$\mathscr{H}_\infty$ Filter Design for Nonlinear Quadratic Systems ⋆

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Abstract: This paper is concerned with the problem of $\mathscr{H}_\infty$ filtering for continuous-time nonlinear quadratic systems. The aim is to design a full order dynamic filter that can also contain quadratic terms. The strategy relies on the use of a quadratic Lyapunov function and an inequality condition that assures an $\mathscr{H}_\infty$ performance bound for the augmented quadratic system, composed by the original system and the filter to be designed, in a regional (local) context. Then, by using the Finsler's lemma, an enlarged parameter space is created, where the Lyapunov matrix appears separated from the system matrices. Imposing structural constraints to the decision variables, theoretical conditions, which can be treated as linear matrix inequality conditions by fixing a grid on a scalar parameter, can be derived for the filter design. As illustrated by numerical experiments, the proposed conditions can improve the $\mathscr{H}_\infty$ performance provided by linear filters by including the quadratic terms in the filter dynamics.

Keywords: Filtering; Quadratic Systems; H-infinity Norm; Linear Matrix Inequalities;

1. INTRODUCTION

The filtering problem for linear systems has received a lot of attention in the last years. Sufficient conditions for the existence of full order filters for uncertain linear systems assuring a prescribed $\mathscr{H}_2$ or $\mathscr{H}_\infty$ performance based on Linear Matrix Inequalities (LMIs) appeared with quadratic stability (Geromel (1999); de Souza and Trofino (2000); Geromel et al. (2000); Geromel and de Oliveira (2001)), parameter dependent Lyapunov functions (Xie et al. (2004); Barbosa et al. (2005); Duan et al. (2006)) and, more recently, with Lyapunov functions with polynomial dependence of degree greater than one (Gao et al. (2008); Lacerda et al. (2011)). In contrast, the study of filter design for systems subject to nonlinearities remains as a challenge in the filtering literature. In the last years, some efforts have been made to solve filter design problems in the context of systems with nonlinearities. In Gao and Wang (2003) the nonlinearities are assumed to satisfy global Lipschitz conditions and, then, a linear filter is designed by means of LMIs. In Coutinho et al. (2009) a linear $\mathscr{H}_\infty$ filter is proposed for a class of nonlinear systems described by a differential-algebraic representation and Basin et al. (2009) tackle the problem of central suboptimal $\mathscr{H}_\infty$ filter design for nonlinear polynomial systems. By applying sum-of-squares (SOS) approaches, Li et al. (2012) propose a convergent iterative algorithm to solve the problem of linear $\mathscr{H}_\infty$ filters for polynomial systems. In most cases, despite the fact that the system has a nonlinear dynamic model, the implemented filter is linear.

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As another aspect of the problem, it is important to underline that the characterization of an estimate of the basin of attraction of the origin for a nonlinear system is a challenging problem (Khalil (2002); Chesi (2011)). Actually, the global stability of the origin can hardly be certified for nonlinear systems in general (Koditschek and Narendra (1982)).

In this paper the problem of $\mathscr{H}_\infty$ filtering for continuous-time nonlinear quadratic systems, i.e., systems whose dynamics depend quadratically on the states, is considered. The filter we want to design has the same structure as the system, i.e., it is a full order dynamic filter with quadratic terms. Firstly, using a quadratic Lyapunov function and LMI based techniques, a sufficient condition that assures an $\mathscr{H}_\infty$ bound to the dynamics of the error system, i.e., original quadratic system and the proposed filter, in a regional (local) context is obtained. This condition can be viewed as an adaptation of recent results of Valmórbida et al. (2010) for state feedback control of saturated quadratic systems. Then, by using the Finsler's lemma and imposing structural constraints to the decision variables, quasi-LMI conditions with a scalar parameter are proposed for the design of the matrices of the quadratic filter assuring an $\mathscr{H}_\infty$ bound to the error dynamic system. As illustrated by the numerical experiments, the proposed condition can provide quadratic filters that assure less conservative $\mathscr{H}_\infty$ bounds when compared to standard linear filters.

The same class of nonlinear quadratic systems has been studied in Amato et al. (2007, 2010). In these papers, sufficient conditions allowing to design state feedback control law or an observer-based control law are proposed. Additionally, being given a polytopic region of the state-space, the closed loop system is made asymptotically stable and the associated region of attraction contains this polytopic region. In the current paper, we propose an alternative way, which, with our filtering objective, prevent to choose an initial polytope.

The paper is organized as follows. Section 2 presents the system under consideration and the problem we intend to solve, Section 3 presents the preliminary results. The main results are presented in Section 4. Section 5 provides numerical experiments that illustrate the advantages of the proposed method and Section 6 concludes the paper.

Notation. Matrices are denoted by capital letters and small letters denote vectors. The elements of a matrix $A \in \mathbb{R}^{m\times n}$ are denoted by $A_{(i,j)}$, $i = 1,\ldots,m$, $j = 1,\ldots,n$. $A_{(i)}$ denotes the ith row of matrix A. For two symmetric matrices, A and B, $A > B$ ($A \geq B$) means that $A - B$ is positive definite (positive semi-definite). For matrices or vectors ($'$) indicates transpose. The block-diagonal matrix obtained from vectors is expressed by $\mathrm{diag}(x_1,\ldots,x_n)$. Similarly, the block-diagonal matrix obtained from matrices, by $\mathrm{diag}(X_1,\ldots,X_n)$. Identity matrices are denoted by I and null matrices are denoted by 0. The symbol $\star$ means a symmetric block in matrices.

2. PROBLEM STATEMENT

Consider the nonlinear quadratic system [1]

$$\begin{aligned} \dot{x} &= Ax + \begin{bmatrix} x'A_{q1}x \\ x'A_{q2}x \\ \vdots \\ x'A_{qn}x \end{bmatrix} + B_1 w \\ z &= C_1 x + D_{11} w \\ y &= C_2 x + D_{21} w \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbb{R}^r$ is the noise input, $z \in \mathbb{R}^p$ is the signal to be estimated and $y \in \mathbb{R}^q$ is the measured output. The matrices that describe the system have the following dimensions: $A \in \mathbb{R}^{n\times n}$, $A_{qi} \in \mathbb{R}^{n\times n}$, $i = 1,\ldots,n$, $B_1 \in \mathbb{R}^{n\times r}$, $C_1 \in \mathbb{R}^{p\times n}$, $D_{11} \in \mathbb{R}^{p\times r}$, $C_2 \in \mathbb{R}^{q\times n}$, $D_{21} \in \mathbb{R}^{q\times r}$.

Furthermore, the signal w is supposed energy bounded, that is $w \in \mathscr{L}_2$. Without loss of generality we assume that the signal w is $\mathscr{L}_2$-normalized, that is, it satisfies:

$$\|w\|_2^2 = \int_0^\infty w(\tau)'w(\tau)d\tau \leq 1 \tag{2}$$

Let us define $A_q \in \mathbb{R}^{n\times n^2}$ and $X \in \mathbb{R}^{n^2\times n}$ being given by

$$A_q = \begin{bmatrix} A_{q1(1)} & A_{q1(2)} & \cdots & A_{q1(n)} \\ \vdots & \vdots & \ddots & \vdots \\ A_{qn(1)} & A_{qn(2)} & \cdots & A_{qn(n)} \end{bmatrix} \tag{3}$$

and

$$X = \begin{bmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{bmatrix} \tag{4}$$

where $A_{qi(j)} \in \mathbb{R}^{1\times n}$ denotes the jth row of matrix $A_{qi} \in \mathbb{R}^{n\times n}$. Then system (1) can be rewritten as

$$\begin{aligned} \dot{x} &= Ax + A_q X x + B_1 w \\ z &= C_1 x + D_{11} w \\ y &= C_2 x + D_{21} w \end{aligned} \tag{5}$$

[1] For simplicity, the dependence on t is omitted.

The aim of this paper is: find a full-order quadratic stable filter described as

$$\begin{aligned} \dot{x}_f &= A_f x_f + \begin{bmatrix} x_f'A_{qf1}x_f \\ x_f'A_{qf2}x_f \\ \vdots \\ x_f'A_{qfn_f}x_f \end{bmatrix} + B_f y \\ z_f &= C_f x_f + D_f y \end{aligned} \tag{6}$$

with $n_f = n$, $A_f \in \mathbb{R}^{n_f\times n_f}$, $A_{qfi} \in \mathbb{R}^{n_f\times n_f}$, $i = 1,\ldots,n_f$, $B_f \in \mathbb{R}^{n_f\times q}$, $C_f \in \mathbb{R}^{p\times n_f}$, $D_f \in \mathbb{R}^{p\times q}$, $x_f \in \mathbb{R}^{n_f}$ the estimated state and $z_f \in \mathbb{R}^p$ the estimated output.

Note that, by using similar definitions (3) and (4) with respect to the filter (6), one can write system (6) as

$$\begin{aligned} \dot{x_f} &= A_f x_f + A_{qf} X_f x_f + B_f y \\ z_f &= C_f x_f + D_f y \end{aligned} \tag{7}$$

The quadratic terms in the filter can be interpreted as a counteraction to the influence of the quadratic terms of the system. Defining the augmented state vector $\tilde{x}' = \begin{bmatrix} x' & x_f' \end{bmatrix}$ and the output error $e = z - z_f$, the augmented system (5)–(7) reads

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{A}_q \tilde{X}\tilde{x} + \tilde{B}w \\ e &= \tilde{C}\tilde{x} + \tilde{D}w \end{aligned} \tag{8}$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ B_f C_2 & A_f \end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \qquad \tilde{A}_q = \begin{bmatrix} A_q & 0 \\ 0 & A_{qf} \end{bmatrix} \in \mathbb{R}^{2n\times 2n^2},$$

$$\tilde{X} = \begin{bmatrix} X & 0 \\ 0 & X_f \end{bmatrix} \in \mathbb{R}^{2n^2\times 2n}, \qquad \tilde{B} = \begin{bmatrix} B_1 \\ B_f D_{21} \end{bmatrix} \in \mathbb{R}^{2n\times r},$$

$$\tilde{C} = [C_1 - D_f C_2 \;\; -C_f] \in \mathbb{R}^{p\times 2n}, \quad \tilde{D} = [D_{11} - D_f D_{21}] \in \mathbb{R}^{p\times r}$$

At this stage, it is important to mention that system (8) with $w = 0$ can be globally asymptotically stable (i.e., asymptotically stable for any initial condition $\tilde{x}(0) \in \mathbb{R}^{2n}$) only for some particular structure of both matrices $\tilde{A}$ and $\tilde{A}_q$ (see, for example, Koditschek and Narendra (1982); Valmórbida et al. (2013) and references therein). Then, the stability of system (8) is studied in a regional (local) context, requiring that $\tilde{A}$ is Hurwitz. The problem addressed in the paper can be summarized as follows.

Problem 1. Determine a full-order quadratic stable filter as (7) and a region $S_0 \subseteq \mathbb{R}^{2n}$ such that:

(1) when $w = 0$, the region S_0 is an estimate of the basin of attraction of the origin for system (8). That means that for any $\tilde{x}(0) \in S_0$, the resulting trajectories of system (8) asymptotically converge towards the origin;
(2) when $w \neq 0$:
   (a) the trajectories of system (8) do not leave the region S_0 for any initial condition $\tilde{x}(0) = 0$;
   (b) the $\mathscr{H}_\infty$ performance between the disturbance w and the output error $e = z - z_f$ is limited by γ for any initial condition $\tilde{x}(0) = 0$, that is: $\|e\|_2^2 \leq \gamma \|w\|_2^2$.

3. PRELIMINARIES

Let us recall the following lemma issued from Valmórbida et al. (2010) on which our results are based.

Lemma 1. (Valmórbida et al. (2010)). Consider a matrix $P \in \mathbb{R}^{n\times n}$, $P = P' > 0$ and a vector v such that $\|v\| = 1$. Every point on the boundary of an ellipsoid, $\partial\mathscr{E}(P) = \{x \in \mathbb{R}^n; x'Px = 1\}$, can be parameterized by $x = P^{-\frac{1}{2}}Tv$, with $T'T = I$.

Based on the parameterization of Lemma 1, we can present the following result on stability analysis for system (8).

Proposition 2. If there exist a matrix $P = P' > 0 \in \mathbb{R}^{2n\times 2n}$ and a positive scalar ξ such that the inequality

$$\begin{bmatrix} \tilde{A}'P + P\tilde{A} + \xi I & P\tilde{B} & \tilde{C}' & P\tilde{A}_q \\ \tilde{B}'P & -I & \tilde{D}' & 0 \\ \tilde{C} & \tilde{D} & -\gamma^2 I & 0 \\ \tilde{A}_q' P & 0 & 0 & -\xi\tilde{P} \end{bmatrix} < 0 \qquad (9)$$

is satisfied with $\tilde{P} = diag(P,\ldots,P) \in \mathbb{R}^{2n^2\times 2n^2}$, then

(1) when $w = 0$, the region $S_0 = \mathscr{E}(P) = \left\{\tilde{x} \in R^{2n}; \tilde{x}'P\tilde{x} \le 1\right\}$ is an estimate of the region of attraction of the origin for system (8);
(2) when $w \neq 0$, the $\mathscr{H}_\infty$ performance between w and e for system (8) is limited by γ, for initial condition $\tilde{x}(0) = 0$.

Proof. Consider the quadratic Lyapunov function $V(\tilde{x}) = \tilde{x}'P\tilde{x}$, $P = P' > 0$. The $\mathscr{H}_\infty$ performance bound between w and e for system (8) can be obtained by satisfying

$$\dot{V}(\tilde{x}) + \frac{1}{\gamma^2}e'e - w'w < 0$$

for energy signals $w \in \mathscr{L}_2$, which can be written as

$$\begin{bmatrix} \tilde{A}'P + P\tilde{A} + P\tilde{A}_q\tilde{X} + \tilde{X}'\tilde{A}_q'P & P\tilde{B} & \tilde{C}' \\ \tilde{B}'P & -I & \tilde{D}' \\ \tilde{C} & \tilde{D} & -\gamma^2 I \end{bmatrix} < 0 \qquad (10)$$

We are interested in finding an ellipsoid

$$\mathscr{E}(P) = \left\{\tilde{x} \in R^{2n}; \tilde{x}'P\tilde{x} \le 1\right\}$$

inside which $\dot{V}(\tilde{x}) < 0$ when $w = 0$ and $\dot{V}(\tilde{x}) + \frac{1}{\gamma^2}e'e - w'w < 0$ when $w \neq 0$. Hence, by applying the parameterization of Lemma 1, for $\tilde{x} \in \partial\mathscr{E}$, the time-derivative $\dot{V}(\tilde{x})$ can be written as

$$\dot{V}(\tilde{x}) = \tilde{x}'\left(\tilde{A}'P + P\tilde{A} + P\tilde{A}_q\tilde{P}^{-\frac{1}{2}}\tilde{T}V + V'\tilde{T}'\tilde{P}^{-\frac{1}{2}}\tilde{A}_q'P\right)\tilde{x} + 2\tilde{x}'P\tilde{B}w$$

with $\tilde{T} = diag(T,\ldots,T) \in \mathbb{R}^{2n^2\times 2n^2}$, and $V = diag(v,\ldots,v) \in \mathbb{R}^{2n^2\times 2n}$, where $\|v\| = 1$. One can write

$$\tilde{x}'\left(P\tilde{A}_q\tilde{P}^{-\frac{1}{2}}\tilde{T}V + V'\tilde{T}'\tilde{P}^{-\frac{1}{2}}\tilde{A}_q'P\right)\tilde{x} \le \tilde{x}'\left(\frac{1}{\xi}P\tilde{A}_q\tilde{P}^{-1}\tilde{A}_q'P + \xi V'\tilde{T}'\tilde{T}V\right)\tilde{x}$$

with $\xi > 0$. As $\tilde{T}'\tilde{T} = I$ and $V'V = I$, it follows:

$$\tilde{x}'\left(P\tilde{A}_q\tilde{P}^{-\frac{1}{2}}\tilde{T}V + V'\tilde{T}'\tilde{P}^{-\frac{1}{2}}\tilde{A}_q'P\right)\tilde{x} \le \tilde{x}'\left(\frac{1}{\xi}P\tilde{A}_q\tilde{P}^{-1}\tilde{A}_q'P + \xi I\right)\tilde{x}$$

Thus if the inequality

$$\begin{bmatrix} \tilde{A}'P + P\tilde{A} + \frac{1}{\xi}PA_q\tilde{P}^{-1}A_q'P + \xi I & P\tilde{B} & \tilde{C}' \\ \tilde{B}'P & -I & \tilde{D}' \\ \tilde{C} & \tilde{D} & -\gamma^2 I \end{bmatrix} < 0$$

holds then inequality (10) is satisfied. By using Schur complement the above inequality is equivalent to:

$$\begin{bmatrix} \tilde{A}'P + P\tilde{A} + \xi I & P\tilde{B} & \tilde{C}' & P\tilde{A}_q \\ \tilde{B}'P & -I & \tilde{D}' & 0 \\ \tilde{C} & \tilde{D} & -\gamma^2 I & 0 \\ \tilde{A}_q' P & 0 & 0 & -\xi\tilde{P} \end{bmatrix} < 0$$

which corresponds to relation (9). Hence, from Lemma 1, if relation (9) is satisfied, then for every $\tilde{x} \in \partial\mathscr{E}(P)$ we have

(1) $\dot{V}(\tilde{x}) \le \dot{V}(\tilde{x}) + \frac{1}{\gamma^2}e'e < 0$ when $w = 0$;
(2) $\dot{V}(\tilde{x}) + \frac{1}{\gamma^2}e'e - w'w < 0$ when $w \neq 0$.

By integrating the last inequality for $\tilde{x}(0) = 0$, one gets:

$$V(\tilde{x}(T)) - V(\tilde{x}(0)) + \frac{1}{\gamma^2}\int_0^T e(\tau)'e(\tau)d\tau - \int_0^T w(\tau)'w(\tau)d\tau < 0$$

or

$$V(\tilde{x}(T)) < \int_0^T w(\tau)'w(\tau)d\tau \le 1, \quad \forall T > 0$$

i.e., the trajectories of the augmented system (8) do not leave the set $\mathscr{E}(P)$. When $w = 0$, we have $\dot{V}(x) < 0$, which ensures that $\tilde{x} \to 0$ as $t \to \infty$ for any $\tilde{x} \in \mathscr{E}(P)$. □

Let us give the following lemma (Finsler's Lemma) that will be useful to derive the conditions for filter design.

Lemma 3. (de Oliveira and Skelton (2001)). Let $w \in \mathbb{R}^n$, $\mathscr{Q} \in \mathbb{R}^{n\times n}$ and $\mathscr{B} \in \mathbb{R}^{m\times n}$ with rank $(\mathscr{B}) < n$ and $\mathscr{B}^\perp$ a basis for the null space of $\mathscr{B}$ ($\mathscr{B}\mathscr{B}^\perp = 0$). Then, the following conditions are equivalent:

i) $w'\mathscr{Q}w < 0, \forall\, w \neq 0 : \mathscr{B}w = 0$;
ii) $\mathscr{B}^{\perp\prime}\mathscr{Q}\mathscr{B}^\perp < 0$;
iii) $\exists\, \mu \in \mathbb{R} : \mathscr{Q} - \mu\mathscr{B}'\mathscr{B} < 0$;
iv) $\exists\, \mathscr{X} \in \mathbb{R}^{n\times m} : \mathscr{Q} + \mathscr{X}\mathscr{B} + \mathscr{B}'\mathscr{X}' < 0$.

4. MAIN RESULTS

By using Lemma 3, Proposition 2 leads to the following proposition.

Proposition 4. If there exist a matrix $P = P' > 0 \in \mathbb{R}^{2n\times 2n}$, matrices $F_1 \in \mathbb{R}^{2n\times 2n}$, $F_2 \in \mathbb{R}^{2n\times 2n}$, $F_3 \in \mathbb{R}^{r\times 2n}$, $F_4 \in \mathbb{R}^{2n^2\times 2n}$, $F_5 \in \mathbb{R}^{p\times 2n}$, and a positive scalar ξ such that the inequality

$$\Theta + \Psi < 0 \qquad (11)$$

is satisfied with

$$\Theta = \begin{bmatrix} \xi I & P & 0 & 0 & \tilde{C}' \\ \star & 0 & 0 & 0 & 0 \\ \star & \star & -I & 0 & \tilde{D}' \\ \star & \star & \star & -\xi\tilde{P} & 0 \\ \star & \star & \star & \star & -\gamma^2 I \end{bmatrix} \qquad (12)$$

$\tilde{P} = diag(P,\ldots,P) \in \mathbb{R}^{2n^2\times 2n^2}$, and

$$\Psi = \begin{bmatrix} F_1\tilde{A} + \tilde{A}'F_1' & -F_1 + \tilde{A}'F_2' & F_1\tilde{B} + \tilde{A}'F_3' & F_1\tilde{A}_q + \tilde{A}'F_4' & \tilde{A}'F_5' \\ \star & -F_2 - F_2' & F_2\tilde{B} - F_3' & F_2\tilde{A}_q - F_4' & -F_5' \\ \star & \star & F_3\tilde{B} + \tilde{B}'F_3' & F_3\tilde{A}_q + \tilde{B}'F_4' & \tilde{B}'F_5' \\ \star & \star & \star & F_4\tilde{A}_q + \tilde{A}_q'F_4' & \tilde{A}_q'F_5' \\ \star & \star & \star & \star & 0 \end{bmatrix} \qquad (13)$$

then,

(1) when $w = 0$, the region $S_0 = \mathscr{E}(P) = \left\{\tilde{x} \in R^{2n}; \tilde{x}'P\tilde{x} \le 1\right\}$ is an estimate of the region of attraction of the origin for system (8);
(2) when $w \neq 0$:
    (a) the trajectories of system (8) do not leave the region S_0 for any initial condition $\tilde{x}(0) = 0$;
    (b) the $\mathscr{H}_\infty$ performance between the disturbance w and the output error $e = z - z_f$ is limited by γ for any initial condition $\tilde{x}(0) = 0$, that is: $\|e\|_2^2 \le \gamma\|w\|_2^2$.

Proof. By considering

$$\mathscr{X}=\begin{bmatrix}F_1\\F_2\\F_3\\F_4\\F_5\end{bmatrix},\ \mathscr{Q}=\begin{bmatrix}\xi I & P & 0 & 0 & \tilde{C}'\\ \star & 0 & 0 & 0 & 0\\ \star & \star & -I & 0 & \tilde{D}'\\ \star & \star & \star & -\xi\tilde{P} & 0\\ \star & \star & \star & \star & -\gamma^2 I\end{bmatrix},\ \mathscr{B}=\begin{bmatrix}\tilde{A}'\\-I\\\tilde{B}'\\\tilde{A}_q'\\0\end{bmatrix}' \tag{14}$$

in condition *iv)* of Lemma 3 with

$$\mathscr{B}^{\perp\prime}=\begin{bmatrix}I & \tilde{A}' & 0 & 0 & 0\\ 0 & \tilde{B}' & I & 0 & 0\\ 0 & \tilde{A}_q' & 0 & I & 0\\ 0 & 0 & 0 & 0 & I\end{bmatrix}$$

and by using condition *ii)* of Lemma 3, one obtains condition (9) (except by the exchange of rows and columns 3 and 4). □

Proposition 4 presents a nonlinear condition because the decision variables of interest (i.e., A_f, A_{qf}, B_f, C_f and D_f) appear in sub-matrices multiplying the extra variables F_i, $i=1,\ldots,5$. To linearize the condition presented in Proposition 4, based on the strategies in Duan et al. (2006); Lacerda et al. (2011), the following structure imposed on matrices F_i, $i=1,\ldots,5$, is considered:

$$F_1=\begin{bmatrix}F_{11} & \hat{K}\\F_{13} & \hat{K}\end{bmatrix},\ F_2=\begin{bmatrix}F_{21} & \hat{K}\\F_{23} & \hat{K}\end{bmatrix},\ F_3=\begin{bmatrix}F_{31} & 0_{r\times n}\end{bmatrix},$$
$$F_4=\begin{bmatrix}F_{41} & 0_{n^2\times n}\\F_{43} & 0_{n^2\times n}\end{bmatrix},\ F_5=\begin{bmatrix}F_{51} & 0_{p\times n}\end{bmatrix} \tag{15}$$

where $\hat{K}\in\mathbb{R}^{n\times n}$. For convenience, matrix P is also partitioned in $n\times n$ blocks

$$P=\begin{bmatrix}P_{11} & P_{12}\\P_{12}' & P_{22}\end{bmatrix} \tag{16}$$

and the following changes of variables are adopted

$$K_1=\hat{K}A_f,\quad K_2=\hat{K}B_f,\quad K_3=\hat{K}A_{qf} \tag{17}$$

With this choice for the decision variables, Proposition 4 can be reformulated in a way that allows the direct determination of the filter matrices presented in the following theorem.

Theorem 5. If there exist a matrix $P=P'>0$ as in (16), matrices F_i, $i=1,\ldots,5$ as in (15), $K_1\in\mathbb{R}^{n\times n}$, $K_2\in\mathbb{R}^{n\times q}$, $K_3\in\mathbb{R}^{n\times n^2}$, $C_f\in\mathbb{R}^{p\times n}$, $D_f\in\mathbb{R}^{p\times q}$, $\gamma>0$ and $\xi>0$ such that the inequality

$$\Theta+\overline{\Psi}<0 \tag{18}$$

is satisfied with Θ as in (12) and $\overline{\Psi}$ given by (20) (top of next page), then,

$$A_f=\hat{K}^{-1}K_1, B_f=\hat{K}^{-1}K_2, A_{qf}=\hat{K}^{-1}K_3, C_f, D_f \tag{19}$$

are the matrices of the quadratic filter solution to Problem 1.

Proof. Following the same steps as those in proof of Proposition 4, if (18) is satisfied with the slack variables as in (15), then the $\mathscr{H}_\infty$ filter that solves Problem 1 is given by (19). □

Theorem 5 provides a sufficient matrix inequality condition for the existence of a nonlinear quadratic $\mathscr{H}_\infty$ filter, derived from Proposition 4 by imposing a particular structure to the slack variables F_i, $i=1,\ldots,5$.

Remark 6. To recover the classical linear filter it suffices to consider $A_{qf}=0$, i.e., simply imposing $K_3=0$ in Theorem 5.

Remark 7. It is important to observe that inequality (18) becomes an LMI when the positive scalar ξ is fixed. By using a griding on ξ, a convex optimization problem can be stated to minimize γ for each fixed value of ξ:

$$\begin{cases}\min\gamma\\ \text{subject to LMI (18)}\end{cases} \tag{21}$$

where the decision variables are P, F_i, $i=1,\ldots,5$, K_1, K_2, K_3, C_f, D_f and γ.

5. NUMERICAL EXPERIMENTS

The objective of the experiments is to illustrate the conditions proposed in this paper and show the potential of the nonlinear quadratic filters in comparison with the linear ones ($A_{qf}=0$). The matrix inequality conditions in both cases depend on a scalar parameter ξ that needs to be searched. In the following experiments a simple linear search with precision 0.01 has been used in ξ. By applying optimization algorithms, as for example *fminsearch* in the optimization toolbox of MATLAB, the conditions could be improved. The routines were implemented in MATLAB, version 7.6.0.324 (R2008a) SP 2 using Yalmip (Löfberg (2004)) and SeDuMi (Sturm (1999)). The computer used was an Intel® Core 2 Duo (2.0 GHz), 3GB RAM, Windows Vista.

Consider the Lorenz attractor, a nonlinear quadratic system [2] also studied in Valmórbida et al. (2010), with matrices

$$A=\begin{bmatrix}-\sigma & \sigma & 0\\ \rho & -1 & 0\\ 0 & 0 & -b\end{bmatrix},$$
$$A_q=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -0.5 & 0 & 0 & 0 & -0.5 & 0 & 0\\ 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$
$$B_1=[1\ 0\ 0]',\quad C_2=[1\ 0\ 0],\quad D_{21}=[0.5],$$
$$C_1=[0.5\ 1\ 1],\quad D_{11}=[0],$$

where σ, ρ and b are positive scalars. By linearizing matrix A around the equilibrium point

$$x^*=\left[\sqrt{b(\rho-1)}\ \sqrt{b(\rho-1)}\ \rho-1\right]'$$

one has

$$A=\begin{bmatrix}-\sigma & \sigma & 0\\ 1 & -1 & -\sqrt{b(\rho-1)}\\ \sqrt{b(\rho-1)} & \sqrt{b(\rho-1)} & -b\end{bmatrix}$$

Figure 1 depicts the $\mathscr{H}_\infty$ bounds obtained with a linear filter ($A_{qf}=0$) obtained through the design conditions of Theorem 5 (in blue), with $K_3=0$, and also using the analysis conditions of Proposition 4 applied to the augmented system (8) with the corresponding filter (in black) with parameters $\sigma=1$, $b=8/3$ and $\rho=4$. The minimum value of γ obtained by using a linear filter for this interval is $\gamma=1.6057$ with $\xi=0.40$ for design, and the minimum γ obtained from the analysis of the augmented system (8) is $\gamma=0.8586$ with $\xi=0.50$.

Figure 2 shows the $\mathscr{H}_\infty$ performance obtained with a nonlinear quadratic filter designed by the conditions of Theorem 5 (blue) and the bounds obtained from the analysis of the augmented system (8) (in black) with parameters $\sigma=1$, $b=8/3$ and $\rho=4$. The minimum achieved with the design condition in Theorem 5 is $\gamma=1.0628$ for $\xi=0.41$, while the minimum γ considering the analysis of the augmented system (8) is $\gamma=0.6428$ obtained for $\xi=0.49$. The nonlinear quadratic filter provides the smallest bounds, both to the design condition

[2] This system can present chaotic behavior when $\sigma=10$, $b=8/3$ and $\rho\geq 25$.

$$
\overline{\Psi}=\begin{bmatrix}
F_{11}A+A'F_{11}'+K_2C_2+C_2'K_2' & K_1+A'F_{13}'+C_2'K_2' & -F_{11}+A'F_{21}'+C_2'K_2' & -\hat{K}+A'F_{23}'+C_2'K_2' & F_{11}B_1+K_2D_{21}+A'F_{31}' & F_{11}A_q+A'F_{41}' & K_3+A'F_{43}' & A'F_{51}' \\
\star & (K_1+K_1') & -F_{13}+K_1' & -\hat{K}+K_1' & F_{13}B_1+K_2D_{21} & F_{13}A_q & K_3 & 0 \\
\star & \star & -F_{21}-F_{21}' & -\hat{K}-F_{23} & F_{21}B_1+K_2D_{21}-F_{31}' & F_{21}A_q-F_{41}' & K_3-F_{43}' & -F_{51}' \\
\star & \star & \star & -\hat{K}-\hat{K}' & F_{23}B_1+K_2D_{21} & F_{23}A_q & K_3 & 0 \\
\star & \star & \star & \star & F_{31}B_1+B_1'F_{31}' & F_{31}A_q+B_1'F_{41}' & B_1'F_{43}' & B_1'F_{51}' \\
\star & \star & \star & \star & \star & F_{41}A_q+A_q'F_{41}' & A_q'F_{43}' & A_q'F_{51}' \\
\star & \star & \star & \star & \star & \star & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & 0
\end{bmatrix} \quad (20)
$$

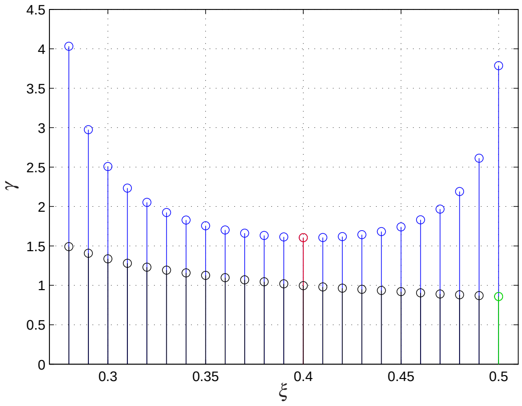

Fig. 1. Behavior of γ with the variation of ξ for a linear filter obtained with Theorem 5.

and for the analysis of the augmented system. Furthermore, it is important to note that for some values of ξ (for example $\xi = 0.25$) the condition from Theorem 5 did not provide a linear filter, while a nonlinear quadratic filter can be obtained.

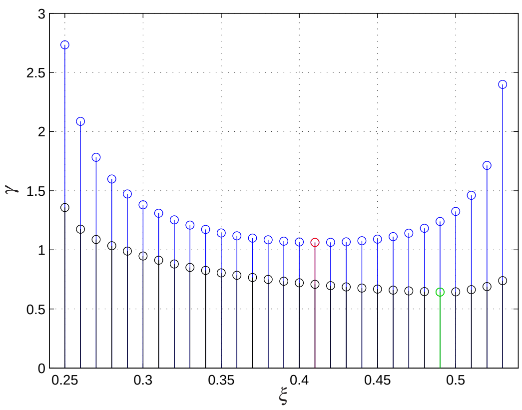

Fig. 2. Behavior of γ with the variation of ξ for a quadratic filter obtained with Theorem 5.

Table 1 presents a comparison between the $\mathcal{H}_\infty$ performance obtained by Theorem 5, with a nonlinear quadratic filter, and the one obtained by means of a linear filter ($A_{qf} = 0$). It can be noted that the nonlinear quadratic filter provides the best results mainly for smaller values of ρ.

Table 1. $\mathcal{H}_\infty$ performance comparison, quadratic filter $\times$ linear filter, with $b = 8/3$.

Parameters		Theorem 5		Linear filter	
σ	ρ	ξ	γ	ξ	γ
1	3.2	0.36	1.7980	–	–
1	3.5	0.39	1.2209	0.37	2.5641
1	3.6	0.39	1.1645	0.38	2.1307
1	3.7	0.40	1.1251	0.39	1.9062
1	3.8	0.41	1.0979	0.39	1.7684
1	3.9	0.41	1.0772	0.40	1.6728
1	4	0.41	1.0628	0.40	1.6057
2	4	0.73	0.5038	0.76	0.6025

In order to provide a time simulation for the filter behavior, consider the input noise signal

$$
w(t) = \sin(0.5t)\exp(-0.1t) \quad (22)
$$

Figure 3 shows the output for the augmented system (8), i.e., the error signal, for the linear filter (blue dashed line) and for the nonlinear quadratic filter (red line), with initial condition $\tilde{x}(0) = 0$, parameters $\sigma = 1$, $\rho = 3.5$ and $b = 8/3$. The values of ξ are indicated in Table 1. It is possible to note that the nonlinear quadratic filter obtained by Theorem 5 provides the smallest error output in view of the noise $w(t)$ in (22). In this case, the nonlinear quadratic filter obtained with Theorem 5 is given by

$$
A_f = \begin{bmatrix} -3.3942 & 0.5254 & 0.0215 \\ -8.4362 & -1.4371 & 1.5404 \\ 11.5077 & 1.2044 & -5.5196 \end{bmatrix},\ B_f = \begin{bmatrix} -2.1227 \\ -3.6466 \\ 4.4287 \end{bmatrix},
$$

$$
C_f = \begin{bmatrix} -1.8073 & -0.8338 & -0.3494 \end{bmatrix},\ D_f = \begin{bmatrix} -0.5114 \end{bmatrix},
$$

$$
A_{qf} = \begin{bmatrix} -0.1427 & 0.0081 & 0.0128 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.0037 & 0.0573 & 0.0900 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.2840 & 0.0161 & 0.0254 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

while the linear filter is given by

$$
A_f = \begin{bmatrix} -3.3588 & 0.4991 & -0.0326 \\ -10.6368 & -0.7225 & 2.0057 \\ 13.7158 & 0.4619 & -6.3106 \end{bmatrix},\ B_f = \begin{bmatrix} -2.1303 \\ -4.7658 \\ 5.3911 \end{bmatrix},
$$

$$
C_f = \begin{bmatrix} -2.0964 & -0.7118 & -0.3283 \end{bmatrix},\ D_f = \begin{bmatrix} -0.6672 \end{bmatrix}
$$

We can verify that for both nonlinear quadratic filter and linear filter, the evolution of the states $\tilde{x}$ remains confined in the region

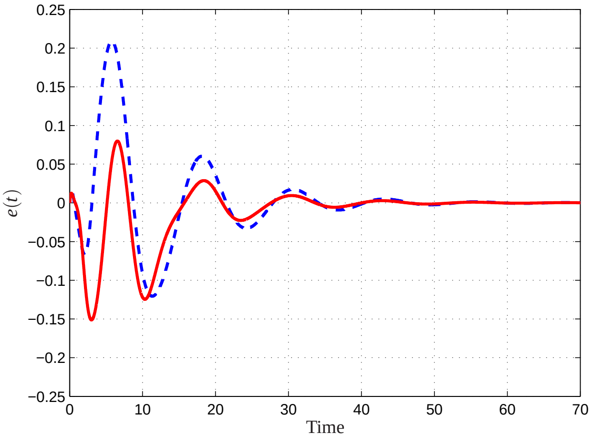

Fig. 3. Error time response for the augmented systems (8) obtained with a linear filter (blue dashed line) and with a nonlinear quadratic system (red line), for initial condition $\tilde{x}(0) = 0$. Parameters $\sigma = 1$, $\rho = 3.5$, $b = 8/3$ and ξ as in Table 1.

$S_0 = \mathscr{E}(P) = \left\{\tilde{x} \in R^{2n}; \tilde{x}' P \tilde{x} \leq 1\right\}$, meaning that system (8) is locally asymptotically stable.

6. CONCLUSION

New matrix inequality conditions for the design of full-order nonlinear quadratic $\mathscr{H}_\infty$ filters have been proposed for continuous-time nonlinear quadratic systems. By numerical experiments it was showed that the filters with quadratic terms can improve the results obtained by linear filters. As future research, the authors are investigating Lyapunov functions of degree greater than two in the state and extensions to cope with $\mathscr{H}_2$ filter design as well.

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