Norm-controllability, or how a nonlinear system responds to large inputs

Matthias A. Müller * **Daniel Liberzon** ** **Frank Allgöwer** *

* *Institute for Systems Theory and Automatic Control (IST), University of Stuttgart, Germany.*
e-mail: {matthias.mueller,frank.allgower}@ist.uni-stuttgart.de
** *Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, USA.*
e-mail: liberzon@illinois.edu

Abstract: The purpose of this paper is to survey and discuss recent results on norm-controllability of nonlinear systems. This notion captures the responsiveness of a nonlinear system with respect to the applied inputs in terms of the norm of an output map, and can be regarded as a certain type of gain concept and/or a weaker notion of controllability. We state several Lyapunov-like sufficient conditions for this property in a simplified formulation, and illustrate the concept with several examples.

Keywords: Nonlinear Systems, Controllability, Lyapunov Methods, Input-to-State Stability

1. INTRODUCTION

When considering dynamical systems with inputs and outputs, a key question is how the applied inputs affect the behavior of the states and outputs of the system. In this respect, various properties are of interest. For example, one fundamental system property is controllability, which is usually formulated as the ability to reach any state from any other state by choosing an appropriate control input (see, e.g., Hespanha [2009], Sontag [1998]). Another important question is whether bounded inputs lead to bounded system states or outputs. This question is dealt with in the context of input-to-state stability (ISS) [Sontag, 1989] and related notions involving outputs such as input-to-output stability (IOS) [Sontag and Wang, 1999] or $\mathcal{L}_\infty$ stability (see, e.g., Khalil [2002]), respectively, where an upper bound on the norm of the system state and the output, respectively, are considered in terms of the infinity norm of the input.

In other settings, a problem complementary to the above is of interest. Namely, one would like to obtain a *lower* bound on the system state or the output in terms of the norm of the applied inputs. This could, e.g., be the case in the process industry, where one wants to determine whether and how an increasing amount of reagent yields an increasing amount of product, or in economics, where certain inputs such as the price of a product or the number of advertisements influence the profit of a company. Furthermore, if the system input constitutes a disturbance, it is interesting to obtain a lower bound for the effect of the worst case disturbance on the system states or the output.

In this paper, we give a simplified and streamlined exposition of the concept of norm-controllability, which was introduced by Müller et al. [2011, 2012] and deals with the questions raised above. As the wording suggests, in contrast to point-to-point controllability, we consider the *norm* of the system state (or, more generally, of an output), and ask how it is affected by the applied inputs. In particular, we are interested in whether this norm can be made large by applying large enough inputs for sufficiently long time. The definition of norm-controllability (see Section 3) is such that the reachable set of the system projected on the output space can be lower bounded in terms of the norm of the applied inputs and the time horizon over which they were applied. In this respect, norm-controllability can be seen as complementary to the concepts of ISS or IOS and $\mathcal{L}_\infty$ stability, respectively. We survey several sufficient conditions for norm-controllability from Müller et al. [2011, 2012] and state them in a simplified way. Furthermore, we illustrate the proposed concept with several examples.

2. PRELIMINARIES AND SETUP

We consider nonlinear control systems of the type

$$\dot{x} = f(x,u), \qquad y = h(x), \qquad x(0) = x_0 \tag{1}$$

with state $x \in \mathbb{R}^n$, output $y \in \mathbb{R}^k$, and input $u \in U \subseteq \mathbb{R}^m$, where the set U of admissible input values can be any closed subset of $\mathbb{R}^m$ (or the whole $\mathbb{R}^m$). Suppose that $f \in C^{\bar{k}-1}$ for some $\bar{k} \geq 1$ and $\partial^{\bar{k}-1} f / \partial x^{\bar{k}-1}$ is locally Lipschitz in x and u. Input signals $u(\cdot)$ to the system (1) satisfy $u(\cdot) \in L^\infty_{loc}(\mathbb{R}_{\geq 0}, U)$, where $L^\infty_{loc}(\mathbb{R}_{\geq 0}, U)$ denotes the set of all measurable and locally bounded functions from $\mathbb{R}_{\geq 0}$ to U. We say that a set $\mathcal{B} \subseteq \mathbb{R}^n$ is rendered

* The work of the first and third author was supported by the German Research Foundation (DFG) within the Priority Programme 1305 "Control Theory of Digitally Networked Dynamical Systems" and within the Cluster of Excellence in Simulation Technology (EXC 310/1) at the University of Stuttgart. The work of the second author was supported by the NSF grants CNS-1217811 and ECCS-1231196, and by a National Research Foundation Grant funded by the Korean Government (NRF-2011-220-D00043).

control-invariant for system (1) by a set $\bar{U} \subseteq U$, if for every $x_0 \in \mathcal{B}$ and every $u(\cdot)$ satisfying $u(\cdot) \in L^\infty_{loc}(\mathbb{R}_{\geq 0}, \bar{U})$ the corresponding state trajectory satisfies $x(t) \in \mathcal{B}$ for all $t \geq 0$. We assume that the system (1) exhibits the *unboundedness observability* property (see Angeli and Sontag [1999] and the references therein), which means that for every trajectory of the system (1) with finite escape time t_{esc}, also the corresponding output becomes unbounded for $t \to t_{esc}$. This is a very reasonable assumption as one cannot expect to measure responsiveness of the system in terms of an output map h (as we will later do) if a finite escape time cannot be detected by this output map. We remark that, for example, all linear systems satisfy this assumption, as do all nonlinear systems with radially unbounded output maps.

For every $b > 0$, denote by $U_b := \{u \in U : 0 \leq |u| \leq b\}$ the set of all admissible input values with norm in the interval $[0, b]$, which we assume to be nonempty. Furthermore, for every $a, b > 0$, denote by

$$\mathcal{U}_{a,b} := \{u(\cdot) : u(t) \in U_b,\ \forall t \in [0, a]\} \subseteq L^\infty_{loc}(\mathbb{R}_{\geq 0}, U) \quad (2)$$

the set of all measurable and locally bounded input signals whose norm takes values in the interval $[0, b]$ on the time interval $[0, a]$. Let $\mathcal{R}^\tau\{x_0, \mathcal{U}\} \subseteq \mathbb{R}^n \cup \{\infty\}$ be the reachable set of the system (1) at time $\tau \geq 0$, starting at the initial condition $x(0) = x_0$ and applying input signals $u(\cdot)$ in some set $\mathcal{U} \subseteq L^\infty_{loc}(\mathbb{R}_{\geq 0}, U)$. The reachable set $\mathcal{R}^\tau\{x_0, \mathcal{U}\}$ contains ∞ if for some $u(\cdot) \in \mathcal{U}$ a finite escape time $t_{esc} \leq \tau$ exists. Define $R^\tau_h(x_0, \mathcal{U})$ as the radius of the smallest ball in the output space centered at $y = 0$ which contains the image of the reachable set $\mathcal{R}^\tau\{x_0, \mathcal{U}\}$ under the output map $h(\cdot)$, or ∞ if this image is unbounded.

3. NORM-CONTROLLABILITY: DEFINITION AND DISCUSSION

In the following, we precisely define and discuss the notion of norm-controllability.

Definition 1. The system (1) is *norm-controllable* from x_0 with gain function γ, if there exists a function $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\gamma(\cdot, b)$ nondecreasing for each fixed $b > 0$ and $\gamma(a, \cdot)$ of class $\mathcal{K}_\infty$ for each fixed $a > 0$, such that for all $a > 0$ and $b > 0$

$$R^a_h(x_0, \mathcal{U}_{a,b}) \geq \gamma(a, b), \quad (3)$$

where $\mathcal{U}_{a,b}$ is defined in (2). □

The above definition of norm-controllability[1] can be interpreted in the following way. It provides a measure for how large the norm of the output y can be made in terms of the maximum magnitude b of the applied inputs and the length of the interval a over which they are applied. This is captured via the gain function γ, which gives a lower bound on the radius of the smallest ball which contains the image of the reachable set under the output map h, when inputs with magnitude in the interval $[0, b]$ are applied over the time interval $[0, a]$. For each fixed time horizon a, $\gamma(a, \cdot)$ is required to be of class $\mathcal{K}_\infty$, which means that with inputs of increasing magnitude one should be able to also increase the norm of the output. On the other hand, for every fixed upper bound b on the input norm, increasing the time horizon a over which such inputs are applied should result in a non-decreasing magnitude of the output. In this respect, norm-controllability captures both the "short-term" as well as the "long-term" responsiveness of the system (1) with respect to the input u in terms of the norm of the output map h, for which the gain γ is a quantitative measure.

Furthermore, as mentioned in the Introduction, we note that the concept of norm-controllability can be seen as somehow complementary to ISS (respectively, to related concepts involving outputs such as IOS and $\mathcal{L}_\infty$ stability). Namely, if a system is ISS, at each time t the norm of the system state can be upper bounded in terms of the $\mathcal{L}_\infty$ norm of the input (plus some decaying term depending on the initial condition). Translated into our setting, this would correspond to an *upper* bound on the radius of the smallest ball which contains the reachable set, i.e., there exist a function $\alpha \in \mathcal{K}_\infty$ and a function $\beta \in \mathcal{KL}$ such that for all $a, b > 0$,

$$R^a_h(x_0, \mathcal{U}_{a,b}) \leq \beta(|x_0|, a) + \alpha(b), \quad (4)$$

with $h(x) = x$. On the other hand, norm-controllability gives a *lower* bound on $R^a_h(x_0, \mathcal{U}_{a,b})$ in terms of the gain function γ in (3). If a system is both norm-controllable (with $h(x) = x$) and ISS, then it follows from (4) that $\gamma(\cdot, b)$ is bounded for each $b > 0$. In this case, $\gamma(\infty, \cdot)$ gives a lower bound for the smallest possible ISS-gain function α of the system. Similar considerations apply to related notions involving general outputs (different than the special choice $h(x) = x$) such as IOS [Sontag and Wang, 1999] or $\mathcal{L}_\infty$ stability (see, e.g., Khalil [2002]).

4. SUFFICIENT CONDITIONS FOR NORM-CONTROLLABILITY

In this section, we formulate several Lyapunov-like sufficient conditions for a system to be norm-controllable. The following Theorems originally appeared in Müller et al. [2011, 2012], and we restate them here in a unified and slightly simplified way. The proofs are omitted in this paper due to space restrictions and they are, modulo some small modifications, identical to those of the respective Theorems in Müller et al. [2011, 2012].

4.1 Sufficient condition based on first-order directional derivatives

The first sufficient condition for norm-controllability uses the notion of lower directional derivatives, which we recall from Studniarski [1991], Clarke et al. [1997]. Namely, for a function $V : \mathbb{R}^n \to \mathbb{R}$, the lower directional derivative of V at a point $x \in \mathbb{R}^n$ in the direction of a vector $h_1 \in \mathbb{R}^n$ is defined as

$$V^{(1)}(x; h_1) := \liminf_{t \searrow 0, \bar{h}_1 \to h_1} (1/t)\big(V(x + t\bar{h}_1) - V(x)\big).$$

Note that at each point $x \in \mathbb{R}^n$ where V is continuously differentiable, it holds that $V^{(1)}(x; h_1) = L_{h_1}V = (\partial V/\partial x)h_1$.

Theorem 1. Suppose there exist a set $\bar{U} \subseteq U$ and a closed set $\mathcal{B} \subseteq \mathbb{R}^n$ which is rendered control-invariant by $\bar{U}$ for system (1). Furthermore, suppose there exist a

[1] We note that while Definition 1 is the same as the originally proposed definition of norm-controllability in Müller et al. [2011, 2012], the set U_b appearing in (2) is defined in a slightly different way. This allows us to also restate the sufficient conditions later on in a slightly simplified way.

continuous function $\omega : \mathbb{R}^n \to \mathbb{R}^q$, $1 \le q \le n$, a function $V : \mathbb{R}^n \to \mathbb{R}$, which is continuously differentiable on $\mathbb{R}^n \setminus W$ with $W := \{x \in \mathbb{R}^n : \omega(x) = 0\}$ and $\partial V/\partial x$ is locally Lipschitz on $\mathbb{R}^n \setminus W$, and functions $\alpha_1, \alpha_2, \chi, \rho, \nu \in \mathcal{K}_\infty$ such that the following holds:

- For all $x \in \mathcal{B}$,
$$\nu(|\omega(x)|) \le |h(x)|, \tag{5}$$
$$\alpha_1(|\omega(x)|) \le V(x) \le \alpha_2(|\omega(x)|). \tag{6}$$
- For each $b > 0$ and each $x \in \mathcal{B}$ such that $|\omega(x)| \le \rho(b)$, there exists some $u \in \bar{U} \cap U_b$ such that
$$V^{(1)}(x; f(x,u)) \ge \chi(b). \tag{7}$$

Then the system (1) is norm-controllable from all $x_0 \in \mathcal{B}$ with gain function
$$\gamma(a,b) = \nu\Big(\alpha_2^{-1}\Big(\min\Big\{a\chi(b) + V(x_0), \alpha_1(\rho(b))\Big\}\Big)\Big). \tag{8}$$

Remark 1. If (7) does not only hold if $|\omega(x)| \le \rho(b)$, but rather for all $x \in \mathcal{B}$, then we can let $\rho \to \infty$ and γ in (8) simplifies to $\gamma(a,b) = \nu\big(\alpha_2^{-1}(a\chi(b) + V(x_0))\big)$. Note that in this case, $\gamma(a,\cdot)$ might not be of class $\mathcal{K}_\infty$, as $\gamma(a,0) \ne 0$ if $V(x_0) \ne 0$. Nevertheless, $\gamma(a,\cdot)$ still satisfies all other properties of a class $\mathcal{K}_\infty$ function, i.e., is continuous, strictly increasing and unbounded. □

In the following, we discuss various aspects of the sufficient condition for norm-controllability presented in Theorem 1. First, the condition expressed by (5)–(6), means that V can be lower and upper bounded in terms of the norm of a function ω which has to be "aligned" with the output map h in the sense given by (5). In the special case of $h(x) = \omega(x) = x$, this reduces to the usual condition that V is positive definite and decrescent.

Second, we need to allow V to be not continuously differentiable for all x where $\omega(x) = 0$ because $V \in C^1$ together with (6) would imply that the gradient of V vanishes for all x where $\omega(x) = 0$, and thus it would be impossible to satisfy (7) there. In the examples given in Section 5, a typical choice will be $V(x) = |\omega(x)|$.

When the output map h is such that $h(x_0) = 0$, norm-controllability captures the system's ability to "move away" from the initial state x_0. This could e.g. be of interest if one wants to know how far one can move away from an initial equilibrium state (x_0, u_0). In other settings, it makes sense to consider $h(x_0) \ne 0$, e.g. in a chemical process where initially already some product is available. This allows us, for fixed h, ω, and V, to vary the initial condition x_0, and the effect of this is given by the term $V(x_0)$ in (8). Also, there might be several possible choices for the functions ω and V satisfying the conditions of Theorem 1. In this case the degrees of freedom in the choice of ω and V can be used to maximize the gain γ in (8). Example 3 will illustrate this in more detail. Furthermore, if systems without outputs are considered, i.e., an output map h is not given a priori, we might first search for functions ω and V satisfying the relevant conditions of Theorem 1. Then, we can quantify the responsiveness of the system with respect to every a posteriori defined output map h satisfying (5). It is also useful to note that increasing the output dimension by appending extra variables to the output cannot destroy norm-controllability (it can only help attain it).

Besides the conceptual complementarity between norm-controllability and ISS as discussed in Section 3, we emphasize that also the sufficient conditions of Theorem 1 are in some sense "dual" to the conditions in typical ISS results [2] [Sontag and Wang, 1995]. Namely, a system is ISS if and only if there exist a continuously differentiable function V and $\mathcal{K}_\infty$-functions $\alpha_1, \alpha_2, \chi, \rho$ such that $\alpha_1(|x|) \le V(x) \le \alpha_2(|x|)$, and $\dot{V} \le -\chi(|x|)$ for all u and all x satisfying $|x| \ge \rho(|u|)$. This means that the decay rate of V can be upper-bounded in terms of $|x|$ if $|x|$ is large enough in comparison to $|u|$ (and this has to hold *for all* u). In contrast to this, for norm-controllability we require in (7) that *there exists* a u such that the growth rate of V can be lower-bounded in terms of b (the upper bound for the input norm) if $|x|$ is small compared to b. However, we note that while it is instructive to highlight the connections and similarities between the sufficient condition for norm-controllability in Theorem 1 and typical Lyapunov-like conditions for ISS, most of the technical ideas used to prove the respective results are very different.

Theorem 1 is stated such that norm-controllability is established on some control-invariant set $\mathcal{B}$. In particular, this includes as a special case $\mathcal{B} = \mathbb{R}^n$, in which case we can take $\bar{U} = U$. In the following, we show that in Theorem 1, the assumption that the set $\mathcal{B}$ is control-invariant under $\bar{U}$ can be relaxed. To this end, define by Λ_b the sublevel set
$$\Lambda_b := \big\{x \in \mathbb{R}^n : V(x) \le \alpha_1(\rho(b))\big\}. \tag{9}$$
Then, for each $b > 0$ and each $x \in \mathcal{B}$ such that $|\omega(x)| \le \rho(b)$, denote by $\hat{U}^b(x) \subseteq U_b$ a set which contains *some* inputs u (at least one) which satisfy (7). Then, let $\widetilde{U}^b := \cup_{x \in \mathcal{B}, |\omega(x)| \le \rho(b)} \hat{U}^b(x)$. If $\mathcal{B}$ is control-invariant under $\widetilde{U} := \cup_{b>0} \widetilde{U}^b$, we have the situation of Theorem 1 (with $\bar{U} = \widetilde{U}$); if not, consider the following.

Proposition 1. In Theorem 1, the assumption that a set $\bar{U}$ exists such that the set $\mathcal{B}$ is control-invariant under $\bar{U}$ can be replaced by the following. For each $b > 0$, there exists a set $H_b \subseteq \mathbb{R}^n$ with $H_b \cap \Lambda_b = \emptyset$ and Λ_b defined by (9) such that if $x_0 \in \mathcal{B}$ and $u(t) \in \widetilde{U}^b$ for all $t \ge 0$, then $x(t) \in \mathcal{B} \cup H_b$ for all $t \ge 0$. Furthermore, instead of only holding for all $x \in \mathcal{B}$, equations (5)–(6) hold for all $x \in \cup_{b>0} H_b \cup \mathcal{B}$. □

The condition in Proposition 1 means that each trajectory $x(\cdot)$ cannot exit $\mathcal{B}$ before exiting Λ_b. In other words, when at some time instant t we have $x(t) \in \Lambda_b$, then also $x(t) \in \mathcal{B}$. This means that at each time t, either $x(t) \in \mathcal{B}$ (which allows us to apply (7)), or $V(x(t)) \ge \alpha_1(\rho(b))$, which allows us to derive the same gain function γ as in (8).

4.2 Sufficient condition based on higher-order directional derivatives

The sufficient condition for norm-controllability presented in Theorem 1 is appealing due to its rather simple structure and its similarity with other Lyapunov-like results such as for ISS. However, this condition can be rather restrictive and is in general not satisfied for systems whose output y has a relative degree greater than one. The sufficient conditions presented in the following sections

[2] Again, similar considerations apply to related notions involving outputs such as IOS.

resolve this issue by relaxing the conditions of Theorem 1, and can be used for systems with arbitrary relative degree. These relaxed conditions are based on higher-order lower directional derivatives, which we define in accordance with [Studniarski, 1991, Equation (3.4a)] (compare also the earlier work [Ben-Tal and Zowe, 1982]). Namely, if $V^{(1)}(x;h_1)$ exists, the second-order lower directional derivative of V at a point x in the directions h_1 and h_2 is defined as [3]

$$V^{(2)}(x;h_1,h_2) := \liminf_{t\searrow 0,\bar{h}_2\to h_2} (2/t^2)\Big(V(x+th_1+t^2\bar{h}_2) - V(x) - tV^{(1)}(x;h_1)\Big).$$

In general, if the corresponding lower-order lower directional derivatives exist, for $k \geq 1$ define the kth-order lower directional derivative as

$$V^{(k)}(x;h_1,\dots,h_k) := \liminf_{t\searrow 0,\bar{h}_k\to h_k} (k!/t^k) \times \Big(V(x+th_1+\cdots+t^k\bar{h}_k) - V(x) - tV^{(1)}(x;h_1) - \cdots - (1/(k-1)!)t^{k-1}V^{(k-1)}(x;h_1,\dots,h_{k-1})\Big). \quad (10)$$

Similar to Section 4.1, we will later consider lower directional derivatives of a function V along the solution $x(\cdot)$ of system (1). For [4] $k \leq \bar{k}$, we obtain the following expansion for the solution $x(\cdot)$ of system (1), starting at time t' at the point $x := x(t')$ and applying some *constant* input u:

$$x(t) = x + (\Delta t)h_1 + \cdots + (\Delta t)^k h_k + o((\Delta t)^{k+1}) \quad (11)$$

with $\Delta t := t - t'$ and

$$\begin{aligned} h_1 &:= \dot{x}(t') = f(x,u),\\ h_2 &:= (1/2)\ddot{x}(t') = (1/2)\partial f/\partial x|_{(x,u)} f(x,u),\\ &\cdots\\ h_k &:= (1/k!)x^{(k)}(t'). \end{aligned} \quad (12)$$

In order to facilitate notation, in the following we write

$$V^{(k)}(x;f(x,u)) := V^{(k)}(x;h_1,\dots,h_k) \quad (13)$$

for the kth-order lower directional derivative of V at the point x along the solution of (1) when a *constant* input u is applied, i.e., with $h_1,\dots,h_k$ given in (12). It is straightforward to verify that at every point where V is sufficiently smooth, $V^{(k)}(x;f(x,u))$ reduces to $L_f^k V|_{(x,u)}$ (compare [Studniarski, 1991, Section 3] and [Ben-Tal and Zowe, 1982, p.73]), where $L_f^k V|_{(x,u)}$ is the kth-order Lie derivative of V along the vector field f. Namely, if V is locally Lipschitz, the kth-order lower directional derivative reduces to the kth-order Dini derivative (i.e., in (10) the direction h_k is fixed (see [Studniarski, 1991, Proposition 3.4(a)])); furthermore, if $V \in C^k$, then the limit in (10) exists (compare [Studniarski, 1991, Proposition 3.4(b)]) which equals $L_f^k V$ [Ben-Tal and Zowe, 1982, p.73].

Theorem 2. Suppose there exist a set $\bar{U} \subseteq U$ and a closed set $\mathcal{B} \subseteq \mathbb{R}^n$ which is rendered control-invariant by $\bar{U}$ for system (1). Furthermore, suppose there exist a continuous function $\omega : \mathbb{R}^n \to \mathbb{R}^q$, $1 \leq q \leq n$, a function $V : \mathbb{R}^n \to \mathbb{R}$, which for some $1 \leq k \leq \bar{k}$ is k times continuously differentiable on $\mathbb{R}^n \setminus W$ with $W := \{x \in \mathbb{R}^n : \omega(x) = 0\}$ and $\partial^k V/\partial x^k$ is locally Lipschitz on $\mathbb{R}^n \setminus W$, and functions $\alpha_1, \alpha_2, \chi, \rho, \nu \in \mathcal{K}_\infty$ such that (5)–(6) is satisfied and the following holds:

For each $b > 0$ and each $x \in \mathcal{B}$ such that $|\omega(x)| \leq \rho(b)$, there exists some $u \in \bar{U} \cap U_b$ such that

$$V^{(j)}(x;f(x,u)) \geq 0 \qquad j = 1,\dots,k-1, \quad (14)$$

$$V^{(k)}(x;f(x,u)) \geq \chi(b). \quad (15)$$

Then the system (1) is norm-controllable from all $x_0 \in \mathcal{B}$ with gain function

$$\gamma(a,b) = \nu\Big(\alpha_2^{-1}\Big(\min\Big\{\frac{1}{k!}a^k\chi(b) + V(x_0), \alpha_1(\rho(b))\Big\}\Big)\Big).$$

Remark 2. For the special case of $k = 1$, Theorem 1 is recovered. Furthermore, Proposition 1 also applies to Theorem 2, i.e., the assumption of control-invariance of the set $\mathcal{B}$ can be relaxed. □

4.3 Sufficient condition based on directional derivatives of different order

In this Section, we generalize the previous results to the case where the control-invariant set $\mathcal{B}$ can be partitioned into several regions where (14)–(15) holds for different k. To this end, for a closed set $\mathbb{X} \subseteq \mathbb{R}^n$ and $\ell \geq 1$, denote by $\mathcal{R}^\ell(\mathbb{X})$ a partition of $\mathbb{X}$ such that $\mathbb{X} = \cup_{i=1}^{\ell}\mathcal{R}_i$ and $\mathcal{R}_i$ is closed for all $1 \leq i \leq \ell$. Note that $\mathcal{R}^1(\mathbb{X}) = \mathbb{X}$.

Theorem 3. Suppose there exist a set $\bar{U} \subseteq U$ and a closed set $\mathcal{B} \subseteq \mathbb{R}^n$ which is rendered control-invariant by $\bar{U}$ for system (1). Furthermore, suppose there exist a partition $\mathcal{R}^\ell(\mathcal{B})$ for some $\ell \geq 1$ with corresponding integer constants $1 \leq k_1 < k_2 < \dots < k_\ell \leq \bar{k}$, a continuous function $\omega : \mathbb{R}^n \to \mathbb{R}^q$, $1 \leq q \leq n$, a function $V : \mathbb{R}^n \to \mathbb{R}$, which is k_ℓ times continuously differentiable on $\mathbb{R}^n \setminus W$ with $W := \{x \in \mathbb{R}^n : \omega(x) = 0\}$ and $\partial^{k_\ell} V/\partial x^{k_\ell}$ is locally Lipschitz on $\mathbb{R}^n \setminus W$, and functions $\alpha_1, \alpha_2, \chi, \rho, \nu \in \mathcal{K}_\infty$, such that (5)–(6) is satisfied and the following holds:

i) For each $b > 0$ and each $x \in \mathcal{R}_i$ for some $1 \leq i \leq \ell$ satisfying $|\omega(x)| \leq \rho(b)$, there exists some $u \in \bar{U} \cap U_b$ such that

$$V^{(j)}(x;f(x,u)) \geq 0 \qquad j = 1,\dots,k_i-1, \quad (16)$$

$$V^{(k_i)}(x;f(x,u)) \geq \chi_i(b). \quad (17)$$

ii) For all $1 \leq i_1, i_2 \leq \ell$ and all $x \in \mathcal{R}_{i_1} \cap \mathcal{R}_{i_2}$, it holds that $U_{i_1}^b(x) \cap U_{i_2}^b(x) \cap \bar{U} \neq \emptyset$, where

$$U_i^b(x) := \{u \in U_b : (16) - (17) \text{ hold}\}$$

Then the system (1) is norm-controllable from all $x_0 \in \mathcal{B}$ with gain function

$$\gamma(a,b) = \nu\Big(\alpha_2^{-1}\Big(\min\Big\{\Psi(a,b) + V(x_0), \alpha_1(\rho(b))\Big\}\Big)\Big),$$

where

$$\Psi(a,b) = \min_{i\in\{1,\dots,\ell\}} \frac{(k_\ell - k_i)!}{k_\ell!}a^{k_i}\chi_i(b). \quad (18)$$

Remark 3. In the special case of $\ell = 1$, Theorem 2 is recovered. Furthermore, condition *ii)* means that at those points where two regions $\mathcal{R}_{i_1}$ and $\mathcal{R}_{i_2}$ overlap, there exists (at least one) u such that (16)-(17) are satisfied with both i_1 and i_2 for the *same* u. □

[3] In contrast to Studniarski [1991], Ben-Tal and Zowe [1982], here we include the factor 2 (and later, in (10), the factor $k!$) into the definition of higher-order lower directional derivatives. We take this slightly different approach such that later, at each point where V is sufficiently smooth, these lower directional derivatives reduce to classical directional derivatives (without any extra factors as in Studniarski [1991], Ben-Tal and Zowe [1982]).

[4] Recall that $f \in C^{\bar{k}-1}$ for some $\bar{k} \geq 1$.

Remark 4. Again, similar to Proposition 1, the assumption of control-invariance of the set $\mathcal{B}$ can be relaxed. Furthermore, we note that the sets $\mathcal{R}_i$ of the partition $\mathcal{R}^\ell(\mathcal{B})$ can also depend on the upper bound b of the applied inputs, i.e., $\mathcal{R}_i = \mathcal{R}_i(b)$. □

5. EXAMPLES

In this Section, we illustrate the concept of norm-controllability as well as the presented sufficient conditions with several simple examples.

Example 1: Consider the system $\dot{x} = -x^3 + u$ with output $h(x) = x$, and take $\omega(x) = x$ and $V(x) = |x|$. For each $x \neq 0$ and each $b > 0$, by choosing u such that $|u| = b$ and $xu \geq 0$, we obtain

$$\begin{aligned}\dot{V} &= -|x|^3 + \mathrm{sign}(x)u \\ &= -|x|^3 + \theta \mathrm{sign}(x)u + (1-\theta)\mathrm{sign}(x)u \\ &\geq (1-\theta)b =: \chi(b), \qquad 0 < \theta < 1\end{aligned}$$

for all $|x| \leq \sqrt[3]{\theta b} =: \rho(b)$. For $x = 0$, choosing u such that $|u| = b$ results in $V'(0, f(0,u)) = b \geq \chi(b)$. Hence we can apply Theorem 1 with $\mathcal{B} = \mathbb{R}^n$ and[5] $\nu = \alpha_1 = \alpha_2 = \mathrm{id}$ to conclude that the considered system is norm-controllable from all $x_0 \in \mathbb{R}$ with gain function $\gamma(r,s) = \min\{(1-\theta)rs + |x_0|, \sqrt[3]{\theta s}\}$.

Example 2: Consider the system $\dot{x} = \frac{u}{1+|u|}$ with $h(x) = x$. This system is not norm-controllable. Namely, it is easy to see that $|\dot{x}| \leq 1$ for all x and u. But this means that we cannot find a function $\gamma(\cdot,\cdot)$ which is a $\mathcal{K}_\infty$ function in the second argument such that (3) holds, as for a given time horizon a, the norm of the output cannot go to infinity as $b \to \infty$. Hence this system lacks the "short-term" responsiveness captured by the norm-controllability property. Nevertheless, one can see that for $a \to \infty$, also $|h(x)| \to \infty$ for every constant control $u > 0$.

Example 3: Consider the system

$$\dot{x} = f(x,u) = \begin{bmatrix} -x_1^3 + x_2 + u \\ -x_2 + x_1 + u \end{bmatrix}, \qquad h(x) = x. \qquad (19)$$

As pointed out in Section 3, with this example we illustrate how different functions ω and V can be used to establish norm-controllability for system (19). To this end, consider the two functions $\omega_1(x) = x_1$ and $\omega_2(x) = x_2$, as well as $V_1(x) = |x_1|$ and $V_2(x) = |x_2|$. It holds that $|h(x)| \geq |\omega_i(x)|$ for $i = 1, 2$; thus in both cases we can choose $\nu_i = \alpha_{1,i} = \alpha_{2,i} = \mathrm{id}$ in (5)–(6). Furthermore, the positive orthant $\mathbb{R}^2_{\geq 0} := \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ is rendered control-invariant by $\bar{U} := \mathbb{R}_{\geq 0}$, which can be easily seen by noting that the vector field f points inside the positive orthant for all x on its boundary and all $u \geq 0$. Considering ω_1 and V_1, by similar calculations as in Example 1 one can show via Theorem 1 that the system (19) is norm-controllable from all $x_0 = [x_{10}\ x_{20}]^T \in \mathbb{R}^2_{\geq 0}$ with gain $\gamma_1(r,s) = \min\{(1-\theta)rs + |x_{10}|, \sqrt[3]{\theta s}\}$. Similar calculations using ω_2 and V_2 yield that the system (19) is norm-controllable from all $x_0 \in \mathbb{R}^2_{\geq 0}$ with gain $\gamma_2(r,s) = \min\{(1-\theta)rs + |x_{20}|, \theta s\}$. Hence we can conclude that the system (19) is norm-controllable from all $x_0 \in \mathbb{R}^2 \geq 0$ with gain $\gamma = \max\{\gamma_1, \gamma_2\}$, which shows how the possible degrees of freedom in the choice of the functions ω and V can be used to maximize the gain γ. Furthermore, by the choice of the functions ω_1, ω_2 and V_1, V_2, we also have proven norm-controllability of the system (19) for the a posteriori defined output maps $h_1(x) = x_1$ and $h_2(x) = x_2$.

[5] $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ denotes the identity function, i.e., $\mathrm{id}(s) = s$ for all $s \in \mathbb{R}^n$.

Example 4: Consider the double integrator system $\dot{x}_1 = x_2$, $\dot{x}_2 = u$ with output $h(x) = x_1$. The relative degree of this output is $r = 2$. Consider again the positive orthant $\mathcal{B}_1 := \mathbb{R}^2 \geq 0$ which is rendered control-invariant by $\bar{U} := \mathbb{R}_{\geq 0}$. Let $\omega(x) := x_1$ and $V(x) := |x_1|$. For all $x \in \mathcal{B}_1$ and $u \in \bar{U}$, we obtain $V^{(1)}(x; f(x,u)) = x_2$ and $V^{(2)}(x; f(x,u)) = u$. Hence for each $b > 0$, by choosing $u = b$ we obtain that $V^{(2)}(x; f(x,u)) = b =: \chi(b)$. We can now apply Theorem 2 with $\mathcal{B}_1 := \mathbb{R}^2 \geq 0$, $\bar{U} := \mathbb{R}_{\geq 0}$, $k = 2$ and $\chi = id$ together with Remark 1 to conclude that the system is norm-controllable from all $x_0 \in \mathcal{B}_1$ with gain function $\gamma(a,b) = (1/2)a^2 b$. Note that we could not apply Theorem 1 (i.e., we need $k = 2 > 1$) as $V^{(1)}(x; f(x,u)) = x_2$ cannot be lower bounded in terms of u. Similar considerations also apply to the negative orthant $\mathcal{B}_2 := \mathbb{R}^2 \leq 0$ which is rendered control-invariant by $\bar{U} := \mathbb{R}_{\leq 0}$.

Example 5: Consider an isothermal continuous stirred tank reactor (CSTR) in which an irreversible, second-order reaction from reagent A to product B takes place [Ogunnaike and Ray, 1994]:

$$\begin{aligned} dC_A/dt &= (q/V)(C_{A_i} - C_A) - kC_A^2 \\ dC_B/dt &= -(q/V)C_B + kC_A^2, \end{aligned} \qquad (20)$$

where C_A and C_B denote the concentrations of species A and B (in $[mol/m^3]$), respectively, V is the volume of the reactor (in $[m^3]$), q is the flow rate of the inlet and outlet stream (in $[m^3/s]$), k is the reaction rate (in $[1/s]$), and C_{A_i} is the concentration of A in the inlet stream, which can be interpreted as the input. Using $x_1 := C_A$, $x_2 := C_B$, $c := q/V$ and $u := C_{A_i}$, one obtains the system

$$\begin{aligned} \dot{x}_1 &= -cx_1 - kx_1^2 + cu =: f_1(x,u) \\ \dot{x}_2 &= kx_1^2 - cx_2 =: f_2(x). \end{aligned} \qquad (21)$$

The physically meaningful states and inputs are $x \in \mathbb{R}^2_{\geq 0}, u \in \mathbb{R}_{\geq 0}$, i.e., nonnegative concentrations of the two species. We are interested in the amount of product B per time unit, i.e. in the output $y = h(x) = qx_2$. Taking $\omega(x) = x_2$ and $V(x) = |\omega(x)|$, one obtains that for all $x \in \mathbb{R}^2_{\geq 0}$ and $u \in \mathbb{R}_{\geq 0}$

$$V^{(1)}(x; f(x,u)) = kx_1^2 - cx_2, \qquad (22)$$

$$V^{(2)}(x; f(x,u)) = -kx_1^2(3c + 2kx_1) + c^2 x_2 + 2kcx_1 u, \qquad (23)$$

$$V^{(3)}(x; f(x,u)) = (-6kcx_1 - 6k^2x_1^2 + 2kcu)f_1(x,u) + c^2 f_2(x). \qquad (24)$$

Now consider the region $\mathcal{B} := \{x : 0 \leq x_2 \leq (k/c)x_1^2\} \subset \mathbb{R}^2_{\geq 0}$. Note that $V^{(1)}(x; f(x,u)) \geq 0$ for all $x \in \mathcal{B}$. Let $0 < \varepsilon, \theta < 1$, and for $b \geq 0$ define $\varphi_1(b;\varepsilon) := (-(3-\varepsilon)c + \sqrt{(3-\varepsilon)^2c^2 + 16ck\theta b})/(4k)$, $\varphi_2(b) := \min\{cb/(8(c+k)), \sqrt{cb/(8(c+k))}\}$ and $\Phi(b) := \min\{\varphi_1(b;0), \varphi_2(b)\}$. Now consider the following partition of $\mathcal{B}$, which is also exemplarily depicted in Figure 1:

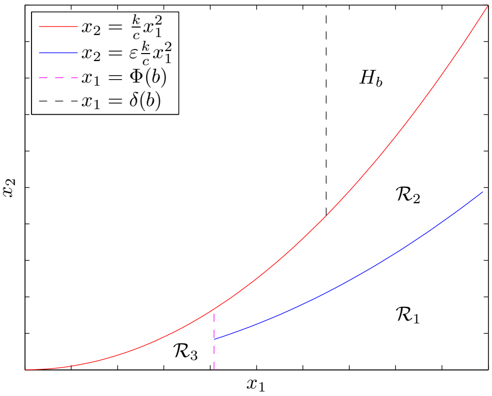

Fig. 1. Partition of the control-invariant region $\mathcal{B}$ and the set H_b in Example 9.

$$\mathcal{R}_1(b) := \{x \in \mathcal{B} : 0 \leq x_2 \leq \varepsilon(k/c)x_1^2, x_1 \geq \Phi(b)\},$$
$$\mathcal{R}_2(b) := \{x \in \mathcal{B} : (\varepsilon k/c)x_1^2 \leq x_2 \leq (k/c)x_1^2, x_1 \geq \Phi(b)\},$$
$$\mathcal{R}_3(b) := \{x \in \mathcal{B} : 0 \leq x_2 \leq (k/c)x_1^2, x_1 \leq \Phi(b)\}.$$

For all $b > 0$ and all $x \in \mathcal{R}_1(b)$, we obtain from (22) that $V^{(1)}(x; f(x,u)) \geq (1-\varepsilon)kx_1^2 \geq (1-\varepsilon)k\Phi^2(b) =: \chi_1(b)$. For all $b > 0$ and all $x \in \mathcal{R}_2(b)$, by choosing $u = b$ we obtain from (23) that $V^{(2)}(x; f(x,u)) \geq 2(1-\theta)kcb\Phi(b) =: \chi_2(b)$, for all $x_1 \leq \varphi_1(b;\varepsilon)$, which holds if $x_2 \leq (\varepsilon k/c)\varphi_1^2(b;\varepsilon) =: \rho(b)$. Finally, for all $b > 0$ and all $x \in \mathcal{R}_3(b)$, by choosing again $u = b$ we obtain from (23) that $V^{(2)}(x; f(x,u)) \geq 0$ and from (24) that $V^{(3)}(x; f(x,u)) \geq kc^2b^2 =: \chi_3(b)$.

In order to be able to apply Theorem 3, it remains to show that the set $\mathcal{B}$ can be made control-invariant. We will use the relaxed form given by Proposition 1. Namely, given the above, for each $x \in \mathcal{B}$ and each $b > 0$ such that $\omega(x) \leq \rho(b)$, we let $\hat{U}^b(x) := \{b\}$ and hence also $\widetilde{U}^b = \{b\} \subseteq \mathbb{R}_{\geq 0}$. Now note that for all x such that $x_2 = 0$ and $x_1 \geq 0$, $f(x,u)$ points inside $\mathcal{B}$ for all $u \in \mathbb{R}_{\geq 0}$, and hence no trajectory can leave the set $\mathcal{B}$ there. At the other boundary, i.e., for all x such that $x_2 = (k/c)x_1^2$, $f(x,u)$ points outside $\mathcal{B}$ only if $x_1 \geq (-c + \sqrt{c^2 + 4cku})/(2k) =: \delta(u)$. However, for each $b > 0$, if $x_1(\tau) \geq \delta(b)$ for some $\tau \geq 0$, then it follows from (21) that also $x_1(t) \geq \delta(b)$ for all $t \geq \tau$ in case that $u(t) \in \widetilde{U}^b$. Hence for each $b > 0$, we define the set $H_b := \{x : x_1 \geq \delta(b), x_2 \geq (k/c)x_1^2\}$ (see also Figure 1). Furthermore, it is straightforward to verify that for each $b > 0$, $H_b \cap \Lambda_b = \emptyset$, where $\Lambda_b = \{x : |x_2| \leq \rho(b)\}$ according to (9).

Summarizing the above, we can apply Theorem 3 with $\ell = 3$, $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, $\alpha_1 = \alpha_2 = \mathrm{id}$ and $\nu = q\mathrm{id}$ together with Proposition 1 to conclude that the system (21) is norm-controllable from all $x_0 \in \mathcal{B}$ with gain function $\gamma(r,s) = q\min\{\Psi(r,s) + V(x_0), \rho(s)\}$ with Ψ defined in (18). An interpretation of this fact is as follows. If $x_2 \leq (k/c)x_1^2$, then a sufficiently large amount of reagent A compared to the amount of product B is present in the reactor in order that the amount of product B can be increased. On the other hand, if $x_2 > (k/c)x_1^2$, then already too much product B is inside the reactor such that its amount will first decrease (due to the outlet stream), no matter how large the concentration of A in the inlet stream (i.e., the input u) is, and hence the conditions of Theorem 3 (in particular (16)) cannot be satisfied there.

6. CONCLUDING REMARKS AND FUTURE WORK

In this paper, we surveyed and discussed the concept of norm-controllability as well as several Lyapunov-like sufficient conditions for ensuring that a nonlinear system possesses this property. Besides its theoretical insights into the input/output behavior of nonlinear systems, we believe that norm-controllability can be an interesting concept in various application contexts such as economics or process engineering. On the theoretical side, future research directions can include investigating necessity of the (relaxed) Lyapunov-like sufficient conditions, as well as weaker variations of norm-controllability such as requiring the estimate (3) only to hold for large enough a and/or b (instead of all $a, b > 0$). On the application side, the next steps would be to advance from rather simple systems such as the presented CSTR example to more realistic case studies in order to further study the potentials and limitations of the norm-controllability concept.

REFERENCES

D. Angeli and E. D. Sontag. Forward completeness, unboundedness observability, and their Lyapunov characterizations. *System & Control Letters*, 38:209–217, 1999.

A. Ben-Tal and J. Zowe. Necessary and sufficient optimality conditions for a class of nonsmooth minimization problems. *Mathematical Programming*, 24:70–91, 1982.

F. H. Clarke, Y. S. Ledyaev, E. D. Sontag, and A. I. Subbotin. Asymptotic controllability implies feedback stabilization. *IEEE Transactions on Automatic Control*, 42(10):1394–1407, 1997.

J. P. Hespanha. *Linear Systems Theory*. Princeton University Press, 2009.

H. K. Khalil. *Nonlinear Systems*. Prentice-Hall, Upper Saddle River, NJ, 2002.

M. A. Müller, D. Liberzon, and F. Allgöwer. On norm-controllability of nonlinear systems. In *Proceedings of the 50th IEEE Conference on Decision and Control and the European Control Conference*, pages 1741–1746, 2011.

M. A. Müller, D. Liberzon, and F. Allgöwer. Relaxed conditions for norm-controllability of nonlinear systems. In *Proceedings of the 51st IEEE Conference on Decision and Control*, pages 314–319, 2012.

B. A. Ogunnaike and W. H. Ray. *Process Dynamics, Modeling, and Control*. Oxford University Press, New York, 1994.

E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, 34:435–443, 1989.

E. D. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Springer, New York, 2nd edition, 1998.

E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. *Systems & Control Letters*, 24:351–359, 1995.

E. D. Sontag and Y. Wang. Notions of input to output stability. *Systems & Control Letters*, 38:235–248, 1999.

M. Studniarski. Second-order necessary conditions for optimality in nonsmooth nonlinear programming. *Journal of Mathematical Analysis and Applications*, 154:303–317, 1991.