CONTROLLABILITY PROPERTIES IN SAFE REGIONS ⋆

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Abstract: For nonlinear control systems in discrete time, the global controllability structure within a safe region of the state space is analyzed. The main results characterize those safe regions, where every point can be steered into a relatively invariant subset of complete approximate controllability. Furthermore, for parameter dependent systems, loss of invariance is analyzed.

1. INTRODUCTION

The purpose of this paper is to analyze controllability properties of nonlinear systems under the additional requirement that a prescribed safe region W in the state space M is not left. The safe region corresponds to the requirement that the system should satisfy certain constraints in order to ensure integrity of the system. Thus, if a trajectory leaves the safe region W, then the system stops. Another interpretatioen is that the complement $H := M \setminus W$ of the world W is a hole in the state space, through which the system may disappear. In the theory of (uncontrolled) dynamical systems, one speaks of "open dynamical systems", or systems with holes in the state space, and there is a considerable body of literature on them, cf. Demers and Young [2006] for a survey. In the present paper we will analyze a class of open control systems in discrete time. A central notion here are control sets, i.e., maximal subsets of complete approximate controllability. For systems in discrete time, Albertini and Sontag were the first to study control sets, cf. Albertini and Sontag [1993]. Control sets and their relations to flows and semiflows have also been analyzed by San Martin and coworkers in the context of semigroups in Lie groups. They elucidate relations between the structure of semisimple Lie groups, semigroup actions, and control sets; cf. e.g. Patrao and San Martin [2007]. Parameter dependence of control sets has been analyzed in Gayer [2004], Graf [2011], and Colonius and Kliemann [2000].

The invariant subsets of complete controllability are called the invariant control sets. If, under small perturbations, invariance is lost, one may expect that the perturbed system still shows similar, although transient behavior. Here the invariant control sets turn into control sets which are no more invariant and one can show that they lose their invariance only if they change discontinuously in the Hausdorff metric; cf. [Gayer, 2004, Corollary 24]. In the present contribution, we show that control sets relative to a safe region, called here W-control sets, are generated. If the safe region is small enough, the generated W-control set is invariant relative to W. Thus the invariant control sets turn into relatively invariant control sets.

The original motivation for the analysis of invariant control sets (in continuous time) is due to the fact, that they often determine the support of invariant measures of associated stochastic systems; cf. Arnold and Kliemann [1987]. First results which indicate that analogously relatively invariant control sets often determine the supports of conditionally invariant measures for associated stochastic system are given in Colonius [2012]. This provides a major motivation for the present paper. It is worth to mention that the analysis and computation of safe regions in control systems also originates from rather different application areas. For example, Tomlin et al. [1998] discuss safe regions motivated by aircraft traffic control problems.

The contents of this paper is as follows: In Section 2, relative control sets and their invariance properties are characterized. Section 3 discusses loss of invariance under parameter changes. Furthermore, a simple example is discussed which illustrates the results.

2. CONTROL SETS AND RELATIVE INVARIANCE

In this section, basic definitions and properties of control systems in discrete time are collected and results on relative invariance for deterministic control systems are proved. Some results in the continuous time case have been given in Colonius and Kliemann [2000], for the discrete time case considered here we rely on Patrao and San Martin [2007] and Colonius et al. [2010]; cf. also Wirth [1998].

Suppose a discrete time control system on a state space M is given which has the form

$$x_{k+1} = f(x_k, u_k),\ u_k \in \Omega, k \in \mathbb{Z}, \tag{1}$$

where M is a subset of $\mathbb{R}^d$ (or a manifold), $\Omega \subset \mathbb{R}^d$ is compact and connected with $\overline{\text{int}\Omega} = \Omega$ and $f : M\times\Omega \to M$ is a continuous map. Throughout we also assume that $f_v := f(\cdot, v)$ is a diffeomorphism on a neighborhood of M for every $v \in \Omega$. Suppose that an open, relatively compact subset $W \subset M$ is fixed such that $f(W \times \Omega) \cap W \neq \emptyset$ and

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$f(W \times \Omega) \not\subset W$. We may think of the prescribed region W as the world in which the system lives.

Sometimes, the following notation will be useful: Let $f_W := f_{|W\times\Omega} : W \times \Omega \to M$, and consider, with a slight abuse of notation, the following "open" control system

$$x_{k+1} = f_W(x_k, u_k),\ u_k \in \Omega, \tag{2}$$

Note that (2) only makes sense, if $x_k \in W$. Thus this system may enter $M \setminus W$, but it cannot leave $M \setminus W$.

For $x \in M$ and a control function $u : \mathbb{N} \cup \{0\} \to \Omega$ we abbreviate $f^0(x,u) := x$ and

$$f^n(x,u) := f_{u_n} \circ f_{u_{n-1}} \circ ... \circ f_{u_0}(x), n \in \mathbb{N}.$$

Analogously, the restricted maps $f_W^n(x,u)$ are defined. Control system (1) is forward accessible in W if for every $x \in W$ and every $n \in \mathbb{N}$ the reachable sets or positive orbits relative to W

$$\mathcal{O}_W^{+,n}(x) = \{f_W^n(x,u) \mid u : \mathbb{N} \cup \{0\} \to \Omega\}$$

have nonvoid interiors. Obviously, this holds iff all reachable sets $\mathcal{O}_W^{+,1}(x), x \in W$, at time $n = 1$ have nonvoid interiors. Furthermore, forward accessibility implies that for every $x \in W$ there is a control u with $f^n(x,u) \in W$ for all $n \in \mathbb{N}$. We also define the negative orbits relative to W

$$\mathcal{O}_W^{-,n}(x) := \{y \in W \mid\ x = f_W^n(y,u), u : \mathbb{N} \cup \{0\} \to \Omega\}$$

Throughout the rest of the paper, we restrict attention to forward accessible control systems of the form (2) with the additional property that the negative relative orbits $\mathcal{O}_W^{-,1}(x), x \in W$, are either empty or have nonvoid interior. For brevity, we just call these systems accessible.in the open, relatively compact world W in a state space M. We also write

$$\mathcal{O}_W^+(x) := \bigcup_{n\in\mathbb{N}} \mathcal{O}_W^{+,n}(x) \text{ and } \mathcal{O}_W^-(x) := \bigcup_{n\in\mathbb{N}} \mathcal{O}_W^{-,n}(x).$$

Restricting attention to the world W, we obtain the following notions.

A subset $D_W \subset W$ with nonvoid interior is called a W-control set (or relative control set with respect to W) if for all $x, y \in D_W$ one has $y \in \overline{\mathcal{O}_W^+(x)}$ and D_W is maximal with this property i.e., if $D_W' \supset D_W$ is a set such that $y \in \overline{\mathcal{O}_W^+(x)}$ for all $x, y \in D_W'$, then $D_W = D_W'$. A W-control set is called relatively invariant, if $x \in D_W$ and $f^k(x,u) \notin D_W$ for some control u and some $k \in \mathbb{N}$, implies $f^k(x,u) \notin W$.

For the sake of brevity, we call relatively invariant W-control sets just relatively invariant control sets, if it is clear from the context, which world W is considered. If $W = M$, we omit the index W and just speak of control sets and invariant control sets. By accessibility, a subset $D \subset M$ is an invariant control set iff $\overline{\mathcal{O}^+(x)} = D$ for all $x \in D$. Furthermore, a control set (with $W = M$) as defined above is also a control set in the sense of [Patrao and San Martin, 2007, Section 4.2] (where a much more general situation is considered) and hence all properties derived in that paper hold for control sets as defined above. In particular,

$$D = \overline{\mathcal{O}^+(x)} \cap \mathrm{int}\mathcal{O}^-(x)$$

for every x in the core (or transitivity set as it is called in [Patrao and San Martin, 2007, Section 4.2]) defined by $\mathrm{core}D := \{y \in D \,|\, \text{there is } z \in D \text{ with } z \in \mathrm{int}\mathcal{O}^+(y)$ and $y \in \mathrm{int}\mathcal{O}^+(z)\}$; the set $\mathrm{core}D$ is open and it is dense in D. Since Ω is connected, a control set is invariant iff it is closed; see [Colonius et al., 2010, Lemma 3]. Relative control sets are, in general, properly contained in control sets, since they need not be maximal with respect to the whole state space. Nevertheless, they enjoy many properties which are analogous to those of control sets.

We define the core of a relatively invariant control set D_W by $\mathrm{core}D_W := \{y \in D_W \,|\, \text{there is } z \in D_W \text{ with } z \in \mathrm{int}\mathcal{O}_W^+(y) \text{ and } y \in \mathrm{int}\mathcal{O}_W^+(z)\}$.

Theorem 1. (i) Relative control sets are pairwise disjoint. (ii) For every relative control set D_W the core $\mathrm{core}D_W$ is an open set and it is dense in D_W. (iii) A relative control set D_W is relatively invariant iff it is closed relative to W. (iv) Let D_W be a relatively invariant control set. Then D_W is an invariant control set iff $\partial D_W \cap \partial W = \varnothing$.

Proof. Assertions (i) to (iii) follow by minor modifications of the proofs in [Patrao and San Martin, 2007, Section 4.2] and [Colonius and Kliemann, 2000, Section 3.3]. If the condition $\partial D_W \cap \partial W = \varnothing$ in (iv) holds, assertion (iii) implies that D_W is a closed control set in M and hence an invariant control set for system (1). The converse follows, since an invariant control set in M is closed.

The main result on existence of relatively invariant control sets is the following.

Theorem 2. Consider a control system of the form (2) which is accessible in an open, relatively compact world W in a state space M. Consider $x \in W$ and assume that there exists a closed set $Q \subset W$ such that for all $y \in \mathcal{O}_W^+(x)$ one has $\overline{\mathcal{O}_W^+(y)} \cap Q \neq \emptyset$. Then there exists a relatively invariant W-control set $D_W \subset \overline{\mathcal{O}_W^+(x)}$. Furthermore, the following assertions are equivalent: (i) There is a closed set $Q \subset W$ such that $\overline{\mathcal{O}_W^+(x)} \cap Q \neq \emptyset$ for all $x \in W$. (ii) For every $x \in W$ there is a relatively invariant control set D with $D \subset \overline{\mathcal{O}_W^+(x)}$. If (i) holds, there are only finitely many relatively invariant control sets.

Proof. For $y \in \overline{\mathcal{O}_W^+(x)}$ let $Q(y) := \overline{\mathcal{O}_W^+(y)} \cap Q$. Consider the family $\mathcal{F}$ of nonvoid and compact subsets in W given by $\mathcal{F} = \{Q(y),\ y \in Q(x)\}$. Then $\mathcal{F}$ is ordered via

$$Q(y) \prec Q(z) \text{ if } z \in \overline{\mathcal{O}_W^+(y)}.$$

Every linearly ordered set $\{Q(y_i),\ i \in I\}$ has an upper bound

$$Q(y) = \bigcap_{i\in I} Q(y_i) \text{ for some } y \in \bigcap_{i\in I} Q(y_i),$$

because the intersection of decreasing compact subsets of the compact set Q is nonempty. Thus Zorn's lemma implies that the family $\mathcal{F}$ has a maximal element $Q(y)$. Now we claim that the set

$$D_W := \overline{\mathcal{O}_W^+(y)}^W$$

is a W-invariant control set; here the closure is taken relative to W. In fact: Note first that $y \in Q \subset W$, hence $y \in D_W$ and by accessibility

$$\emptyset \neq \mathrm{int}\mathcal{O}_{W,\leq t}^+(y) \subset D_W.$$

Thus $\mathrm{int}D_W \neq \emptyset$. Furthermore, every $z \in D_W$ is approximately reachable from y within W, i.e., $D_W \subset \overline{\mathcal{O}_W^+(y)}^W$.

Conversely, $y \in \overline{\mathcal{O}^+_W(z)}^W$, because otherwise $y \notin Q(z) = \overline{\mathcal{O}^+_W(z)}^W \cap Q(x)$, hence this is a proper subset of $Q(y)$ contradicting the maximality of $Q(y)$. Thus approximate controllability in D_W holds. In order to show that D_W is a W-control set, we have to verify that D_W is maximal with this property: Otherwise there is a W-control set $D'_W \supset D_W$ containing a point $z \notin D_W = \overline{\mathcal{O}^+_W(y)}^W$. Because approximate controllability holds in D'_W it follows that $z \in \overline{\mathcal{O}^+_W(y)}^W$ contradicting the choice of z. Hence D_W is a W-control set. Now consider $\varphi(t,z,u) \in W$ with $t > 0$, $u \in \mathcal{U}$, and $z \in D_W$. Then $\varphi(t,z,u) \in \overline{\mathcal{O}^+_W(y)}^W = D_W$. Hence D_W is W-invariant. Finally, note that for every y in a W-invariant control set $D_W \subset \overline{\mathcal{O}^+_W(x)}^W$ one has

$$\emptyset \neq \overline{\mathcal{O}^+_W(y)}^W \cap Q \subset W.$$

Hence W-invariance implies $\overline{D_W}^W \cap Q \neq \emptyset$. The arguments above also show that assertion (i) implies assertion (ii). Next we will show that the number of relatively invariant W-control sets is finite, if (i) holds. Assume that there are infinitely many W-invariant control sets $D_{W,n}, n \in \mathbb{N}$. Then $\overline{D_{W,n}} \cap Q \neq \emptyset$ for all n. By compactness of Q it follows that there is a sequence $x_n \in \overline{D_{W,n}} \cap Q$ converging to some $x \in Q \subset W$. Then one finds a relatively invariant W-control set D_W with $D_W \subset \overline{\mathcal{O}^+(x)}$. Because the core of D_W is nonvoid and $\overline{D_n} = \overline{\text{core} D_n}$, we obtain a contradiction to relative invariance of $D_{W,n}$ for n large enough. Hence there are only finitely many relatively invariant W-control sets. It remains to show that assertion (ii) implies assertion (i). Choose for each of the finitely many relatively invariant W-control set $D_{W,i}$ a point $x_i \in \text{core} D_{W,i}$, and let Q be this finite set. Then Q is a compact subset of W and satisfies the condition in (i).

As a final observation in this section, note that every control system of the form (1) generates a discrete-time dynamical system (a skew product system) in the following way: Denote the space of control functions $u : \mathbb{Z} \to \Omega$ by $\Omega^{\mathbb{Z}}$, define $\vartheta : \Omega^{\mathbb{Z}} \to \Omega^{\mathbb{Z}}$ as the shift

$$\vartheta(..., u_{-1}, u_0, u_1, ...) := (..., u_0, u_1, u_2, ...),$$

and let

$$F : (x,u) \mapsto (f(x,u_0), \vartheta u) : M \times \Omega^{\mathbb{Z}} \to M \times \Omega^{\mathbb{Z}}. \quad (3)$$

Then F is invertible and its iterations define a continuous dynamical system in discrete time, if we endow $\Omega^{\mathbb{Z}}$ with the metric

$$d(u,v) := \sum_{i\in\mathbb{Z}} 2^{-|i|} \|u_i - v_i\| \text{ for } u,v \in \Omega^{\mathbb{Z}}.$$

For the restriction $F_W : W \times \Omega^{\mathbb{Z}} \to M \times \Omega^{\mathbb{Z}}$ the iteration $(F_W \circ F_W)(x,u)$ is only defined, if $F_W(x,u) \in W \times \Omega^{\mathbb{Z}}$, i.e., $f(x,u_0) \in W$. This construction shows that open control systems may be viewed as a special case of open dynamical systems as defined in the literature. Note also that an analogous construction with $\Omega^{\mathbb{N}}$ yields a semi-dynamical system.

3. PARAMETER DEPENDENCE

In this section, we discuss the fate of invariant control sets, when, under a parameter variation, they lose their invariance. It will turn out, that for an appropriately defined small world W, relatively invariant W-control sets are generated.

Suppose that f depends on a real parameter $\alpha \in I$, where I is an open interval in $\mathbb{R}$, and consider a family of control systems of the form

$$x_{n+1} = f^{\alpha}(x_n, u_n), u_n \in \Omega, \alpha \in I. \quad (4)$$

We assume that for every α the assumptions for system (1) are satisfied and we denote the corresponding objects for the α-system by $D^{\alpha}, \mathcal{O}^{\alpha,+}(x)$, etc. Then [Colonius et al., 2010, Proposition 3 and Theorem 2] shows the following continuity properties of control sets.

Theorem 3. Suppose that system (4) with $\alpha = \alpha_0$ is accessible in M, that f is continuous with respect to (x,u,α), and that the map $v \mapsto f^{\alpha}(x,v)$ is a diffeomorphism for all x, α. Let D^{α_0} be a compact invariant control set for f_{α_0}. (i) There are $\varepsilon_0 > 0$ and a family of control sets D^{α} for $|\alpha - \alpha_0| < \varepsilon_0$ with the following property: For all compact subsets $K \subset \text{core} D^{\alpha_0}$ there is $\varepsilon_K \in (0, \varepsilon_0)$ such that $K \subset \text{core} D^{\alpha}$ for all α with $|\alpha - \alpha_0| < \varepsilon_K$. The map $\alpha \mapsto \overline{D^{\alpha}}$ is lower semicontinuous in the Hausdorff metric. (ii) Suppose that the map $\alpha \mapsto \overline{D^{\alpha}}$ is continuous at $\alpha = \alpha_0$. Then there is $\varepsilon_2 > 0$ such that for all α with $|\alpha - \alpha_0| < \varepsilon_2$ the D^{α} are invariant control sets.

This theorem shows that under small perturbations, one finds near an invariant control set D^{α_0} a control set for the perturbed system. While there may be other control sets near D^{α_0} for the perturbed system, the condition in (i) picks a "big" control set. Since we are interested in situations, where an invariant control set loses its invariance, assertion (ii) shows that for us only the case is relevant, where $\overline{D^{\alpha}}$ changes discontinuously at $\alpha = \alpha_0$. The following theorem gives more information on this situation. It shows that, in addition to the control sets D^{α}, relative control sets are generated. While the D^{α} are not invariant control sets, these relative control sets are relatively invariant, if the world W is chosen small enough.

Theorem 4. Consider the family (4) of control systems and suppose that the assumptions of Theorem 3 are satisfied. (i) For every open neighborhood W of D^{α_0} there are $\varepsilon_1 = \varepsilon_1(W) > 0$ and a family of relative control sets D^{α}_W for $|\alpha - \alpha_0| < \varepsilon_0$ with the following property: For all compact subsets $K \subset \text{core} D^{\alpha_0}$ there is $\varepsilon_K \in (0, \varepsilon_1)$ such that $K \subset \text{core} D^{\alpha}_W$ for all α with $|\alpha - \alpha_0| < \varepsilon_K$. In particular, $D^{\alpha_0}_W = D^{\alpha_0}$ and the map $\alpha \mapsto \overline{D^{\alpha}_W}$ is lower semicontinuous in the Hausdorff metric. (ii) There are a neighborhood W of D^{α_0}, a constant $\varepsilon_1 > 0$ and $x^0 \in \text{core} D^{\alpha_0}$ such that for all α with $|\alpha - \alpha_0| < \varepsilon_1$ one has $W \subset \mathcal{O}^{\alpha,-}_W(x^0)$, i.e., for every $x \in W$ there are $k \in \mathbb{N}$ and a control u such that $f^{\alpha,k}_W(x,u) = x^0$. (iii) Suppose that W is as in (ii). Then for $|\alpha - \alpha_0|$ small enough, the relative control sets D^{α}_W from assertion (i) are relatively invariant.

Proof. (i) Let $K \subset \text{core} D^{\alpha_0}$ be a compact set with nonvoid interior and let $x_0 \in \text{int} K$. By Theorem 3, the control set $D^{\alpha} \subset M$ satisfies $K \subset \text{core} D^{\alpha}$ for α near α_0. Furthermore, for any two points $x, y \in K$ there are $n \in \mathbb{N}$ and a control u such that

$$y = f^{\alpha_0,n}_W(x,u).$$

By compactness of K and accessibility, we may assume that there is $N \in \mathbb{N}$ with $n \leq N$ for all $x, y \in K$. As stated in Theorem 3, $y \in \mathcal{O}^{\alpha,+}(x)$ for all α near α_0. Inspection of the proof of this result in Colonius et al. [Colonius et al., 2010, Theorem 2] (which is based on the implicit function theorem) shows that for α near α_0 the trajectories from x to y are uniformly close to those for α_0, hence $y \in \mathcal{O}_W^{\alpha,+}(x)$ for α near α_0. Thus there is a relative control set D_W^α containing K. Now consider a sequence of compact subsets K_n with nonvoid interior satisfying $K_n \subset K_{n+1} \subset \mathrm{core}D^{\alpha_0}$ with $\overline{\bigcup_{n\in\mathbb{N}} K_n} = \overline{\mathrm{core}D^{\alpha_0}}$. Note that for every compact subset of coreD^{α_0} there is n with $K \subset K_n$. Then the argument above yields the desired family of relative control sets D_W^α. (ii) For $y_0 \in D^{\alpha_0}$ there is a control value $v \in \Omega$ with $f^{\alpha_0}(y_0, v) \in D^{\alpha_0}$. Using that $f^{\alpha_0}(\cdot, v)$ is a diffeomorphism and that coreD^{α_0} is dense in D^{α_0} one may assume that there is a compact subset $K \subset \mathrm{core}D^{\alpha_0}$ with $f^{\alpha_0}(y_0, v) \in \mathrm{int}K$. By continuity, one has for all y in a neighborhood of y_0 that $f^\alpha(y, v) \in \mathrm{core}D^{\alpha_0}$. By compactness of D^{α_0}, finitely many of these neighborhoods cover D^{α_0} and their union defines a neighborhood W of D^{α_0}. Then $K \subset \mathrm{core}D_W^\alpha$ for the control sets according to (i) and one finds by controllability in coreD_W^α a point $x^0 \in K$ with $W \subset \mathcal{O}_W^{\alpha,-}(x^0)$. (iii) Suppose, contrary to the assertion, that there are $x \in D_W^\alpha$ and $k \in \mathbb{N}$ such that for some control u one has $f^k(x, u) \in W \setminus D$ and $f^j(x, u) \in W$ for $j = 1, ..., k-1$. The assumption on W implies that $f^k(x, u) \in W \subset \mathcal{O}_W^{\alpha,-}(x^0)$ for some $x^0 \in \mathrm{core}D^{\alpha_0}$. By maximality of relative control sets it follows that $f^k(x, u) \in D_W^\alpha$, a contradiction. Hence D_W^α is relatively invariant.

We observe that for $\alpha = \alpha_0$ every neighborhood of D^{α_0} and hence every world W as in Theorem 4(ii) may have nonvoid intersection with control sets different from D^{α_0}. This would occur if the control set D^{α_0} intersects the closure of another control set $\hat{D}^{\alpha_0}$.

We briefly discuss a simple example illustrating the results above. Here a control system on the circle $M = \mathbb{R}/\mathbb{Z}$ is considered.

Let $f : \mathbb{R}/\mathbb{Z} \times [-1, 1] \to \mathbb{R}/\mathbb{Z}$ be given by

$$f_\alpha(x, u) = x + \frac{\sigma}{2\pi}\cos(2\pi x) + Au + \alpha \mod 1, \qquad (5)$$

where $0 < \sigma < 1$. The control u takes values in $\Omega := [-1, 1]$. Consider a small positive value of the control amplitude A. For $\alpha_0 = -\frac{\sigma}{2\pi} - A$ the extremal graph $f_{\alpha_0}(\cdot, 1)$ is tangent to the diagonal at a point b. Here f_{α_0} admits an invariant control set $D^{\alpha_0} = [b, c]$ with $b < c$. For α below α_0 there is an invariant control set D^α which is an interval that varies continuously with α. For $\alpha > \alpha_0$ the system is completely controllable on $\mathbb{R}/\mathbb{Z}$. Take the safe region W as an open set containing D^{α_0} with $W \subset \mathcal{O}_W^{\alpha,-}(x^0)$ for some $x^0 \in \mathrm{core}D^{\alpha_0}$, e.g., let $W := (0.1, 0.6)$. Then for $\alpha > \alpha_0$ the only control set is the (invariant) control set $D^\alpha = \mathbb{R}/\mathbb{Z}$. There is a unique relatively invariant W-control set D_W^α, which has the form $D_W^\alpha = [b(\alpha), 0.6)$, where $b(\alpha)$ is given by the intersection of the lower sinusoidal curve (depending on α) with the diagonal; hence $b(\alpha_0) = b$ is the left boundary point of D^{α_0}. Thus this interval is closed relative to $W = (0.1, 0.6)$.

One can easily modify this example, so that to the left of D_W^α there is a second relative control set in W which then is open and not relatively invariant.

4. CONCLUSIONS

This paper has introduced relatively invariant control sets as a generalization of the notion of invariant control sets. While they share many properties with invariant control sets, this notion sheds new light on the perturbation theory of invariant control sets: If the the safe region or world W around an invariant control set is small enough, then an invariant control set D^{α_0} always generates a family of relatively invariant control sets D_W^α in W. This is of interest when the perturbed control sets D^α in M have lost invariance.

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