{i_2} = C_{i_3} = 1.7$, $C_{i_4} = C_{i_5} = C_{i_6} = 1.6$, $C_{i_7} = C_{i_8} = C_{i_9} = 1.5$, and $C_{i_{10}} = C_{i_{11}} = C_{i_{12}} = 1.4$.

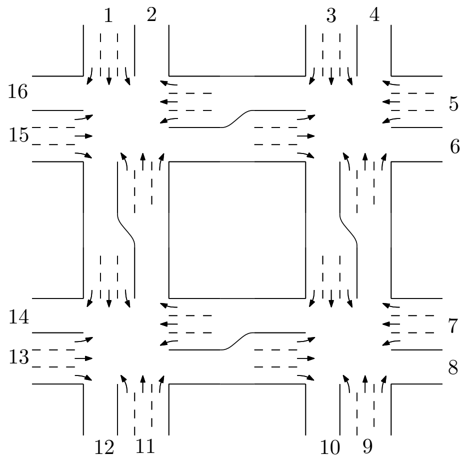

Figure 3. The grid-like network used in the simulations.

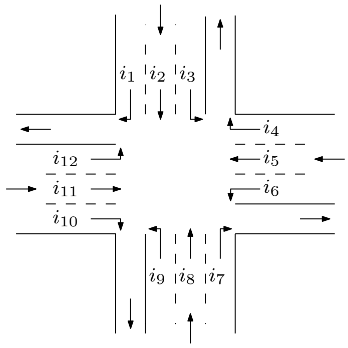

Figure 4. The structure of each of the four intersections in the network in Figure 3.

$\tilde{\lambda}_{\mathcal{M}_{p_i}}$	N-W	N-E	S-E	S-W
$i=1$	0.1190	0.1111	0.1042	0.1190
$i=2$	0.3571	0.3125	0.3571	0.3333
$i=3$	0.1111	0.1042	0.1190	0.1111
$i=4$	0.3333	0.2941	0.3333	0.3571
$\sum_{i=1}^4 \tilde{\lambda}_{\mathcal{M}_{p_i}}$	0.9206	0.8219	0.9137	0.9206

Table 1. Critical scaled phase occupancies at the four intersections. N-W refers to the North-West intersection in Figure 3; N-E, S-E and S-W are to be interpreted similarly.

Referring to Figure 3, let the departure rates be given by $\theta_e = 1$ on all the 8 outgoing roads in the network, that is, for $e \in \{2,4,6,8,10,12,14,16\}$, and $\theta_e = 0$ otherwise. We also set $\lambda_e = 1$ for all the 8 incoming roads in the network, that is, for $e \in \{1,3,5,7,9,11,13,15\}$.

For each intersection, we consider four disjoint phases, as illustrated in Figure 2: p_1 corresponds to the left turn movements in the east-west direction, p_2 corresponds to the through and right turn movements in the east-west direction, p_3 corresponds to the left turn movements in the north-south direction, and p_4 corresponds to the through and right turn movements in the north-south direction. The values of critical scaled induced flows at the four intersections is provided in Table 1. Notice that, at every intersection, $\sum_{i=1}^4 \tilde{\lambda}_{\mathcal{M}_{p_i}} < 1$ which corresponds to the necessary (Proposition 1) and one of the sufficient conditions (Theorem 1) for stability.

We compare the evolution of lane occupancies under cooperative green light policy and back pressure policies. For the cooperative green light policy, we use (11) with $\kappa = 0.025$, and for the back-pressure policy, we consider the following continuous time continuous space approximation for every intersection:

$$h_{p_i}^{BP} = \frac{e^{\eta P_{p_i}}}{\sum_{j=1}^{4} e^{\eta P_{p_j}}}, i \in \{1,2,3,4\}, \eta = 10, \qquad (15)$$

where the pressure associated with the phases is computed as follows. For a lane i, if $g(i)$ is the road which vehicles on i turn into, then the pressure associated with the movement $(i, g(i))$ is $P_i := C_i(\rho_i - \sum_{k \in \mathcal{E}^+_{\tau_{g(i)}}} \beta_j^{\tau_{g(i)}} \rho_j)$, and the total pressure associated with phase p_j for $j \in \{1,2,3,4\}$ is $P_p := \sum_{i \in \phi(p_j)} P_i$.

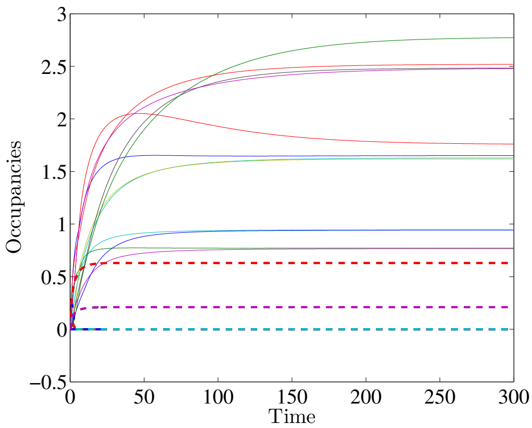

Figure 5. Comparison of the evolution of lane occupancies at the N-W intersection in Figure 3, under cooperative green light and back pressure policies. Solid lines correspond to back pressure policy, and dashed lines correspond to cooperative green light policy.

Figure 5 shows the comparison between the occupancy levels on the lanes in the North-West intersection in Figure 3 under the cooperative green light and the back pressure policies, as described above. As implied by Theorem 1, only the lanes belonging to $\mathcal{M}$ exhibit a nonzero equilibrium occupancy. In this specific example, lanes 7, 9, 10 and 12 belong to $\mathcal{M}$ at the North-West intersection. Note that, due to symmetry, the trajectories of the occupancy levels in lanes 7 and 10 are identical, and so are occupancies on lanes 9 and 12. Figure 5, which is for relatively high occupancy values as described in Table 1, suggests that the maximum network throughput under a cooperative green light policies that does not require information about turn ratio is the same as that under a back pressure policy that does require information about turn ratio, cf. (15).

5. CONCLUSION

In this paper, we designed adaptive traffic signal control policies for urban traffic networks under multi-movement phase architecture. We proposed a class of distributed traffic control policies, called cooperative green light policies, that neither require the explicit knowledge of the turn ratios nor do they require to estimate them, and yet can be proven to be maximally stabilizing in some scenarios. Future research directions include extending these results to more general phase architectures, cyclic network topologies, finite capacities on lane occupancies and dynamic route choice behavior, and extending the formulation to other metrics of performance such as delay.

REFERENCES

Aboudolas, K., Papageorgiou, M., Kouvelas, A., Kosmatopoulos, E., 2010. A rolling-horizon quadratic-programming approach to the signal control problem in large-scale congested urban road networks. Transportation Research Part C: Emerging Technologies 18 (5), 680 – 694.

Angeli, D., Sontag, E., October 2003. Monotone control system. IEEE Transactions on Automatic Control 48 (10).

Bretherton, R., Wood, K., Bowen, G., 1998. SCOOT version 4. In: Proceedings of 9th International Conference on Road Transport Information and Control.

Cai, C., Wong, C., Heydecker, B., 2009. Adaptive traffic signal control using approximate dynamic programming. Transportation Research Part C: Emerging Technologies 17 (5), 456 – 474.

Como, G., Lovisari, E., Savla, K., 2013. Throughput optimality and overload behavior of dynamical flow networks under monotone distributed routing. IEEE Transactions on Control of Networked SystemsUnder review. Available at `http://arxiv.org/abs/1308.1993`.

Le, T., Kovacs, P., Walton, N., Vu, H. L., Andrew, L. L. H., Hoogendorn, S. S. P., 2013. Decentralized signal control for urban road networks, submitted.

Papacostas, C. S., Prevedouros, P. D., 2001. Transportation Engineering and Planning, 3rd Edition. Prentice Hall.

Papageorgiou, M., Ben-Akiva, M., Bottom, J., Bovy, P., Hoogendoorn, S., Hounsell, N., Kotsialos, A., McDonald, M., 2007. Chapter 11: ITS and traffic management. In: Barnhart, C., Laporte, G. (Eds.), Transportation. Vol. 14 of Handbooks in Operations Research and Management Science. Elsevier, pp. 715 – 774.

Robertson, D., 1997. The TRANSYT method of coordinating traffic signals. Traffic Engineering & Control 38 (2), 76–77.

Savla, K., Lovisari, E., Como, G., 2013. On maximally stabilizing adaptive traffic signal control. In: Allerton Conference on Communication, Control and Computing, (Monticello, IL).

Savla, K., Lovisari, E., Como, G., 2014. On maximally stabilizing traffic signal control with unknown turn ratios, available at `http://www-bcf.usc.edu/~ksavla/IFAC14.pdf`.

Stevanovic, A., 2010. Adaptive traffic control systems: domestic and foreign state of practice. No. Project 20-5 (Topic 40-03).

Varaiya, P., 2013. The max-pressure controller for arbitrary networks of signalized intersections. In: Advances in Dynamic Network Modeling in Complex Transportation Systems. Springer, pp. 27–66.

Wongpiromsarn, T., Uthaicharoenpong, T., Wang, Y., Frazzoli, E., Wang, D., 2012. Distributed traffic signal control for maximum network throughput. In: IEEE Conference on Intelligent Transportation Systems. Anchorage, AK, pp. 588–595.