

# Robust Fixed-order Discrete-time LPV Controller Design

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**Abstract:** A new method for the design of fixed-order dynamic output-feedback Linear Parameter Varying (LPV) controllers for discrete-time LPV systems with bounded scheduling parameter variations is presented. Sufficient conditions for stability and induced  $l_2$ -norm performance of an LPV system are given through a set of Linear Matrix Inequalities (LMIs) and exploited for design. Controller parameters appear directly as decision variables in the convex optimisation program, which enables preserving a desired controller structure in addition to the low order. Efficiency of the proposed method is illustrated on a simulation example, with an iterative convex optimisation scheme used for the control system performance improvement.

*Keywords:* Linear Parameter-Varying; Fixed-order Controller Design; Parameter Dependent Lyapunov Function; Induced  $l_2$ -Norm Performance.

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## 1. INTRODUCTION

The LPV system modelling and control paradigm arises naturally as a successor of classical gain-scheduling controller design approaches (Shamma and Athans (1991), Leith and Leithead (2000)). It allows modelling a wide class of nonlinear systems and the use of many tools from linear systems theory for analysis and control. Recently, a number of applications have been treated in the LPV framework; modelling and control of turbofan engines (W. Gilbert et al. (2010)), active braking control (G. Panzani et al. (2012)) and semi-active vehicle suspension design (C. Poussot-Vassal et al. (2008)), to name just a few.

Over the last 20 years, different continuous-time LPV controller design strategies for LPV systems with state-space description were proposed (e.g. F. Wu et al. (1996), Apkarian and Adams (1998), Sato (2011), Wu (2001)). Some important results for the stability analysis of uncertain and LPV polytopic discrete-time systems are presented in M. C. de Oliveira et al. (1999), R. C. L. F. Oliveira and P. L. D. Peres (2005), J. Daafouz and J. Bernussou (2001). These ideas establish a good starting point for an LPV controller synthesis. A few recent publications cover the case of controller synthesis for discrete-time LPV systems affected by scheduling parameters with limited variations (R. C. L. F. Oliveira and P. L. D. Peres (2009), F. Amato et al. (2005), J De Caigny et al. (2012)).

All enlisted methods result in a controller in either state-feedback or full-order output-feedback form. For online reconstruction of the full-order controller, time-consuming linear algebraic operations need to be employed. Moreover, the order of the controller may be too high since it depends on the augmented plant model order. Some methods for the LPV controller reduction are available (Beck (2006)),

but there is no guarantee of preserving stability or performance of the original LPV system with reduced controller. On the other side, a state-feedback LPV controller demands state estimation, which is a non-trivial task for LPV systems. In both cases controller is restored from the optimisation results by a nonlinear change of variables, which ruins user requested structure in the controller. As well, in most practical applications, resources available for control are highly limited. This is why a method for the direct design of low-order output-feedback LPV controllers, which are easier to implement and with accordingly lower execution times, is highly needed.

Some methods for the fixed-order LPV controller design in the transfer function setting are presented in W. Gilbert et al. (2010), S. Formentin et al. (2013) and Z. Emedi and A. Karimi (2012). The use of transfer function models is very well aligned with industrial practice and modelling paradigm in the SISO case (Tóth (2010)). However, the extension to the MIMO case can be highly non-trivial comparing to its simplicity in the state-space setting.

The importance of the discrete-time LPV controller design methods comes from the fact that the LPV models produced by identification procedures are usually in discrete-time (e.g. Toth et al. (2009), V. Cerone et al. (2012), Verdult (2002)). As well, control is anyway performed using digital computers in practice. The problem is that preservation of the closed-loop stability under the discretisation of a continuous-time LPV system could require too high sampling frequency (R. Toth et al. (2008)).

To the best of our knowledge, there is no fixed-order discrete-time state-space LPV controller design method presented in the literature. In this paper a class of discrete-time LPV state-space plants, affine in the scheduling parameter vector, is considered. User imposed controller structure is preserved since controller parameters appear directly as decision variables in the convex optimisation

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program. The realistic case of limited scheduling parameter variations is treated through the use of Parameter Dependent Lyapunov Functions (PDLF) affine in the scheduling parameter vector. Upper bound on the induced  $l_2$ -norm performance of a control system is enhanced through the use of iterative convex optimisation procedure.

The paper is organised as follows. First, preliminaries about the LPV system stability and performance are given in Section 2. Stabilising LPV controller design procedure is proposed in Section 3. Extension of the procedure to induced  $l_2$ -norm performance design are given in Section 4. An illustrative simulation example is given in Section 5 and the concluding remarks in Section 6.

## 2. PRELIMINARIES

### 2.1 LPV plant and controller

The class of LPV discrete-time systems considered in this paper can be represented by the following model:

$$\begin{aligned} \mathbf{x}_g(k+1) &= A_g(\boldsymbol{\theta})\mathbf{x}_g(k) + B_u(\boldsymbol{\theta})\mathbf{u}(k) + B_w(\boldsymbol{\theta})\mathbf{w}(k) \\ \mathbf{z}(k) &= C_z(\boldsymbol{\theta})\mathbf{x}_g(k) + D_{zu}(\boldsymbol{\theta})\mathbf{u}(k) + D_{zw}(\boldsymbol{\theta})\mathbf{w}(k) \\ \mathbf{y}(k) &= C_y\mathbf{x}(k) + D_{yw}\mathbf{w}(k). \end{aligned} \quad (1)$$

Here  $\mathbf{x}_g(k) \in \mathbb{R}^n$  represents the state vector,  $\mathbf{u}(k) \in \mathbb{R}^{n_u}$  is the control input vector,  $\mathbf{z}(k) \in \mathbb{R}^{n_z}$  is the vector of controlled outputs and  $\mathbf{y}(k) \in \mathbb{R}^{n_y}$  is the vector of measured outputs. The time-varying scheduling parameter vector  $\boldsymbol{\theta} = [\theta_1(k), \dots, \theta_{n_\theta}(k)]^T$  is assumed to belong to a hyper-rectangle  $\Theta \in \mathbb{R}^{n_\theta}$ , or equivalently

$$\theta_i(k) \in [-\bar{\theta}_i, \bar{\theta}_i], \quad i = 1, \dots, n_\theta. \quad (2)$$

where, without loss of generality, symmetric bounds around  $\theta_i = 0$  is assumed. The scheduling parameters  $\theta_i$  are assumed to be independent.

Strict properness of the plant model is a non-restricting assumption, since in discrete-time systems there is always a delay of at least one sampling period. For a technical reason matrices  $C_y$  and  $D_{yw}$  are assumed to be independent of the scheduling parameter vector. However, similar results could be obtained for the case of  $C_y$  and  $D_{yw}$  depending on  $\theta$ , and  $B_u$  and  $D_{zu}$  being constant.

Affine dependence on the scheduling parameter vector is assumed for all  $\theta$ -dependent matrices. This can be represented, for example for  $A_g$ , as

$$A_g(\boldsymbol{\theta}(k)) = A_{g_0} + \sum_{i=1}^{n_\theta} \theta_i(k) A_{g_i}. \quad (3)$$

The following fixed-order LPV dynamic output feedback controller structure is considered:

$$\begin{aligned} \mathbf{x}_c(k+1) &= A_c(\boldsymbol{\theta})\mathbf{x}_c(k) + B_c(\boldsymbol{\theta})\mathbf{y}(k) \\ \mathbf{u}(k) &= C_c\mathbf{x}_c(k) + D_c\mathbf{y}(k), \end{aligned} \quad (4)$$

where  $\mathbf{x}_c(k) \in \mathbb{R}^{n_c}$  represents the controller state vector. The choice of controller order  $n_c$  is fully left to user.

Matrices  $A_c(\boldsymbol{\theta})$  and  $B_c(\boldsymbol{\theta})$  are supposed to have an affine dependency on scheduling parameter vector. This implies that the closed-loop matrices are as well affine in the scheduling parameters. Closed-loop system can be written as

$$\begin{aligned} \mathbf{x}(k+1) &= A_{cl}(\boldsymbol{\theta})\mathbf{x}(k) + B_{cl}(\boldsymbol{\theta})\mathbf{w}(k) \\ \mathbf{y}(k) &= C_{cl}(\boldsymbol{\theta})\mathbf{x}(k) + D_{cl}(\boldsymbol{\theta})\mathbf{w}(k), \end{aligned} \quad (5)$$

where  $\mathbf{x}(k) = [\mathbf{x}_g(k) \ \mathbf{x}_c(k)]^T$  and

$$\begin{aligned} A_{cl}(\boldsymbol{\theta}) &= \begin{bmatrix} A_g(\boldsymbol{\theta}) + B_y(\boldsymbol{\theta})D_cC_y & B_g(\boldsymbol{\theta})C_c \\ B_c(\boldsymbol{\theta})C_y & A_c(\boldsymbol{\theta}) \end{bmatrix} \\ B_{cl}(\boldsymbol{\theta}) &= \begin{bmatrix} B_w(\boldsymbol{\theta}) + B_y(\boldsymbol{\theta})D_cD_{yw} \\ B_c(\boldsymbol{\theta})D_{yw} \end{bmatrix} \\ C_{cl}(\boldsymbol{\theta}) &= [C_z(\boldsymbol{\theta}) + D_{zu}(\boldsymbol{\theta})D_cC_y \quad D_{zu}(\boldsymbol{\theta})C_c] \\ D_{cl}(\boldsymbol{\theta}) &= [D_{zw}(\boldsymbol{\theta}) + D_{zu}(\boldsymbol{\theta})D_cD_{yw}] \end{aligned} \quad (6)$$

*Remark 1.* The closed-loop matrices in (6) are affine in  $\boldsymbol{\theta}$  as some plant matrices are limited to be  $\boldsymbol{\theta}$ -independent. If this was not the case, the problem could be treated using the homogenous polynomials relaxations (e.g. as in J De Caigny et al. (2012)). However, for the simplicity of presentation we continue with this assumption.

### 2.2 Discrete-time LPV system stability conditions

Assessing the stability of an LPV system through the use of a Lyapunov function quadratic in the state is well treated in the literature (e.g. M. C. de Oliveira et al. (1999)). In the discrete-time case, keeping the Lyapunov matrix  $P$  constant over  $\Theta$  is too restrictive even if the scheduling parameters can change from one extremal value to another over the course of one sampling period (J. Daafouz and J. Bernussou (2001)). Usually in practical applications the maximum possible variation of a scheduling parameter is bounded as in

$$\theta_i^+ - \theta_i \in [-\bar{\delta}_i, \bar{\delta}_i], \quad 0 < \bar{\delta}_i < 2\bar{\theta}_i, \quad i = 1, \dots, n_\theta, \quad (7)$$

where  $\boldsymbol{\theta}^+ = \boldsymbol{\theta}(k+1)$ . To exploit the bounds on scheduling parameter variation, Lyapunov matrix affine in the scheduling parameter vector is considered:

$$P(\boldsymbol{\theta}) = P_0 + \sum_{i=1}^{n_\theta} \theta_i P_i > 0, \quad \forall \boldsymbol{\theta} \in \Theta. \quad (8)$$

Using (8), well-known stability condition for a discrete-time LPV system can be written as

$$P(\boldsymbol{\theta}) - A_{cl}^T(\boldsymbol{\theta})P(\boldsymbol{\theta}^+)A_{cl}(\boldsymbol{\theta}) > 0. \quad (9)$$

This condition has to be satisfied for all admissible values of  $(\boldsymbol{\theta}, \boldsymbol{\theta}^+)$ . The limits on scheduling parameters (2) and their variations (7) imply that  $(\theta_i, \theta_i^+)$  belongs to a set presented by filling on Fig. 1. The set of vertices of hexagon  $A_i B_i D_i E_i F_i H_i$  will be denoted by  $\Omega_{v_i}$ . This means that the pair  $(\boldsymbol{\theta}, \boldsymbol{\theta}^+)$  always belongs to the polytope  $\Omega$  whose vertex set  $\Omega_v$  is given by  $\Omega_v = \Omega_{v_1} \times \Omega_{v_2} \times \dots \times \Omega_{v_{n_\theta}}$ . The logic behind Fig. 1 is rather intuitive. For example, point  $H_i$  comes from the fact that if  $\theta_i = -\bar{\theta}_i$ , then  $\theta_i^+ \leq -\bar{\theta}_i + \bar{\delta}_i$ , since  $\bar{\delta}_i$  is maximum possible increase of  $\theta_i$  over one sample. Points  $B_i$ ,  $D_i$  and  $F_i$  can be obtained in a similar manner.

*Remark 2.* There are two limit cases that are covered by this setup. First is the fixed scheduling parameter case, which is defined by  $\bar{\delta}_i = 0$ . In this case the hexagon  $A_i B_i D_i E_i F_i H_i$  collapses into a line  $A_i E_i$ . In the case of maximum possible variations, defined by  $\bar{\delta}_i = 2\bar{\theta}_i$ , the filled hexagon degenerates into a square  $A_i C_i E_i G_i$ . Here, however the primary focus is on the non-degenerate case, taking its importance into account.

*Remark 3.* The case of non-symmetric variation bounds could be treated straightforwardly. Symmetric bounds are assumed for the simplicity of presentation.

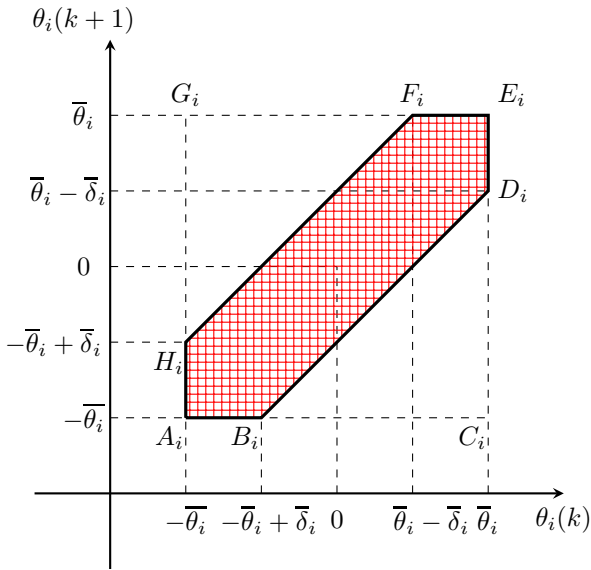


Fig. 1. Admissible  $(\theta_i, \theta_i^+)$  space (filled).

Equivalent representation of (9) in the literature is

$$\begin{bmatrix} P(\theta) & A_{cl}^T(\theta)P(\theta^+) \\ P(\theta^+)A_{cl}(\theta) & P(\theta^+) \end{bmatrix} > 0. \quad (10)$$

As controller variables appearing in  $A_{cl}$  multiply unknown Lyapunov matrix  $P$  in (10), the controller synthesis is a Bilinear Matrix Inequality (BMI) optimisation program. As it is a non-convex optimisation problem, obtaining even (good) local solution is far from trivial. Another issue is that multiplication of  $\theta$  and  $\theta^+$  produces the infinite number of constraints. This can be substituted by the finite number of constraints by application of some relaxation technique (see Scherer (2006)). The idea applied in this publication is to substitute the given infinite set of non-convex constraints on design variables by a finite number of linear matrix inequalities in the controller and Lyapunov function parameters.

### 3. STABILISING FIXED-ORDER DISCRETE-TIME LPV CONTROLLER SYNTHESIS

Over the last 15 years, stability of uncertain and LPV systems is treated using different “slack matrix variable” approaches (e.g. M. C. de Oliveira et al. (1999), R. C. L. F. Oliveira and P. L. D. Peres (2005), J. Daafouz and J. Bernussou (2001)). Similar conditions are developed in M. S. Sadabadi and A. Karimi (2013) and applied to robust fixed-order controller design for uncertain polytopic systems. The following lemma based on the theory from M. S. Sadabadi and A. Karimi (2013) represents a basis for this LPV fixed-order controller synthesis approach.

*Lemma 1.* Strictly Positive Real (SPR) transfer functions  $H(z)$  and  $H^{-1}(z)$  satisfy discrete-time Kalman-Yakubovic-Popov (KYP) lemma with a common Lyapunov matrix  $P$ .

*Lemma 2.* Matrix inequalities

$$\begin{bmatrix} P - M^T P M & M^T P - M^T + T^T A_{cl}^T T^{-T} \\ P M - M + T^{-1} A_{cl} T & 2I - P \end{bmatrix} > 0 \quad (11)$$

and

$$\begin{bmatrix} P_T - A_{cl}^T P_T A_{cl} & A_{cl}^T P_T - A_{cl}^T X + M_T^T \\ P_T A_{cl} - X A_{cl} + M_T & 2X - P \end{bmatrix} > 0, \quad (12)$$

where

$P_T = T^{-T} P T^{-1}$ ,  $M_T = T^{-T} M T^{-1}$ ,  $X = T^{-T} T^{-1}$ , are equivalent.

*Proof.* This lemma is a consequence of Lemma 1. Inequality (11) represents the KYP lemma inequality for

$$H(z) = \left[ \frac{M}{M - T^{-1} A_{cl} T} \middle| \frac{I}{I} \right] \quad (13)$$

Inequality (12) represents the KYP lemma inequality for

$$H^{-1}(z) = \left[ \frac{T^{-1} A_{cl} T}{T^{-1} A_{cl} T - M} \middle| \frac{I}{I} \right] \quad (14)$$

which is pre- and post-multiplied by the block-diagonal matrix  $\text{diag}(T^{-T}, T^{-T})$  and its transpose.  $\square$

Alternatively, the equivalence of (11) and (12) can be proven using the matrix

$$L = \begin{bmatrix} T^{-1} & 0 \\ M T^{-1} - T^{-1} A_{cl} & T^{-1} \end{bmatrix}. \quad (15)$$

Namely, (12) is obtained as (11) pre- and post-multiplied by  $L^T$  and  $L$ . Since pre- and post-multiplication of matrix by the invertible matrix and its transpose does not change its positive definiteness, the matrix inequalities (11) and (12) are equivalent.

*Remark 4.* It can be noticed that Schur stability of both matrices  $A$  and  $M$  is implied through the positive definiteness of the upper left blocks of given matrix inequalities.

#### 3.1 Fixed-order LPV Controller Design Conditions

Using Lemma 2, a sufficient condition for the fixed-order LPV controller synthesis is proposed.

*Theorem 1.* Assume that are given a discrete-time LPV plant affine in scheduling parameter vector  $\theta$ , bounds on the scheduling parameter vector and its variation as in Preliminaries. Furthermore, assume an LPV controller structure (4). Given matrices  $M$  and  $T$ , there exists an LPV controller stabilising the given LPV plant for all admissible scheduling parameter trajectories if

$$\begin{bmatrix} P(\theta) - M^T P(\theta^+) M & (*) \\ P(\theta^+) M - M + T^{-1} A_{cl}(\theta) T & 2I - P(\theta^+) \end{bmatrix} > 0, \quad (16)$$

$$P(\theta) > 0, \quad \forall(\theta, \theta^+) \in \Omega_v,$$

with  $(*)$  representing the terms completing the symmetric matrix.

*Proof.* First it can be observed that the left-hand side of (16) is affine in pair  $(\theta, \theta^+)$ . This means that its validity for  $\forall(\theta, \theta^+) \in \Omega$  can be proven using an appropriate convex combination of vertex inequalities.

Next, it has to be proven that validity of (16) implies stability condition for the closed-loop system  $\forall(\theta, \theta^+) \in \Omega$ . Similarly to the alternative proof of Lemma 2, the following matrix can be considered:

$$L(\theta) = \begin{bmatrix} T^{-1} & 0 \\ M T^{-1} - T^{-1} A_{cl}(\theta) & T^{-1} \end{bmatrix}. \quad (17)$$

Pre- and post-multiplication of (16) by  $L^T(\theta)$  and  $L(\theta)$  imply positive-definiteness of

$$\begin{bmatrix} P_T(\theta) - A_{cl}^T(\theta)P_T(\theta^+)A_{cl}(\theta) & (*) \\ P_T(\theta^+)A_{cl}(\theta) - XA_{cl}(\theta) + M_T & 2X - P_T(\theta^+) \end{bmatrix} \quad (18)$$

for  $\forall(\theta, \theta^+) \in \Omega$ , with the same shorthands as in Lemma 2. The top left block of (18) represents the stability condition (9) for the closed-loop LPV system. Since its positivity for  $\forall(\theta, \theta^+) \in \Omega$  is guaranteed by the Schur complement lemma, stability of the closed-loop system is guaranteed for all allowable scheduling parameter trajectories.  $\square$

*Remark 5.* The total number of constraints in the non-degenerate case corresponds to the cardinality of the set  $\Omega_v$ , which equals  $6^{n_\theta}$ . Considering that in realistic applications there are rarely more than 3 scheduling parameters (F. Wu et al. (1996)), this number of LMIs should be numerically tractable in acceptable execution time.

### 3.2 Fixed-order LPV Controller Synthesis Algorithm

In the continuous-time LPV controller design method presented in Z. Emedi and A. Karimi (2013), the idea for choosing matrix  $M$  is based on the design of initial controllers for all vertices of  $\Theta$ , and solving the inverse of the synthesis problem. Similar idea can be applied here to find appropriate values for  $M$  and  $T$ .

*Remark 6.* It is important to emphasise that the fixed-order controller design is not a trivial task even for an LTI plant, being a non-convex optimisation problem as well. A few approaches are available in the form of Matlab<sup>®</sup> toolbox for  $\mathcal{H}_\infty$  and  $\mathcal{H}_2$  controller design, for example `hinfstruct` (Apkarian and Noll (2006)), `HIFOO` (Burke et al. (2006)) and `FDRC` (Karimi (2013)). Since we need an LTI controller just to initialise the algorithm (not necessarily an optimal one, in any sense), one of these or similar methods should suffice.

Suppose that initial controllers  $K_i, i = 1, \dots, 2^{n_\theta}$  correspond to the vertices of hyper-rectangle  $\Theta$ . This means that for each LTI system obtained by fixing  $\theta_v \in \Theta_v$  one of the above-mentioned fixed-order LTI controller design methods is used to design appropriate stabilising LTI controller  $K_i$ . The next step is the choice of matrices  $M$  and  $T$ . Based on  $K_i, i = 1, \dots, 2^{n_\theta}$ , closed-loop matrices  $A_{cl}(\theta_v)$  can be calculated. By introducing  $A_{cl}(\theta_v)$  into (18), feasible  $X, M_T$  and  $P_T$  can be obtained. Then, from matrix  $X$  the similarity transform matrix  $T$  can be reconstructed by Cholesky factorisation, and from  $M_T$  and  $T$  rises  $M = T^T M_T T$ . Now the controller design phase can be performed using  $M$  and  $T$  in (16).

Evidently, this kind of method could fail already in the first phase, having no feasible  $X, M_T$  and  $P_T$  for given choice of initial controllers. For this reason, we replace (16) and (18) by

$$\begin{bmatrix} \sigma^2 P(\theta) - M^T P(\theta^+) M & (*) \\ P(\theta^+) M - M + T^{-1} A_{cl}(\theta) T & 2I - P(\theta^+) \end{bmatrix} > 0, \quad (19)$$

$$\begin{bmatrix} \sigma^2 P_T(\theta) - A_{cl}^T(\theta) P_T(\theta^+) A_{cl}(\theta) & (*) \\ P_T(\theta^+) A_{cl}(\theta) - X A_{cl}(\theta) + M_T & 2X - P_T(\theta^+) \end{bmatrix} > 0. \quad (20)$$

The idea is now to iterate between (20) and (19) until minimal  $\sigma$  is not obtained. This corresponds to the exponential decay minimisation, and  $\sigma \leq 1$  guarantees stability of the closed-loop system.

This algorithm can be summarised in the 4 following steps:

- step 1:** choose small  $\epsilon > 0$ ; set  $j = 0$ ; design the initial controllers  $K_i, i = 1, \dots, 2^{n_\theta}$  for  $\theta_v \in \Theta_v$
- step 2:** for  $\forall \theta_v \in \Theta_v$  calculate  $A_{cl}(\theta_v)$  using  $K^{j-1}$  (use initial controllers if  $j = 0$ ); set (20) for  $\forall(\theta, \theta^+) \in \Omega_v$  using  $A_{cl}(\theta)$  and find feasible  $X, M_T$  and  $P_T(\theta)$  while minimising  $\sigma$  by bisection; reconstruct  $T$  from  $X = T^{-T} T^{-1}$  and subsequently  $M = T^T M_T T$ .
- step 3:** set (19) for  $\forall(\theta, \theta^+) \in \Omega_v$  using  $M$  and  $T$  obtained in **step 2** and search for feasible  $K^{j+1}(\theta), P_T(\theta)$  and minimal (by bisection)  $\sigma^j$ ;
- step 4:** if  $\sigma^j - \sigma^{j-1} > \epsilon$  set  $j = j + 1$  and jump to the **step 2**; otherwise stop.

Equivalence of (19) and (20) ensures that at worst case in **step 3** we will obtain exactly the same controller and  $\sigma^j$  as those applied in **step 2**. Therefore stability indicator (and exponential decay parameter)  $\sigma^j$  is monotonically non-increasing in this synthesis procedure.

## 4. INDUCED $L_2$ -NORM PERFORMANCE DESIGN

While ensuring stability of the controlled system, it is important to optimise some performance index of the closed-loop system. A widely used performance measure for the LPV control systems is the induced  $l_2$ -norm performance, an extension of the  $\mathcal{H}_\infty$  norm of LTI systems. In general, it gives a good upper bound on the ratio of “energy” of the performance output and external excitation, as observed in a formal definition (J De Caigny et al. (2012)):

*Definition 1.* Suppose that the external input  $w(k)$  belongs to  $l_2$ , the set of all discrete-time signals with bounded 2-norm. Then,  $\gamma$  is an upper bound on the induced  $l_2$ -norm performance of the LPV system (5) if

$$\sup_{w \neq 0} \frac{\|z\|_2^2}{\|w\|_2^2} < \gamma \quad (21)$$

for all allowable scheduling parameter trajectories.

Induced  $l_2$ -norm performance of an LTI system can be characterised through the well-known Bounded Real Lemma. Its extension to the LPV system case can be found in the literature in the following form:

*Lemma 3.*  $\gamma$  is the upper bound on the induced  $l_2$ -norm performance of the LPV system (5) if

$$P - A_{cl}^T P^+ A_{cl} - \gamma^{-1} C_{cl}^T C_{cl} - E^T (I - \gamma^{-1} D_{cl}^T D_{cl} - B_{cl}^T P B_{cl})^{-1} E < 0 \quad (22)$$

is satisfied for  $\forall(\theta, \theta^+) \in \Omega$ , where

$$E = B_{cl}^T P^+ A_{cl} + \gamma^{-1} D_{cl}^T C_{cl} \quad (23)$$

and dependence on  $\theta$  and  $\theta^+$  is omitted.

Our goal is to propose a method for fixed-order discrete-time LPV controller design, guaranteeing good induced  $l_2$ -norm performance for a given LPV system. Similarly to the stabilising LPV controller design problem, constraints (22) define a non-convex set in the space of design variables. The following theorem proposes an inner convex approximation of the non-convex solution set.

*Theorem 2.* Assume that are given a discrete-time LPV plant affine in scheduling parameter vector  $\theta$ , bounds

on the scheduling parameter vector and its variation as in Preliminaries. Furthermore, suppose that the LPV controller structure is given by (4). Given decoupling matrix  $M$  and state transformation matrix  $T$ , there exists an LPV controller stabilising the given LPV plant and ensuring the induced  $l_2$ -norm performance to be at most  $\gamma$  for all admissible scheduling parameter trajectories if

$$\begin{bmatrix} P - M^T P^+ M & (*) & (*) & (*) \\ P^+ M - M + T^{-1} A_{cl} T & 2I - P^+ & (*) & (*) \\ 0 & B_{cl}^T T^{-T} & I & (*) \\ C_{cl} T & 0 & D_{cl} & \gamma I \end{bmatrix} > 0, \quad (24)$$

$$P > 0, \quad \forall(\theta, \theta^+) \in \Omega_v.$$

*Proof.* As the expression (24) is affine in the pair  $(\theta, \theta^+)$ , we can conclude that its validity for  $\forall(\theta, \theta^+) \in \Omega_v$  guarantees the validity for  $\forall(\theta, \theta^+) \in \Omega$  as well. Next, we will prove that validity of (24) for  $\forall(\theta, \theta^+) \in \Omega$  implies the satisfaction of (22). Consider the non-singular matrix

$$L_1(\theta) = \begin{bmatrix} T^{-T} & T^{-T} M^T - A_{cl}^T(\theta) T^{-T} & 0 & -\gamma^{-1} C_{cl}^T(\theta) \\ 0 & B_{cl}^T(\theta) T^{-T} & -I & \gamma^{-1} D_{cl}^T(\theta) \end{bmatrix}.$$

Pre- and post-multiplication of (24) by  $L_1(\theta)$  and  $L_1^T(\theta)$ , and then immediate application of Schur complement lemma around the bottom-right block, produces exactly (22) with  $P_T = T^{-T} P T^{-1}$  instead of  $P$ . This guarantees the upper bound  $\gamma$  on the induced  $l_2$ -norm performance for all possible scheduling parameter trajectories.  $\square$

To be able to choose  $M$  and  $T$ , we propose a matrix inequality equivalent to (24) in which matrices  $M$ ,  $T$  and  $P$  are decoupled.

*Lemma 4.* The matrix inequality

$$\begin{bmatrix} P_T - A_{cl}^T P_T^+ A_{cl} & (*) & (*) & (*) \\ P_T^+ A_{cl} - X A_{cl} + M_T & 2X - P_T^+ & (*) & (*) \\ B_{cl} M_T - B_{cl}^T X A_{cl} & B_{cl}^T X & I & (*) \\ C_{cl} & 0 & D_{cl} & \gamma I \end{bmatrix} > 0 \quad (25)$$

is equivalent to (24) for  $\forall(\theta, \theta^+) \in \Omega$ .

*Proof.* Observe the matrix

$$L_2(\theta) = \begin{bmatrix} T^{-T} & T^{-T} M^T - A_{cl}^T(\theta) T^{-T} & 0 & 0 \\ 0 & T^{-T} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \quad (26)$$

Pre- and post-multiplication of (24) by  $L_2(\theta)$  and  $L_2^T(\theta)$  gives exactly (25). Since the matrix  $L(\theta)$  is non-singular, these two matrix inequalities are equivalent by the same argument as in Lemma 2.  $\square$

Now, similar algorithm to the one in Section 3 can be developed. Here the initialisation can be performed directly using the previously designed stabilising LPV controller. The optimal cost  $\gamma_i$  will be monotonically non-increasing for the reason of equivalence of (25) and (24).

## 5. SIMULATION RESULTS

To illustrate the potential of the proposed method, an LPV controller is designed for a random 4<sup>th</sup> order discrete-time LPV system. Generated plant matrices are :

$$A(\theta) = \begin{bmatrix} 0.5216 & -0.1788 & 0.6895 & -0.4840 \\ 0.4259 + 0.5412\theta & 0.4998 & -0.8022 & 0.1666 \\ -0.6085 & 0.8867 & 0.4388 & -0.0190 \\ 0.4358 & -0.1857 & 0.1947 + 0.1725\theta & 0.6140 \end{bmatrix},$$

$$\begin{aligned} B_u^T &= [-2.0259 \quad -4.5084 \quad 1.9318 \quad 1.5011], \\ B_w^T &= [0.1629 \quad 0.1812 \quad 0.0254 \quad 0.1827], \\ C_y = C_z &= [4.8299 \quad 0.5267 \quad -0.9993 \quad -3.0121], \\ D_{yw} = D_{zw} &= 0.1897, \quad D_{yu} = D_{zu} = 0.0. \end{aligned}$$

Bounds on the scheduling parameter and its variation are assumed as  $\theta \in [-1, 1]$  and  $\delta \in [-1/3, 1/3]$ . It is important to notice that the given system is unstable even for frozen values of  $\theta$ . The LPV controller order is chosen equal to 2, and all controller matrices are assumed to be  $\theta$ -dependent (this causes no problem as only  $A_g$  depends on  $\theta$ ).

First, random initial controllers of order 2 are designed for two vertices of the scheduling parameter interval. Motivation for this comes from the initialization procedure in the HIFOO toolbox ( $H_\infty$  Fixed-Order Optimization, Burke et al. (2006)). For this purpose the function *fminunc* from the Matlab<sup>®</sup> Optimization Toolbox<sup>™</sup> is used. The cost function is chosen as the spectral radius of the closed-loop state matrix. For each of the two vertices, 50 runs of *fminunc* with different randomly chosen initial points are performed. As a result, 4 stabilising LTI controllers are found for the first vertex, and 11 stabilising LTI controllers for the second one. Obtained closed-loop spectral radii all belong to  $[0.9, 1]$ .

Next, a search for the stabilising LPV controller of order 2 can be performed using the algorithm presented in Section 3. In total, 44 experiments are performed for each possible combination of 4 initial controllers for the first vertex and 11 for the second one. As a convex optimisation solver, SDPT3 (Toh et al. (1999)) is used. For any initial controller combination, algorithm stalls after 15 to 25 iterations. In only one out of 44 cases the final controller is not stabilising (spectral radius of 1.0478). In 35 out of 44 cases the final spectral radius is in  $[0.74, 0.75]$ , a great improvement from initial radius valid just for vertices. It is interesting to notice that in the first few iterations of the algorithm, obtained LPV controllers do not stabilise the system, spectral radius begin larger than 1. The execution time depends on the initial controller and is in the interval  $[150, 250]$ s, but there may be a way to reduce this for one order of magnitude by avoiding bisection over  $\sigma$ .

Finally, obtained stabilising LPV controllers can be used as starting points for the induced  $l_2$ -norm performance controller design. The execution time here is much smaller (around 20 seconds) since no bisection algorithm is involved. For around the half of the controllers, the final  $\gamma$  is between 63 and 64. The optimal  $\gamma$  is 63.7044, and the optimal controller:

$$\begin{aligned} A_k(\theta) &= \begin{bmatrix} -1.8304 & -1.2880 \\ -3.1562 & -0.9414 \end{bmatrix} + \theta \begin{bmatrix} 0.2477 & 0.2138 \\ 0.3821 & -0.0529 \end{bmatrix} \\ B_k(\theta) &= \begin{bmatrix} 0.4548 \\ 0.6232 \end{bmatrix} + \theta \begin{bmatrix} -0.1210 \\ -0.1126 \end{bmatrix} \\ C_k^T(\theta) &= \begin{bmatrix} -0.3958 \\ -0.2988 \end{bmatrix} + \theta \begin{bmatrix} 0.1101 \\ 0.0003 \end{bmatrix} \\ D_k(\theta) &= 0.0762 - 0.0380\theta. \end{aligned}$$

## 6. CONCLUSION

In this paper a method for designing fixed-order dynamic output-feedback Linear Parameter Varying (LPV) controllers for discrete-time LPV systems with bounded scheduling parameter variations is presented. Proposed controller design scheme can iteratively improve induced  $l_2$ -norm performance of the controlled system. Provided simulations illustrate that a wide range of initialisation values leads to good performance of the controlled system. In the future work, comparison with existing LTI controller design methods for fixed values of scheduling parameter could be performed, to approximately measure the quality of obtained induced- $l_2$  performance.

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