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Adaptive phasor control of a Duffing oscillator with unknown parameters

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Abstract: Weakly damped electrical and mechanical oscillators that contain a cubic nonlinearity are described mathematically using Duffing's equation. In particular microelectromechanical systems (MEMS) exhibit mechanical structures that are characterized by nonlinear elasticity due to a hardening or softening spring constant. Driving these nonlinear oscillators by sinusoidal excitation with varying frequency and amplitude we observe typical characteristics of nonlinear oscillators in the frequency response as jump phenomena at bifurcation points. Especially in applications with varying physical parameters due to changes in environmental conditions or external disturbances a stable operation in resonance cannot be guaranteed by applying conventional Phase Locked Loops (PLLs). In this work we propose an adaptive phasor control approach that is based on a phasor representation of the linearized Duffing oscillator in order to control the amplitude and phase of the oscillation separately. The Duffing oscillator is linearized adaptively by using a dynamic parameter estimator for the unknown nonlinear spring constant.

1. INTRODUCTION

Duffing's equation has been originally introduced by Duffing [1918] and describes mathematically the motion of a mechanical, single degree-of-freedom system with harmonic excitation and a nonlinear restoring force. In this work we consider the damped Duffing oscillator with a general excitation function $u(t)$ given by:

$$\ddot{x}(t) + 2d\omega_0\, \dot{x}(t) + \omega_0^2\, x(t) + \alpha\, x(t)^3 = b\, u(t). \qquad (1)$$

$x(t), \dot{x}(t)$ denote general state variables with initial conditions $x(0)$, $\dot{x}(0)$, that for instance represent electrical or mechanical quantities as voltage or angular deflection and their time derivatives. In accordance to the linear case ($\alpha = 0$) ω_0, d and b refer to the undamped eigenfrequency, the damping ratio and the input gain of the oscillating system, respectively. For a simple mass-spring-damper system with mass M, that is suspended on a parallel combination of a spring with constant K and a dashpot with damping factor C these parameters are calculated using:

$$\omega_0 = \sqrt{\frac{K}{M}},\ d = \frac{C}{2\sqrt{MK}},\ b = \frac{1}{M}. \qquad (2)$$

The cubic nonlinearity is parametrized by the constant factor α. In the mass-spring-damper system α accounts for a nonlinear spring constant that decreases ($\alpha < 0$) or increases ($\alpha > 0$) proportional to the square of the displacement. This type of elasticity is often referred to as a softening or hardening spring, respectively. A general solution of (1) for arbitrary inputs $u(t)$ does not exist but assuming a periodic excitation $u(t) = \cos(\omega t)$ analytical as well as numerical solutions of Duffing's equation have been presented in Holmes and Rand [1976], Nayfeh and Mook [1979] and Brennan et al. [2008]. A typical frequency response function of a harmonically forced Duffing oscillator is demonstrated in Fig. 1. The response amplitude of the resulting oscillation is shown qualitatively for the spring softening ($\alpha < 0$), the linear ($\alpha = 0$) as well as the spring hardening ($\alpha > 0$) case. Oscillating systems that exhibit cubic elasticity are quite prevalent in technical systems, particularly in MEMS. A broad compilation of resonant MEMS with Duffing-like behaviour is presented in Rhoads et al. [2010]. The vacuum encapsulated torsional microscanners presented in Hofmann et al. [2012] also possess a nonlinear spring characteristic and mainly motivated this work. These devices are primarily supposed to be applied in mobile laser displays that project video data on arbitrary surfaces by scanning a synchronously modulated RGB laser. In order to achieve large screen diameters and low power consumption simultaneously the micromirrors have to be driven in resonance at constant deflection amplitude. Since the laser source intensity is fluctuating continuously according to the projected video data the spring characteristic of the scanner is changing as well resulting in time varying parameters α and ω_0. To attain a high video resolution a precise deflection tracking is required as well. Using standard PLLs to operate these micromirrors in resonance is usually insufficient. Due to the vacuum packaging the scanners possess very low damping (typically $d \approx 10^{-5}$) causing excessive transition times when driven harmonically to resonance by sweeping the

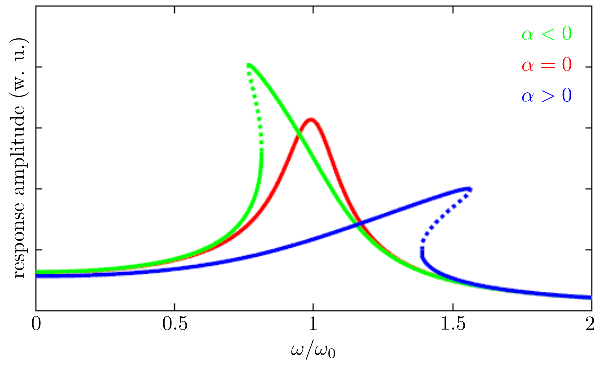

Fig. 1. Frequency response function of a harmonically forced Duffing oscillator (Kanamaru [2008])

excitation frequency. Owing to the unstable path in the frequency response function near the resonance peak (cf. Fig. 1) even small variations of the spring characteristic or the frequency of the driving signal induce a jump to the lower stable branch. The general problem of controlling a Duffing system has been studied extensively in literature, for example in Agrawal et al. [1998] and Harb et al. [2007]. For systems with uncertain or time varying parameters adaptive control strategies have been reported as well in Dong et al. [1997], Loria et al. [1998], Cao [2000] and Yu and Zhang [2004]. In this work we propose a new adaptive phasor control approach to drive a weakly damped Duffing oscillator. It is based on the phasor representation for linear underdamped second order systems, that has been formerly introduced in Koschmieder and Röck [2009] to control a Coriolis Mass Flow Meter (CMFM). Analogous to the investigated Duffing oscillator the tube oscillation of the CMFM is characterized by very low damping as well as unknown and time varying parameters. The model-based phasor control of the CMFM presented in Röck and Koschmieder [2009] exhibits excellent performance in real world experiments. Detailed theoretical as well as practical results regarding the phasor control of linear second order systems will be published in a PhD thesis in 2014. In this work we adopt the phasor control approach to the Duffing oscillator. A nonlinear parameter estimation is used to adaptively linearize the Duffing oscillator by compensating the cubic nonlinearity. In section 2 the estimation algorithm and the reduced order observer for the unmeasured state variable are presented in detail. In order to determine the apriori unknown and time varying quantities eigenfrequency ω_0 and damping ratio d a dynamic parameter model as well as an online parameter estimator are derived in section 3. The derivation of the phasor model as well as the amplitude and frequency controller are covered in section 4. To demonstrate the performance of the proposed control scheme the results of numerical simulations realized in MatLab/Simulink are discussed in section 5. Theoretical considerations regarding stability and robustness of the complete control method as well as a practical evaluation are not part of this paper and will be the subject of future work. In the following sections the time variable t is omitted for reasons of clarity and comprehensibility.

2. ADAPTIVE LINEARIZATION

Reordering (1) gives

$$\ddot{x} + 2d\omega_0\,\dot{x} + \omega_0^2\,x = b\,[u - \frac{\alpha}{b}\,x^3]. \tag{3}$$

From (3) it is easy to see that by choosing

$$u = \nu + \frac{\hat{\alpha}}{b}\,x^3 \text{ with } \hat{\alpha}(t \to \infty) = \alpha \tag{4}$$

an exact linearization of the Duffing oscillator is achieved. The result is the linear second order differential equation

$$\ddot{x} + 2d\omega_0\,\dot{x} + \omega_0^2\,x = b\nu \tag{5}$$

with virtual input ν. As the parameter α is unknown apriori due to tolerances in the manufacturing process and varies over time caused by the fluctuating laser beam intensity a nonlinear parameter estimator is used to determine α online. Following Friedland [1997] a nonlinear dynamical estimator is designed that guarantees asymptotical stability for the dynamics of the estimation error $e = \alpha - \hat{\alpha}$ by incorporating Lyapunov's stability theory. The initial estimation error $e_0 = \alpha - \hat{\alpha}_0$ vanishes for $t \to \infty$. Starting from the general nonlinear system in state space form

$$\underline{\dot{x}} = \underline{f}(\underline{x}, u, \underline{\theta}) \tag{6}$$

with the unknown parameter vector $\underline{\theta}$ the following parameter estimator is chosen:

$$\underline{\hat{\theta}} = \underline{\phi}(\underline{x}) + \underline{z}, \tag{7}$$

$$\underline{\dot{z}} = -\Phi(\underline{x})\,\underline{f}(\underline{x}, u, \underline{\hat{\theta}}), \text{ with } \Phi(\underline{x}) = \left[\frac{\partial\,\phi_i(\underline{x})}{\partial\,x_j}\right]. \tag{8}$$

Assuming $\underline{\theta}$ is constant and the dynamics of the system (6) are affine in the parameter vector $\underline{\theta}$, i.e.

$$\underline{f}(\underline{x}, u, \underline{\theta}) = F(\underline{x}, u)\underline{\theta} + \underline{g}(\underline{x}, u), \tag{9}$$

$$\text{with } F(\underline{x}, u) = \left[\frac{\partial\,f_i(\underline{x}, u, \underline{\theta})}{\partial\,\theta_j}\right] \tag{10}$$

the differential equation describing the error propagation reduces to:

$$\underline{\dot{e}} = -\underline{\dot{\hat{\theta}}} = -\Phi(\underline{x})\,F(\underline{x}, u). \tag{11}$$

By appropriately choosing $\phi(\underline{x})$ the error equation (11) finally results in

$$\underline{\dot{e}} = -L\,\underline{e}, \tag{12}$$

with L being a positive semi-definite matrix and thus guaranteeing asymptotical stability in accordance to Lyapunov's stability theory. For a detailed derivation of the parameter estimator refer to Friedland [1997]. In order to apply this methodology to the Duffing oscillator the system equation in (1) is represented in state space form by defining the state variables $x_1 = x,\ x_2 = \dot{x}$. This gives:

$$\underline{f}(\underline{x}, u, \underline{\theta}) = \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}x_2\\ -2d\omega_0 x_2 - \omega_0^2 x_1 - \alpha x_1^3 + bu\end{bmatrix}$$
$$= \begin{bmatrix}0\\ -x_1^3\end{bmatrix}\alpha + \begin{bmatrix}0 & 1\\ -2d\omega_0 & -\omega_0^2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ b\end{bmatrix}u. \tag{13}$$

Defining

$$\underline{\phi}(\underline{x}) = -l x_1 x_2, \text{ with } l > 0 \tag{14}$$

$\Phi(\underline{x})$ can be calculated directly using (8):

$$\Phi(\underline{x}) = [-l x_2\ \ -l x_1]\,. \tag{15}$$

The matrix L reduces to a positive scalar given by

$$L = \Phi(\underline{x})F(\underline{x}, u) = l x_1^4 > 0 \tag{16}$$

and guarantees that the differential error equation

$$\dot{e} = L\,e = l x_1^4\,e \tag{17}$$

is asymptotical stable. The parameter estimator in (14) requires both state variables $x_1 = x,\ x_2 = \dot{x}$. Assuming

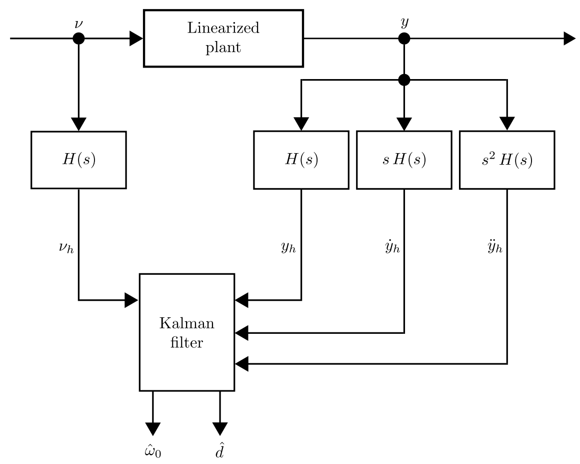

Fig. 2. Parameter estimation

that the first quantity is measured directly, $y = x_1$, a nonlinear reduced order Luenberger observer, that can be found for example in Kailath [1980], is used to estimate the second state x_2.

3. PARAMETER ESTIMATION

Analogous to the quantity α the parameters ω_0 and d vary for individual devices due to tolerances in the manufacturing process. Additionally the linear undamped eigenfrequency ω_0 is also affected by temperature gradients due to the fluctuating laser intensity and varies over time. Again an online estimation algorithm is needed to determine the parameters during operation. As all attempts to find an appropriate function $\phi(\underline{x})$ for estimating the three unknown parameters with the method presented in the previous section have failed, a separate linear state observer is used to determine ω_0 and d. For this purpose we define the discrete time state vector

$$\underline{\eta}_k = \underline{\eta}(t_k) = \begin{bmatrix} \omega_0^2(t_k) \\ 2d\omega_0(t_k) \end{bmatrix} = \begin{bmatrix} \omega_0^2 \\ 2d\omega_0 \end{bmatrix}_k \tag{18}$$

with $t_k = k\,T$, $k \in \mathbb{N}$ and the constant time interval T. Assuming slowly varying parameters ω_0 and d, the linear time varying parameter model is given by:

$$\begin{aligned} \begin{bmatrix} \omega_0^2 \\ 2d\omega_0 \end{bmatrix}_{k+1} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \omega_0^2 \\ 2d\omega_0 \end{bmatrix}_k + \underline{w}_{k+1}\,, \\ \underline{\eta}_{k+1} &= A_k\;\underline{\eta}_k + \underline{w}_{k+1}\,, \end{aligned} \tag{19}$$

$$\begin{aligned} [b\,\nu_h(t_k) - \ddot{y}_h(t_k)]_k &= [y_h(t_k)\;\dot{y}_h(t_k)]_k \begin{bmatrix} \omega_0^2 \\ 2d\omega_0 \end{bmatrix}_k + v_k\,, \\ z_k &= \underline{c}_k^T\;\underline{\eta}_k + v_k. \end{aligned} \tag{20}$$

The stochastic variables $\underline{w}_k$, v_k model random, additive system or measurement noise and represent zero mean, uncorrelated noise processes with covariance Q_k, r_k, respectively. The quantities $\nu_h, y_h, \dot{y}_h$ and $\ddot{y}_h$ are calculated from the input ν and the output y of the linearized system by using the state variable filter topology demonstrated in Fig. 2. $H(s)$ is designed as a linear second order filter with stationary gain $k_s = 1$ and real poles $s_\infty = s_{\infty,1} = s_{\infty,2}$. The parameter state vector $\underline{\eta}_k$ is estimated by applying a discrete time Kalman filter. The required filter equations can be found in Gelb [1974].

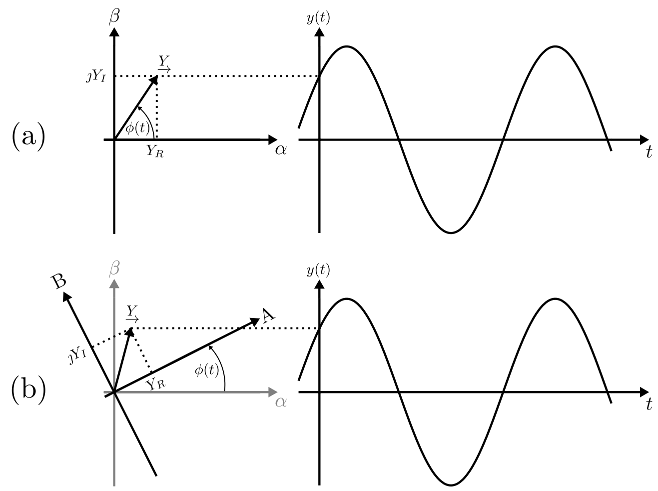

Fig. 3. Definition of the complex phasor in the (a) fixed and (b) with $\dot{\phi}$ rotating complex plane

4. PHASOR CONTROL

Defining the complex, time varying phasor

$$\underset{\rightarrow}{Y} = \mathrm{Re}\{\underset{\rightarrow}{Y}\} + \jmath\,\mathrm{Im}\{\underset{\rightarrow}{Y}\} = Y_R + \jmath Y_I \tag{21}$$

a harmonically oscillating signal y according to Fig. 3 is defined by:

$$\begin{aligned} y &= \mathrm{Im}\{\underset{\rightarrow}{Y} e^{\jmath\phi}\} \\ &= \mathrm{Im}\left\{\Big(Y_R + \jmath Y_I\Big)\Big(\cos\phi + \jmath\sin\phi\Big)\right\} \\ &= Y_R\,\sin\phi + Y_I\,\cos\phi. \end{aligned} \tag{22}$$

By using this phasor representation the amplitude and phase information of a sinusoidal signal can be separated according to (22) and Fig. 3(b). In order to derive a phasor model of the linearized oscillator we differentiate (22) to compute $\dot{y}$, $\ddot{y}$ and define a phasor $\underset{\rightarrow}{U} = U_R + \jmath U_I$ representing the virtual input $\nu = U_R\,\sin\phi + U_I\,\cos\phi$. Substituting these quantities in (5) and setting the state, input and output vectors to

$$\underline{X} = \left[Y_R\;Y_I\;\dot{Y}_R\;\dot{Y}_I\right]^T, \underline{U} = \left[U_R\;U_I\right]^T, \underline{Y} = \left[Y_R\;Y_I\right]^T$$

finally results in the phasor model of the linearized oscillator given by

$$\dot{\underline{X}} = A\,\underline{X} + B\,\underline{U} \tag{23}$$

$$\underline{Y} = C\,\underline{X}. \tag{24}$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (\dot{\phi}^2 - \omega_0^2) & (\ddot{\phi} + 2d\omega_0\dot{\phi}) & -2d\omega_0 & 2\dot{\phi} \\ -(\ddot{\phi} + 2d\omega_0\dot{\phi}) & (\dot{\phi}^2 - \omega_0^2) & -2\dot{\phi} & -2d\omega_0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ b & 0 \\ 0 & b \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The phasor model is represented by a linear fourth order time varying system that enables a separate amplitude and phase control of the sinusoidal oscillation y. To continuously calculate the phasor state vector $\underline{X}$ a Kalman-Bucy-Filter is implemented using the system (23) and measurement (22). The filter equations as well as the mathematical derivation is shown in Gelb [1974].

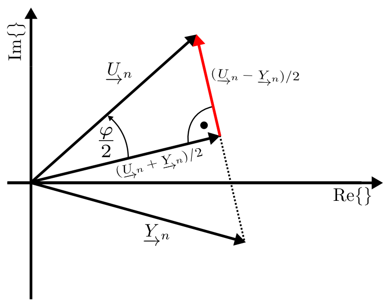

Fig. 4. Definition of the phase error ε in steady state

4.1 Phase control

Using the phasor notation a measurement of the phase shift φ between input and output in steady state can easily be introduced, and used to adjust the stimulating frequency. The objective of phase control is to guarantee that the oscillator is stimulated in its linear eigenfrequency ω_0 at any time. As the linearized model is a second order delay the phase shift between the virtual control input ν and controlled variable y has to be adjusted to $\varphi = \varphi(\omega_0) = -\frac{\pi}{2}$ in steady state. If the corresponding phasors $\underrightarrow{U}$ and $\underrightarrow{Y}$ are represented in the complex plane rotating with $\dot{\phi}$, a generalized phase error ε can be introduced that will vanish, if the phase conditions are met. Using normalized phasors

$$\underrightarrow{U}_n = \frac{\underrightarrow{U}}{|\underrightarrow{U}|}, \underrightarrow{Y}_n = \frac{\underrightarrow{Y}}{|\underrightarrow{Y}|}, |\underrightarrow{U}_n| = |\underrightarrow{Y}_n| = 1$$

the phase angle φ is calculated from Fig. 4 and reads

$$\tan\frac{\varphi}{2} = \frac{|(\underrightarrow{U}_n - \underrightarrow{Y}_n)/2|}{|(\underrightarrow{U}_n + \underrightarrow{Y}_n)/2|}. \tag{25}$$

When stimulating in the linear eigenfrequency $\dot{\phi} = \omega_0$ we get according to the control objective

$$|(\underrightarrow{U}_n - \underrightarrow{Y}_n)| = |(\underrightarrow{U}_n + \underrightarrow{Y}_n)|.$$

Thus we can define the generalized phase error as

$$\varepsilon = \frac{1}{2}|(\underrightarrow{U}_n + \underrightarrow{Y}_n)| - \frac{1}{2}|(\underrightarrow{U}_n - \underrightarrow{Y}_n)|. \tag{26}$$

In steady state the error ε will be zero, if the oscillator is stimulated in its eigenfrequency ω_0. To control the stimulating frequency $\dot{\phi}$, a simple Integral Controller

$$\ddot{\phi} = k_I\,\varepsilon \tag{27}$$

is used. Filtering the control input with a first order delay and cut off frequency ω_f,

$$G_f(s) = \frac{1}{\frac{s}{\omega_f} + 1},$$

finally results in a more smooth control. As the phase control drives the oscillator in its natural eigenfrequency we have $\dot{\phi} = \omega_0$ in steady state. Hence, this controller provides an alternative method to determine ω_0 independently of the parameter observer in section 3.

4.2 Amplitude control

For amplitude control, a 2 DOF (degree of freedom) control scheme is used, consisting of feedforward control and trajectory control in the feedback loop (Horowitz [1963]).

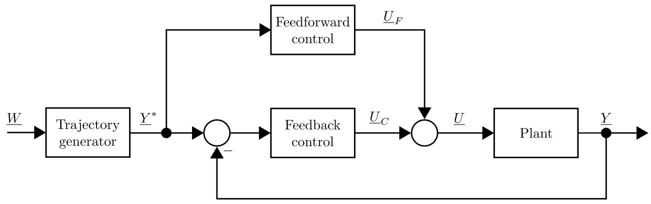

Fig. 5. 2 DOF control scheme for the oscillation amplitude

The basic control approach is depicted in Fig. 5. The amplitude $\underline{Y}$ of the oscillator is tracked via feedforward control to the desired trajectory $\underline{Y}^* = [Y_R^*\ Y_I^*]^T$ that is calculated from the amplitude setpoint $\underline{W} = [W_R\ W_I]^T$ by an appropriate trajectory generator. As the time varying phasor model has full relative degree the feedforward control $\underline{U}_F$ can be calculated very easily using system inversion. In order to account for model uncertainties, changes in parameters and noise a LQ-controller is used in the feedback loop for trajectory control. For the feedforward control $\underline{U}_F = [U_{FR}\ U_{FI}]^T$ with predefined setpoint trajectory $\underline{Y}^*$ we have

$$\begin{aligned} U_{FR} = \frac{1}{b}\Big[&-(\dot{\phi}^2 - \omega_0^2)Y_R^* + 2d\omega_0\dot{Y}_R^* + \ddot{Y}_R^* \\ &+(\ddot{\phi} + 2d\omega_0\dot{\phi})Y_I^* - 2\dot{\phi}\dot{Y}_I^*\Big], \end{aligned} \tag{28}$$

$$\begin{aligned} U_{FI} = \frac{1}{b}\Big[&(\ddot{\phi} + 2d\omega_0\dot{\phi})Y_R^* + 2\dot{\phi}\dot{Y}_R^* \\ &-(\dot{\phi}^2 - \omega_0^2)Y_I^* + 2d\omega_0\dot{Y}_I^* + \ddot{Y}_I^*\Big]. \end{aligned} \tag{29}$$

From this immediately results that the trajectories have to be at least two times continuously differentiable, i.e. $Y_R^*, Y_I^* \in \mathbb{C}^2$. This is realized by using the critically damped second order filters

$$G_{W,i}(s) = \frac{s^i\,Y_R^*}{W_R} = \frac{s^i\,Y_I^*}{W_I} = \frac{s^i\,\omega_W^2}{s^2 + 2\omega_W\,s + \omega_W^2}, \quad i = 0, 1, 2 \tag{30}$$

with unity gain in steady state as the trajectory generator. With the free parameter ω_W, the dynamics of the trajectories can be adjusted and thus the maximum control action can be limited. The feedback controller is designed according to the resulting error model of the feed forward controlled oscillator. For the trajectory error

$$\underline{E}_F = \begin{bmatrix} X_R - Y_R^* \\ X_I - Y_I^* \\ \dot{X}_R - \dot{Y}_R^* \\ \dot{X}_I - \dot{Y}_I^* \end{bmatrix} = \underline{X} - \begin{bmatrix} \underline{Y}^* \\ \underline{\dot{Y}}^* \end{bmatrix} \tag{31}$$

we get by using the phasor model (23)

$$\underline{\dot{E}}_F = A\,\underline{E}_F + B\underline{U}_C, \tag{32}$$

$$\underline{Y}_F = C\underline{E}_F \tag{33}$$

with $\underline{U}_C = [U_{CR}\ U_{CI}]^T$. For feedback control of the trajectory error a LQ-controller with integral action is used, guaranteeing zero steady state error. The feedback loop for the extended system is depicted in Fig. 6. The free parameters of the PI controller can be chosen arbitrarily in order to meet specific requirements. Fig. 7 illustrates the complete phasor control scheme for the Duffing oscillator including adaptive linearization as well as parameter and phasor estimation. The control equation in time domain is then given by:

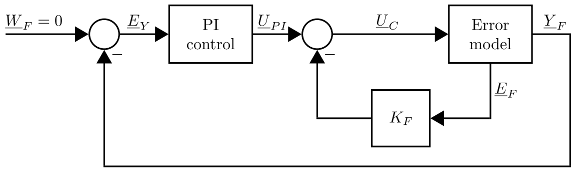

Fig. 6. Extended feedback loop for the error model

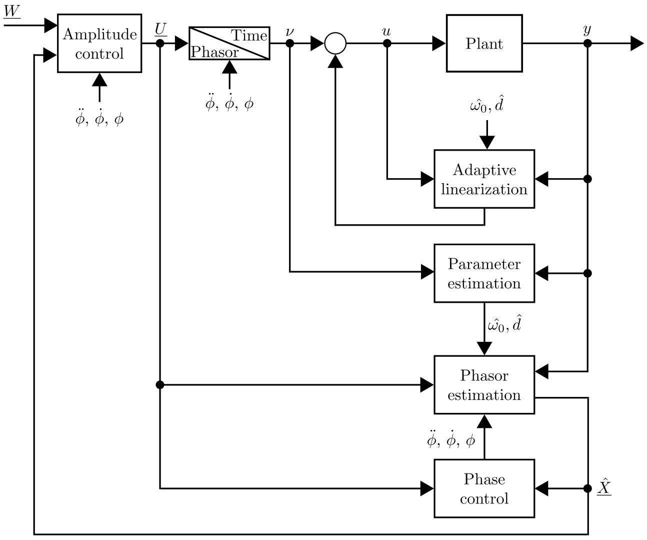

Fig. 7. Complete phasor control scheme for the Duffing oscillator

$$u = \frac{\hat{\alpha}}{b}\, y^3 + \nu\,, \tag{34}$$

$$\begin{aligned} \nu &= U_R\, \sin\phi + U_I\, \cos\phi \\ &= (U_{FR} + U_{CR})\, \sin\phi + (U_{FI} + U_{CI})\, \cos\phi. \end{aligned} \tag{35}$$

5. SIMULATION

To demonstrate the performance of the proposed approach we present numerical simulations that have been realized using MatLab/Simulink. For reasons of numerical stability the simulations are performed using the normalized non-dimensional form

$$\frac{d}{d\tau}\,\eta_1 = \eta_2, \tag{36}$$

$$\frac{d}{d\tau}\,\eta_2 = -2d\omega_n\,\eta_2 - \omega_n^2\,\eta_1 - \alpha_n\,\eta_1^3 + b_n\,u_n, \tag{37}$$

$$y_n = \eta_1, \tag{38}$$

$$\tau = \omega_N\, t,\ \eta_1 = \frac{x_1}{x_{1N}},\ \eta_2 = \frac{x_2}{x_{2N}},\ u_n = \frac{u}{u_N}, \tag{39}$$

$$\omega_n = \frac{\omega_0}{\omega_N},\ \alpha_n = \alpha\frac{x_{1N}^2}{\omega_N^2},\ b_n = b\frac{u_N}{\omega_N^2 x_{1N}}. \tag{40}$$

of the Duffing oscillator. The unknown parameters of the simulated plant are set to $\alpha_n = -0.2$, $\omega_n = 2$, $d = 10^{-5}$ and the input gain is given by $b_n = 3$. In Fig. 8 and 9 the simulation results of an amplitude set point tracking are demonstrated. By choosing W_{Rn}, W_{In} as depicted in Fig. 8 the reference trajectory w_n is set to a sinusoidal oscillation with varying amplitude W_{Rn} and frequency ω_n. The time behaviour of y_n and $\dot{\phi}_N$ clearly reveals that the desired amplitude as well as the frequency of the output y_n are achieved within one oscillation period. Considering

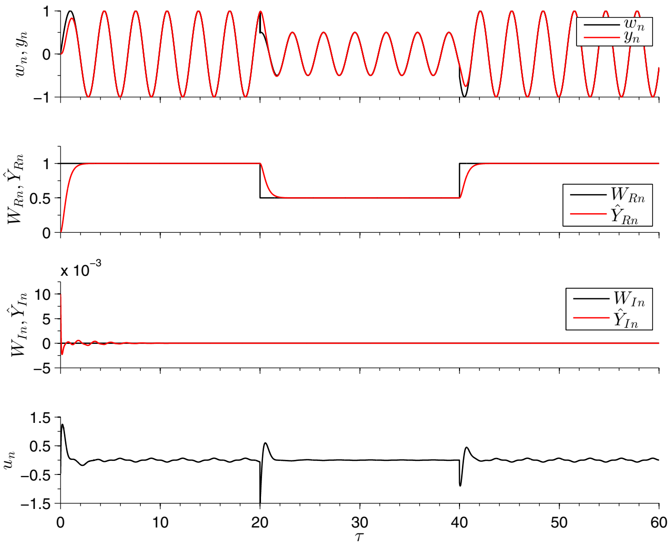

Fig. 8. Simulation of an amplitude setpoint tracking. Reference oscillation w_n, controlled output y_n, reference trajectories W_{Rn}, W_{In}, controlled phasors Y_{Rn}, Y_{In} and control input u_n.

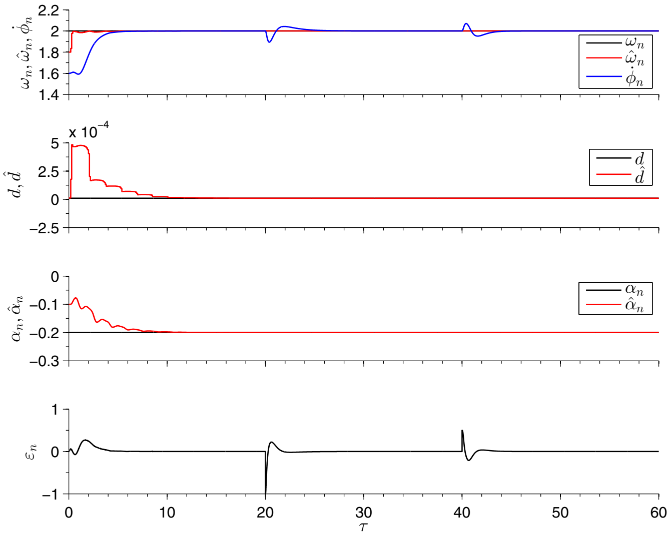

Fig. 9. Simulation of an amplitude set point tracking. Real and estimated parameters $\omega_n, \hat{\omega_n}$, $d, \hat{d}$, $\alpha_n, \hat{\alpha}_N$, controlled excitation frequency $\dot{\phi}_n$ and phase error ε_n.

the low damping ($d = 10^{-5}$) of the simulated system the realized transitions of the amplitude are remarkably fast. However, this requires a relatively large control input u as depicted in Fig. 8. For real systems with constrained actuation force the maximum required control u can be effectively reduced by increasing the amplitude transition time in the trajectory generator (30). The apriori unknown system parameters are also estimated correctly within a few oscillation periods. It is worth noting that the estimation process is not influenced by changes in the oscillation amplitude. The simulation shown in Fig. 10 and 11 demonstrates a controlled Duffing oscillator with time varying system parameters. For this purpose the eigenfrequency ω_n is increased by 10% and the nonlinearity parameter α_n is increased by 50% during the simulation (cf. Fig. 11).

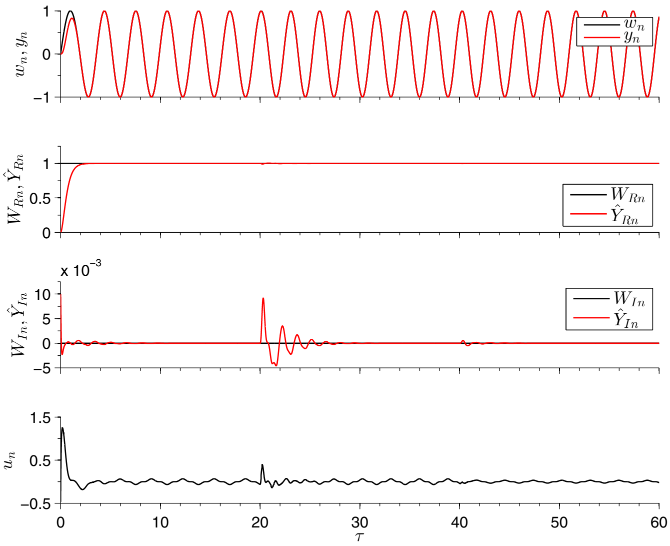

Fig. 10. Simulation of time varying system parameters ω_n and α_n. Reference oscillation w_n, controlled output y_n, reference trajectories W_{Rn}, W_{In}, controlled phasors Y_{Rn}, Y_{In} and control input u_n.

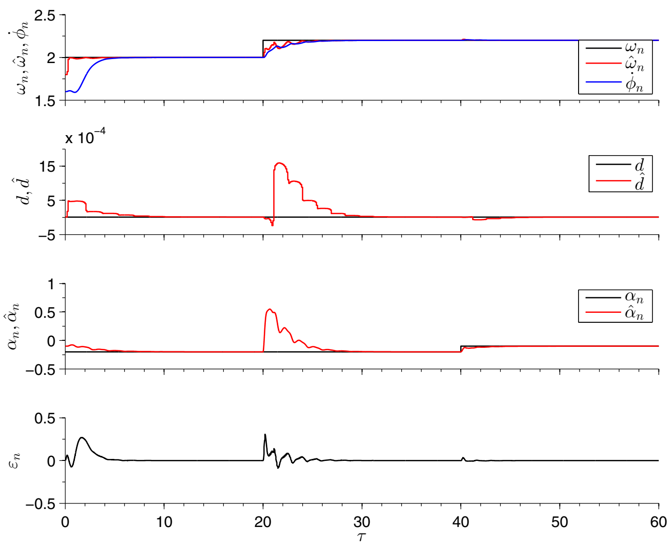

Fig. 11. Simulation of time varying system parameters ω_n and α_n. Real and estimated parameters $\omega_n, \hat{\omega_n}$, $d, \hat{d}$, $\alpha_n, \hat{\alpha}_n$, controlled excitation frequency $\dot{\phi}_n$ and phase error ε_n.

The reference trajectory w_n is set to a sinusoidal oscillation with constant amplitude and frequency equal to the linear eigenfrequency ω_n of the plant. The estimated parameters are presented in Fig. 11 together with the real system parameters. The time responses show that the simulated parameter changes are detected correctly within a few oscillation periods but are subject to strong fluctuations. However, the impact on the controlled output oscillation is negligible.

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