Nonlinear Feedback Types in Impulse and Fast Control [⋆]

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Abstract: It is well-known that a system with linear structure subjected to bounded control inputs for optimal closed-loop control yields nonlinear feedback of discontinuous bang-bang type. This paper investigates new types of nonlinear feedback in the case of optimal impulsive closed-loop control which may naturally generate discontinuous trajectories. The realization of such feedback under impulsive inputs that are allowed to use δ-functions with their higher derivatives requires physically realizable approximations. Described in this paper is a new class of realizable feedback inputs that also allows to produce smooth approximation of controls. Such approach also applies to problems in micro time scales that require so-called fast or ultra-fast controls.

Keywords: Impulse control, fast control, nonlinear feedback, hybrid systems.

1. INTRODUCTION

Present interest in impulse control theory, created earlier for problems of open-loop control (see Krasovski [1957], Neustadt [1964], Kurzhanski and Osipov [1969]), is now confined to closed-loop control solutions under various types of available feedback. This demand is driven by investigation of system models motivated by applied problems that range from economics to hybrid systems, biology and control in micro time (see Bensoussan and Lions [1982], Branicky et al. [1998], Kurzhanski and Varaiya [2009], Ganesan and Tarn [2010]). Detailed descriptions of impulse feedback control for multidimensional systems is given by Kurzhanski and Daryin [2008]. The emphasis of this paper is on describing possible dynamic programming schemes for such problems in the class of "ideal" inputs that are allowed to use δ-functions with their higher derivatives, but not only that. Indicated are problems of closed-loop control under double type of constraints: both soft, integral and hard, "geometric" bounds. The last case produces a physically realizable scheme of approximating ideal solutions by ordinary functions. The novelty of this paper consists firstly in introducing an alternative approximation scheme by substituting, from the beginning, the problem of impulse control with high derivatives of δ -functions by one with continuous trajectories. But now this is done through a generalization of the time-space transformation used earlier by Motta and Rampazzo [1995], Miller and Rubinovich [2003] for "ordinary" impulses. Such generalization (see Daryin and Minaeva [2011]) allows to approximate higher derivatives which is crucial for generating fast controls. Secondly, the suggested scheme allows to cope with bounded uncertain inputs, attenuating their effect.

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2. IMPULSE CONTROL SYSTEM

On a fixed finite time interval $t \in [t_0, t_1]$ consider a control system with two control inputs – the first $U(\cdot)$ is of impulse type and the second $v(t)$ is bounded:

$$dx(t) = A(t)x(t)dt + B(t)dU(t) + C(t)v(t), \qquad (1)$$

with initial condition $x(t_0) = x^0$.

The control $U(\cdot)$ and the trajectory $x(\cdot)$ are functions of bounded variation [1]. Known matrix functions $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, and $C(t) \in \mathbb{R}^{n\times k}$ are continuous.

The control $v(t) \in \mathbb{R}^k$ is a piecewise-continuous function taking values in a given non-empty convex compact set $\mathscr{Q}(t)$. The multivalued mapping $\mathscr{Q}(t)$ is upper semicontinuous with respect to inclusion.

The function $v(\cdot)$ may be interpreted as either an unknown disturbance or an external control independent of $U(\cdot)$, unknown in advance. In any case, the goal of the first control is to minimize the functional of Mayer–Bolza type

$$J(U(\cdot), v(\cdot)) = \mathrm{Var}_{[t_0,t_1]}\, U(\cdot) + \varphi(x(t_1 + 0))$$

in view of coping with any possible realization of $v(\cdot)$. Here Var is the total variation on the indicated interval, and $\varphi(\cdot)$ is a given proper closed convex terminal function. The time interval $[t_0, t_1]$ is fixed in advance.

The state of system (1) is the pair $(t, x) \in \mathbb{R} \times \mathbb{R}^n$.

Problem 1. Find a feedback control $\mathscr{U}$ minimizing the functional

[1] See Riesz and Szökefalvi-Nagy [1990]. The space of m-vector functions of bounded variations is denoted by $BV([t_0, t_1]; \mathbb{R}^m)$. Remind that although such functions need not be continuous, at each point they have finite left and right limits denoted by $x(t-0)$ and $x(t+0)$ respectively. Here we use the convention that such functions are left-continuous, i.e. $x(t) = x(t-0)$.

$$\mathscr{J}(\mathscr{U}) = \max_{v(\cdot)\in\mathscr{Q}(\cdot)} J(U(\cdot), v(\cdot)),$$

where maximum is taken over all admissible of $v(\cdot)$ and $U(\cdot)$ is the realized impulse control.

3. DYNAMIC PROGRAMMING

In order to find the optimal feedback control, we apply the generalization of Dynamic Programming scheme of Motta and Rampazzo [1996] as given by Daryin and Minaeva [2011].

3.1 Min-Max Value Function

The min-max value function is defined as

$$V^-(t_0, x_0) = \min_{U(\cdot)} \max_{v(\cdot)} [\operatorname{Var} U(\cdot) + \varphi(x(t_1+0)) \mid x(t_0) = x_0].$$

Here $x(t)$ is the trajectory of system (1) corresponding to a fixed control $U(\cdot)$ and disturbance $v(\cdot)$.

The function V^- may be calculated as follows. First, we take the maximum over $v(\cdot)$. Note that $\operatorname{Var} U(\cdot)$ does not depend on $v(\cdot)$, and the right end of the trajectory $x(t_1+0)$ may be expressed as

$$x(t_1+0) = \underbrace{X(t_1,t_0)x_0 + \int_{t_0}^{t_1+0} X(t_1,t)B(t)dU(t)}_{\hat{x}(t_1+0)} + \underbrace{\int_{t_0}^{t_1} X(t_1,t)C(t)v(t)dt}_{\mathbf{v}(t_1)} = \hat{x}(t_1+0) + \mathbf{v}(t_1).$$

Here $X(t,\tau)$ is the solution to the following linear matrix ODE: $\partial X(t,\tau)/\partial t = A(t)X(t,\tau)$, $X(\tau,\tau) = I$.

The vector $\hat{x}(t_1+0)$ is the right end of the trajectory with no disturbance. The vector $\mathbf{v}(t_1)$ belongs to the set

$$\mathbf{Q} = \int_{t_0}^{t_1} X(t_1,t)C(t)\mathscr{Q}(t)dt.$$

Now, employing convex analysis [2] (see Rockafellar [1972]) we first get

$$\max_{v(\cdot)\in\mathscr{Q}(\cdot)} \varphi(x(t_1+0)) = \max_{\mathbf{v}\in\mathbf{Q}} \varphi(\hat{x}(t_1+0)+\mathbf{v}) =$$
$$\max_{\mathbf{v}\in\mathbf{Q}} \max_{p\in\mathbb{R}^n} \{\langle \hat{x}(t_1+0)+\mathbf{v}, p\rangle - \varphi^*(p)\} =$$
$$\max_{p\in\mathbb{R}^n} \{\langle \hat{x}(t_1+0), p\rangle + \rho(p \mid \mathbf{Q}) - \varphi^*(p)\} = \psi(\hat{x}(t_1+0)),$$

where $\psi(\hat{x}(t_1+0))$ is a convex function whose conjugate is $\psi^*(p) = \operatorname{conv}\{\varphi^*(p) - \rho(p \mid \mathbf{Q})\}$.

Secondly, we calculate the minimum over $U(\cdot)$. Since now

$$V^-(t_0,x_0) = \min_{U(\cdot)} [\operatorname{Var} U(\cdot) + \psi(\hat{x}(t_1+0)) \mid x(t_0) = x_0],$$

Hence this is an impulse control problem without disturbance. The value function is

$$V(t_0,x_0) =$$
$$= \max_{p\in\mathbb{R}^n} \left\{ \left\langle X^T(t_1,t_0)p, x\right\rangle - \psi^*(p) - \mathscr{I}(p \mid \mathscr{B}_V[t_0,t_1]) \right\},$$

(see Kurzhanski and Daryin [2008]), where

$$\mathscr{B}_V[t_0,t_1] = \{p \mid \|p\|_V \le 1\},$$
$$\|p\|_V = \max\{\|B^T(\tau)X^T(t_1,\tau)p\| \mid \tau\in[t_0,t_1]\},$$

where $\|\ell\|$ is the Euclidean norm. $\mathscr{B}_V[t_0,t_1]$ is a unit ball in $\mathbb{R}^n$ whose defined for the the interval $[t_0,t_1]$.

3.2 Value Function with Corrections

For the min-max value function calculated in the previous section, we shall use an extended notation $V^-(t_0,x_0) = V^-(t_0,x_0;t_1,\varphi(\cdot))$.

Let $t_0 = \tau_N < \tau_{N-1} < \cdots < \tau_1 < \tau_0 = t_1$ be some partition of the interval $[t_0,t_1]$. It will be denoted by $\mathscr{T}$, and $\operatorname{diam}\mathscr{T}$ is $\max\{\tau_k - \tau_{k+1}\}$.

Define the value function with corrections $V^-_{\mathscr{T}}(t,x)$ by the following recurrent relations:

$$V^-_{\mathscr{T}}(\tau_0,x) = V^-(t_1,x;t_1,\varphi(\cdot));$$
$$V^-_{\mathscr{T}}(\tau_{k+1},x) = V^-(\tau_{k+1},x;\tau_k,V^-_{\mathscr{T}}(\tau_k,x)).$$

Function $V^-_{\mathscr{T}}(t,x)$ may be interpreted as the value function for the sequential min-max problem, when at instants τ_k the control obtains information on the current state $x(t)$.

Note that if $\mathscr{T}'$ is a subpartition of $\mathscr{T}$, then clearly $V^-_{\mathscr{T}'}(t,x) \le V^-_{\mathscr{T}}(t,x)$.

3.3 Closed-Loop Value Function

Denote

$$\mathscr{V}^-(t,x) = \inf_{\mathscr{T}} V^-_{\mathscr{T}}(t,x).$$

It may be proven (similar to Kurzhanski and Daryin [2008]) that the value function $\mathscr{V}^-(t,x)$ satisfies a Hamilton–Jacobi–Bellman–Isaacs type quasi-variational inequality:

$$\min\{\mathscr{H}_1, \mathscr{H}_2\} = 0, \qquad (2)$$
$$\mathscr{H}_1(t,x) = \mathscr{V}^-_t + \max_{v\in\mathscr{Q}} \left\langle \mathscr{V}^-_x, A(t)x + C(t)v\right\rangle,$$
$$\mathscr{H}_2(t,x) = \min_{\|h\|=1} \{\|h\| + \left\langle \mathscr{V}^-_x, B(t)h\right\rangle\},$$
$$\mathscr{V}^-(t_1,x) = V(t_1,x;t_1,\varphi(\cdot)).$$

Here the Hamiltonian $\mathscr{H}_1$ corresponds to the motion without control ($dU = 0$), and $\mathscr{H}_2$ corresponds to the jumps generated by control impulses.

4. NONLINEAR FEEDBACK

The HJBI variational inequality (2) may be interpreted as follows (see Kurzhanski and Daryin [2008]): if for $x(t)$ we have $\mathscr{H}_1 = 0$, then the control may be equal to zero, and if $\mathscr{H}_2 = 0$, then the control must have a jump to a state where $H_1 = 0$.

In this section we present three possible approaches how to formalize the described control law and the corresponding closed-loop control systems.

4.1 Formal Definition

Definition 2. Impulse feedback control for system (1) is a set-valued function $\mathscr{U}(t,x) : [t_0,t_1] \to \mathbb{R}^m$, upper semicontinuous in (t,x), taking non-empty convex compact values.

[2] We remind that *support function* of a set X is given by $\rho(p \mid X) = \sup_{x\in X}\langle p,x\rangle$, and that *Fenchel conjugate* to a function $f(x)$ is $f^*(p) = \sup_{x\in\mathbb{R}^n}\{\langle p,x\rangle - f(x)\}$.

Elements h of $\mathscr{U}(\tau, x)$ are interpreted as following: if $h \neq 0$, then the open-loop control $U(t)$ may have a term $h\chi(t-\tau)$. The latter is mathematically formulated by the next definition.

Definition 3. An open-loop control

$$U(t) = \sum_{j=1}^{K} h_j \chi(t - t_j)$$

conforms with the closed-loop strategy $\mathscr{U}(t, x)$ under disturbance $v(t)$ if

(1) for $t \neq t_j$ the set $\mathscr{U}(t, x(t))$ contains the origin;
(2) $h_j \in \mathscr{U}(t_j, x(t_j))$, $j = \overline{1, K}$.
(3) $\mathscr{U}(t_1, x(t_1 + 0)) = \{0\}$.

Here $x(t)$ is the trajectory of (1) generated by $U(t)$ and $v(t)$.

Definition 4. A state (t, x) is called *relaxed* if one of the following is true:

- either $t < t_1$ and $H_1 = 0$,
- or $t = t_1$ and $V(t, x) = \varphi(x)$.

The set of all relaxed states is denoted by $\mathscr{R}$.

From the HJBI variational inequality (2) it follows that

$$\mathscr{U}(t, x) = \{h \mid (t, x + Bh) \in \mathscr{R}, \\ \mathscr{V}^-(t, x + Bh) = \mathscr{V}^-(t, x) - \|h\|\}.$$

4.2 Time-Space Transformation

Another possible way to formalize the impulse feedback control lies in using the extended space-time system (see Motta and Rampazzo [1995], Miller and Rubinovich [2003]):

$$\begin{cases} dx/dt = (A(t(s))x(s) + C(t(s))v(s)) \cdot u^t(s) + \\ \qquad\qquad B(t(s))u^x(s), \\ dt/ds = u^t(s). \end{cases} \tag{3}$$

Here s is the parameterizing variable for trajectories of x and t, $s \in [0, S]$, and the right end S is not fixed. The extended control $u(s) = (u^x(s), u^t(s)) \in \mathbb{R}^m \times \mathbb{R}$ is restricted by hard bound $u(s) \in \mathscr{B}_1 \times [0, 1]$. The original impulse control problem 1 corresponds to the following problem for system (3):

$$\begin{cases} \mathscr{J}(u(\cdot)) = \max\limits_{v(\cdot)} \left\{ \int_0^S \|u^x(s)\| \, ds + \varphi(x(S)) \right\} \to \inf, \\ t(0) = t_0, \quad t(S) = t_1. \end{cases} \tag{4}$$

It is known (Motta and Rampazzo [1995]) that any impulse control and its corresponding state trajectory of the original system (1) may be presented as similar elements of the extended system (3), and that the set of trajectories of (1) is dense in the set of trajectories of (3).

The value function of the problem (4) is the solution to the the Hamilton–Jacobi–Bellman–Isaacs equation

$$\min_{\substack{u^t \in [0,1] \\ u^x \in \mathscr{B}_1}} \max_{v \in \mathscr{Q}(t(s))} H(t, x, V_t, V_x, u^t, u^x, v) = 0, \tag{5}$$

$$H(t, x, \tau, \xi, u^t, u^x, v) = \{[\tau + \langle \xi, A(t)x + C(t)v \rangle] \, u^t + \\ + [\langle \xi, B(t)u^x \rangle + \|u^x\|]\} = 0,$$

which is equivalent to the HJBI equation (2) for the impulse control problem.

Now using (5) it is possible to define control synthesis for (4) as the set of minimizing control vectors in (5):

$$\mathscr{U}^*(t, x) = \bigcup_{(\tau, \xi) \in \partial_C V} \{u \mid H(t, x, \tau, \xi, u^t, u^x) = 0\}, \tag{6}$$

Here $\partial_C V$ is the Clarke subdifferential (see Clarke [1975]) of the value function with respect to both variables (t, x).

Since (3) describes all the trajectories of (1), the control (6) may be regarded as a control synthesis for (1).

The closed-loop system under control (6) is a differential inclusion:

$$\frac{d}{ds}\begin{pmatrix} x \\ t \end{pmatrix} \in \left\{ \begin{pmatrix} (A(t)x + C(t)v)u^t + B(t)u^x \\ u^t \end{pmatrix} \,\middle|\, (u^x, u^t) \in \mathscr{U}^*(t, x), \ v \in \mathscr{Q}(t) \right\}. \tag{7}$$

Since $\mathscr{U}^*(t, x)$ is an upper semicontinuous set-valued function with non-empty compact convex values (this follows from the properties of ∂_C), the solutions to (7) exist and are extendable within the region $(t, x) \in [t_0, t_1] \times \mathscr{R}^n$ (see Filippov [1988]). Any optimal control and the corresponding state trajectory of (1) satisfies (7). In other words, (7) generates all possible optimal trajectories.

4.3 Hybrid System

System (1) under impulse feedback control may be interpreted as a *hybrid system.* In terms of Branicky et al. [1998] it is a "continuous- controlled autonomous-switching hybrid system".

In the region $\mathscr{M} = \{(t, x) \mid H_1 = 0\}$ system has the continuous dynamics

$$\dot{x}(t) = A(t)x(t) + C(t)v(t), \quad (t, x) in \mathscr{M}.$$

The complement to the set $\mathscr{M}$ is the autonomous switching set, and the autonomous transition map is

$$x^+(t) = x(t) + Bh.$$

Here vector h is such that

$$V(t, x(t) + B(t)h) = V(t, x(t)) + \|h\|$$

and $(t, x^+(t))$ is a relaxed state.

A discussion on modeling hybrid system controls using impulses is given by Kurzhanski and Tochilin [2009].

4.4 Example

For the one-dimensional case, the value function may be calculated explicitly.

Consider a 1D linear system

$$dx = (1 - t^2)dU + v(t)dt$$

with $[t_0, t_1] = [-1, 1]$, where the disturbance $v(t) \in [-1, 1]$. It has to be steered from its initial state $x(-1) = x$ by the control that delivers a minimum to functional

$$\mathrm{Var}_{[-1,1]} U(\cdot) + 2|x(t_1 + 0)| \to \inf. \tag{8}$$

Here the value function is $\mathscr{V}^-(t, x) = \alpha(t)|x|$, where

$$\alpha(t) = \min\left(2, \min_{\tau \in [t,1]} \frac{1}{1 - \tau^2}\right).$$

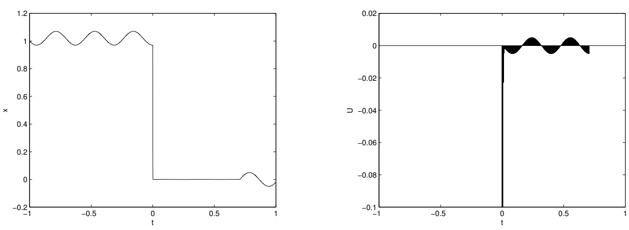

Fig. 1. Trajectory of the system, starting from $x(-1) = 1$, and corresponding control. Disturbance is $v(t) = \sin(20t)$.

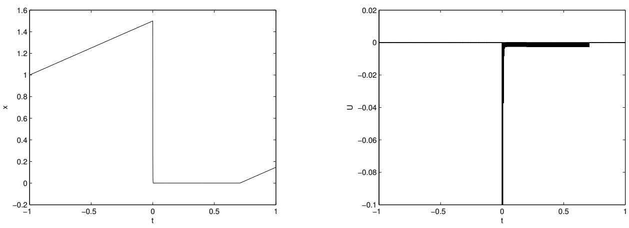

Fig. 2. Trajectory of the system, starting from $x(-1) = 1$, and corresponding control. Disturbance is constant $v(t) = 1$.

We calculate the Hamiltonian functions:

$$\mathscr{H}_1 = \begin{cases} \dfrac{tx}{1-t^2}, & \text{if } 0 \le t \le 1/\sqrt{2}, \\ 0, & \text{if } -1 \le t < 0, \text{ and } 1/\sqrt{2} < t \le 1. \end{cases}$$

$$\mathscr{H}_2 = \begin{cases} t^2, & \text{if } -1 \le t < 0, \\ 2t^2 - 1, & \text{if } 1/\sqrt{2} < t \le 1, \\ 0, & \text{if } 0 \le t \le 1/\sqrt{2}. \end{cases}$$

There are three cases:

(1) if $t < 0$ we have $\mathscr{H}_1 = 0$, $\mathscr{H}_2 \neq 0$, then we do not apply control;
(2) if $0 \le t \le 1/\sqrt{2}$, we have $\mathscr{H}_1 \neq 0$, $\mathscr{H}_2 = 0$ and we steer our system with an impulse control;
(3) if $1/\sqrt{2} < t \le 1$, we have $\mathscr{H}_1 = 0$, $\mathscr{H}_2 \neq 0$, then we do not apply control.

Figs. 1, 2 show trajectories $x(t)$ and control U for different disturbance $v(t)$. Note that we apply impulse control when $0 \le t^* \le \dfrac{1}{\sqrt{2}}$ and the trajectory reaches zero. After that we do not apply control, and the trajectory drifts away from zero, because of the disturbance. This is the trajectory and the control that deliver minimum to functional (8).

5. FAST CONTROLS

Impulse control is an "ideal" one. Bounded functions approximating impulse controls are known as *fast controls*, since they are physically realizable and may steer a system to a given state in arbitrary small time. Such controls may be found, for example, in the following form (see Kurzhanski and Daryin [2008]):

$$u_\Delta(t) = \sum\nolimits_{j=0}^{m} u_j \Delta_{h_j}^{(j)}(t-\tau), \tag{9}$$

where $\Delta_h^{(j)}(t)$ approximate the derivatives of delta-function:

$$\Delta_h^{(0)}(t) = h^{-1}\mathbf{1}_{[0,h]}(t),$$
$$\Delta_h^{(j)}(t) = h^{-1}\left(\Delta_h^{(j-1)}(t) - \Delta_h^{(j-1)}(t-h)\right).$$

The next problem is how to choose the parameters of control (9) — the coefficients h_j and vectors u_j. These should be chosen following physical requirements on the realizations of the control.

5.1 Discontinuous Approximations

We first consider fast controls with various restrictions:

(1) bounded time of control:
$$\max_j\{(j+1)h_j\} \le H;$$
(2) hard bounds on control:
$$\|u_\Delta(t)\| \le \mu;$$
(3) separate hard bounds on approximations of generalized functions of all orders included in the control:
$$\|u_{\Delta,j}(t)\| \le \mu_j,$$
$$u_{\Delta,j}(t) = u_j\Delta_{h_j}^{(j)}(t-\tau).$$

The indicated restrictions lead to moment problems of similar type.

$$\mu \to \inf,$$
$$\left|\Delta_h^{(n)}(t)\right| \le \mu, \quad t \in [-h, h]. \tag{10}$$

We impose extra restrictions to ensure that the approximations $\Delta_h^{(n)}(t)$ affect polynomials of degree n in the same way that $\delta^{(n)}(t)$.

$$\int_{-h}^{h} \Delta_h^{(n)}(t)t^k dt = 0, \quad k = 0, \ldots, n-1,$$
$$\int_{-h}^{h} \Delta_h^{(n)}(t)t^n dt = (-1)^n n! \tag{11}$$

The moment problem (10) with restrictions (11) has the following solution:

$$\Delta_h^{(n)}(t) = \tfrac{1}{4}(-1)^n n! \left(\tfrac{2}{h}\right)^{(n+1)} \operatorname{sign} U_n(ht), \tag{12}$$

where $U_n(t)$ is the Chebyshev polynomial of the second kind: $U_n(t) = \cos(n \arccos t)$.

Approximation (12) is piecewise constant (and hence discontinuous), equal to $\pm\frac{1}{4}n!\left(\frac{2}{h}\right)^{(n+1)}$ between Chebyshev points $t_k = h\cos\frac{\pi j}{n+1}$, $j = 0, \ldots, n+1$. See Fig. 3.

5.2 Smooth Approximations

Apart from discontinuous, we also consider continuous or smooth approximations. To do this, we impose bounds on the k-th derivatives of the approximation:

$$\Delta_{h,k}^{(n)}(t) = \int_{-h}^{t}\int_{-h}^{t_1}\ldots\int_{-h}^{t_{k-1}} g_k^n(t_k)dt_k dt_{k-1}\ldots dt_1,$$
$$|g_k^n(t)| \le \mu.$$

And we add similar restrictions on related polynomials of degree n, that were used for discontinuous approximations:

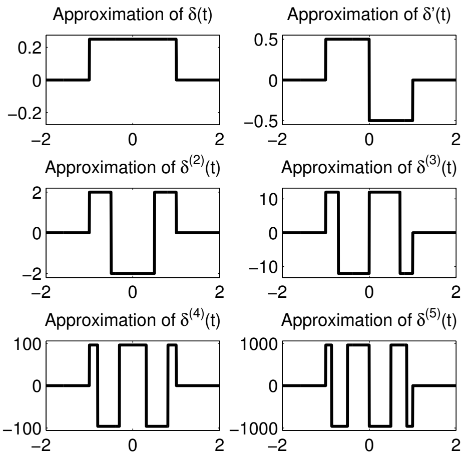

Fig. 3. Discontinuous approximations of $\delta(t)$, $\delta'(t), \ldots, \delta^{(5)}(t)$ with minimal modulus on fixed time interval.

$$\int_{-h}^{h} \Delta_{h,k}^{(n)}(t) t^j dt = 0, \quad j = 0, \ldots, n-1,$$
$$\int_{-h}^{h} \Delta_{h,k}^{(n)}(t) t^n dt = (-1)^n n!$$

This leads to moment problems for the k-th derivative $g_k^n(t)$ of approximation $\Delta_{h,k}^{(n)}(t)$:

$$\mu \to \inf,$$
$$|g_k^n(t)| \le \mu, \quad t \in [-h, h],$$

$$\int_{-h}^{h} g_k^n(t) t^j dt = 0, \quad j = 0, \ldots, n+k-1,$$
$$\int_{-h}^{h} g_k^n(t) t^{n+k} dt = (-1)^{n+k}(n+k)!$$

It turns out that a $(k-1)$-times smooth approximation of $\delta^{(n)}(t)$, $\Delta_{h,k}^{(n)}(t)$, is a normalized k-fold integral of $\Delta_h^{(n+k)}(t)$:

$$\Delta_{h,k}^{(n)}(t) = \frac{1}{(k-1)!} \int_{-h}^{t} g_k^n(\tau)(t-\tau)^{k-1} d\tau,$$

$$g_k^n(t) = \Delta_h^{(n+k)}(t) =$$
$$= \tfrac{1}{4}(-1)^{n+k} \left(\tfrac{2}{h}\right)^{n+k+1} (n+k)! \operatorname{sign} U_{n+k}(ht).$$

Here $k = -1$ corresponds to discontinuous approximations $\Delta_h^{(n)}(t)$, and $k = 0$ leads to continuous (but not smooth) approximations.

Approximations $\Delta_{h,k}^{(n)}(t)$ are piecewise polynomials of order k, with $k-1$ derivatives continuous at the junction points. The coefficients of these polynomials may be calculated recurrently through explicit formulae.

In Fig. 4 we show ontinuously differentiable approximations of $\delta(t)$ and its derivatives.

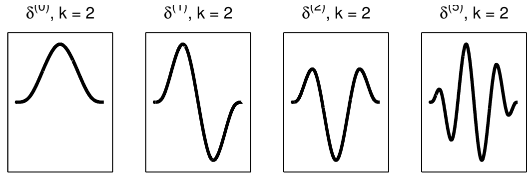

Fig. 4. Continuously differentiable approximations of $\delta(t)$ and its derivatives.

6. FAST FEEDBACK CONTROL

The problem of fast feedback control under uncertainty is to select the control input as a linear combination of functions $\Delta_{h,k}^{(s)}(t)$ given current state (t, x) of the system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t)v(t).$$

This problem may be reduced to the one of impulse feedback control studied earlier if we use the following transformation.

Introduce the matrix functions

$$B_s(t) = \int_{-h}^{h} X(t, t+\tau) B(t+\tau) \Delta_{h,k}^{(s)}(t+\tau)\, d\tau$$

and the block matrix made up of them

$$\mathscr{B}(t) = (B_0(t)\ B_1(t)\ \ldots\ B_S(t)).$$

Consider the corresponding impulse control system

$$dx(t) = A(t)x(t)dt + \mathscr{B}(t)dU(t) + C(t)v(t)dt. \qquad (13)$$

It is of type (1) and the above theory may be applied to it.

If the realized control for system (13) is

$$U(t) = \sum_{j=1}^{K} \sum_{s=0}^{S} h_{j,s} \chi(t - \tau_j),$$

then the corresponding fast control input for system (6) is

$$u(t) = \sum_{j=1}^{K} \sum_{s=0}^{S} h_{j,s} \Delta_{h,k}^{(s)}(t - \tau_j).$$

7. CONCLUSION

This paper gives new insights for the problem of designing realistic feedback control strategies in models based on using impulse control. The suggested solution schemes may be used for problems of control within a broad time scale up to micro levels. They also allow to cope with unknown but bounded disturbances ensuring guaranteed results.

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