Quantum reservoir engineering and single qubit cooling

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Abstract: Stabilizing a quantum system in a desired state has important implications in quantum information science. In control engineering, stabilization is usually achieved by the use of feedback. The closed-loop control paradigm consists of measuring the system in a non-destructive manner, analyzing in real-time the measurement output to estimate the dynamical state and finally, calculating a feedback law to stabilize the desired state. However, the rather short dynamical time-scales of most quantum systems impose important limitations on the complexity of real-time output signal analysis and retroaction. An alternative control approach for quantum state stabilization, bypassing a real-time analysis of output signal, is called reservoir engineering.

In this paper, we start with a general description of quantum reservoir engineering. We then apply this method to stabilize the ground state (lowest energy state) of a single two-level quantum system. Applying the averaging theorem and some simple Lyapunov techniques, we prove the convergence of our proposed scheme. This scheme has recently been successfully implemented on a superconducting qubit and has led to a fast and reliable reset protocol for these qubits.

Keywords: Quantum systems, Reservoir engineering, Control by inter-connection, Lyapunov stabilization, Averaging.

1. INTRODUCTION

While feedback loops are the most important ingredient in classical control theory, their application for the control of quantum systems had been longly considered as counter-intuitive or even impossible. This is due to two major difficulties. The first one comes from the subtleties in the theory of quantum measurements: any measurement implies an instantaneous strong perturbation to the system's state. The concept of *quantum non-demolotion* (QND) measurement has played a crucial role in understanding and resolving this difficulty Braginski and Khalili [1992]. In the context of cavity quantum electro-dynamics (cavity QED) with Rydberg atoms Haroche and Raimond [2006], a first experiment on continuous QND measurements of the number of microwave photons was performed by the group at Laboratoire Kastler-Brossel (ENS) Guerlin et al. [2007]. Later on, this ability of performing continuous measurements allowed the same group to perform the first experi-

ment where a continuous quantum feedback protocol stabilized highly non-classical states of the microwave field in the cavity, the so-called photon number states Sayrin et al. [2011]. In the context of circuit quantum electrodynamics (circuit QED) Devoret et al. [2004], recent advances in quantum-limited amplifiers Roch et al. [2012], Vijay et al. [2012] have opened doors to high-fidelity non-demolition measurements and real-time feedback for superconducting qubits.

The second difficulty is due to the rather short dynamical time scales for these systems. This imposes important limitations on the complexity of real-time analysis that one can perform on the measurement output. Indeed, the time-consuming data acquisition and post-treatment of the output signal, lead to an important latency in the feedback procedure.

An alternative stabilization approach, bypassing a real-time analysis of measurement output signal and therefore avoiding the effect of the feedback latency, is called reservoir engineering. It consists of coupling the system of interest $\mathcal{S}$ to another quantum system, which we will call reservoir $\mathcal{R}$. The latter is a strongly dissipative system with many degrees of freedom. By engineering the coupling between $\mathcal{S}$ and $\mathcal{R}$, it has been shown that one can stabilize interesting quantum states in $\mathcal{S}$ Poyatos et al. [1996].
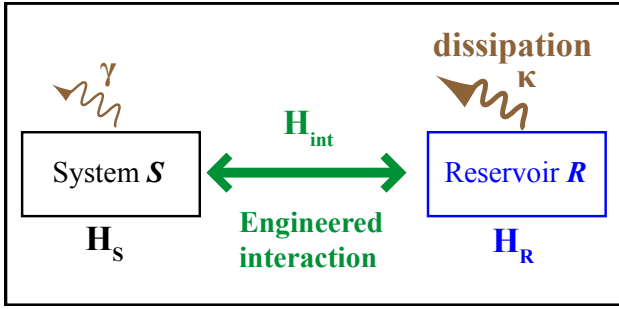
Fig. 1. Reservoir engineering : the aim is to stabilize system $\mathcal{S}$ in a target state. This is achieved by coupling $\mathcal{S}$ to another dissipative quantum system $\mathcal{R}$. By adequately engineering the interaction between $\mathcal{S}$ and $\mathcal{R}$, one can use the dissipation of $\mathcal{R}$ to stabilize a predefined target state in $\mathcal{S}$.

This idea is extremely counter-intuitive since dissipation is usually associated to the loss of quantum features.

In the next section, we give a general model for quantum reservoir engineering. Next, in Section 3, we apply this idea to stabilize the state of lowest energy (ground state) of a single qubit. This process is sometimes referred to as qubit cooling. We will show how this is achieved by coupling the qubit (system $\mathcal{S}$) to a cavity (reservoir $\mathcal{R}$) and by applying some appropriate driving fields. Section 4 is devoted to a proof of the convergence of this scheme and finally, in Section 5 we finish by some concluding remarks and further directions.

2. RESERVOIR ENGINEERING

Consider a quantum system $\mathcal{S}$, of state ρ_S (called density operator, defined in a Hilbert space $\mathcal{H}_S$) and Hamiltonian $\mathbf{H}_S$. The state space for the system is given by the space of all Hermitian, semi-definite positive, trace-class operators ρ_S, defined on $\mathcal{H}_S$ and of unit trace. The system is inevitably coupled to an undesired environment $\mathcal{E}$ which induces a dissipation of rate γ. The dynamics of ρ_S are given by the following Lindblad master equation.

$$\frac{d}{dt}\rho_S = -i[\mathbf{H}_S(t), \rho_S] + \gamma \mathcal{L}[\mathbf{L}_S]\rho_S, \qquad (1)$$

where the commutator is defined as $[A, B] = AB - BA$,

$$\mathcal{L}[\mathbf{L}_S]\rho_S = \mathbf{L}_S \rho_S \mathbf{L}_S^\dagger - \frac{1}{2}\mathbf{L}_S^\dagger \mathbf{L}_S \rho_S - \frac{1}{2}\rho_S \mathbf{L}_S^\dagger \mathbf{L}_S ,$$

and $\mathbf{L}_S$ is an operator which reflects how the system is coupled to the environment $\mathcal{E}$. Usually, ρ_S will converge to a state in thermodynamical equilibrium with $\mathcal{E}$, which does not represent the quantum features that are of interest to us.

Now consider another quantum system called reservoir $\mathcal{R}$, of which the state is defined in a Hilbert space $\mathcal{H}_R$ and of Hamiltonian $\mathbf{H}_R$. We assume that this reservoir is very dissipative: its dissipation rate $\kappa \gg \gamma$. Now we couple the system and the reservoir through an interaction Hamiltonian $\mathbf{H}_{\text{int}}$. The state space of the joint system-reservoir is defined on the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_R$ ($\otimes$ symbolizes the tensor product). Its Hamiltonian $\mathbf{H}$ is given by

$$\mathbf{H} = \mathbf{H}_S \otimes \mathbf{I}_R + \mathbf{I}_S \otimes \mathbf{H}_R + \mathbf{H}_{\text{int}},$$

where $\mathbf{I}_{S,R}$ are simply the identity operators for the system and reservoir. The system-reservoir state ρ follows the dynamics

$$\frac{d}{dt}\rho = -i[\mathbf{H}(t), \rho] + \gamma \mathcal{L}[\mathbf{L}_S \otimes \mathbf{I}_R]\rho + \kappa \mathcal{L}[\mathbf{I}_S \otimes \mathbf{L}_R]\rho. \quad (2)$$

The idea of reservoir engineering is to engineer $\mathcal{R}$ and its coupling to $\mathcal{S}$ in such a way that $\mathcal{S}$ converges close to some predefined target state $\bar{\rho}_S$, i.e. the dynamics (2) converges to a close neighborhood of a state of the form $\rho_\infty = \bar{\rho}_S \otimes \rho_R$, where ρ_R is an arbitrary state in the reservoir. In practice we usually assume $\gamma = 0$, and design the interaction such that (2) converges exactly towards $\bar{\rho}_S \otimes \rho_R$. Then taking $\gamma \neq 0$ but assuming $\kappa \gg \gamma$, the steady state of (2) will remain close to $\bar{\rho}_S \otimes \rho_R$. The power of this method is that it is a "plug and play" stabilization scheme: once the interaction has been engineered, no further action is required, and in particular there is no need for real-time output signal acquisition, processing and retroaction.

Such reservoir engineering schemes are particularly appealing in circuit QED systems, where one can easily engineer a large class of quantum systems (reservoirs) and interaction Hamiltonians. While reservoir engineering is usually used for the stabilization of non-classical states such as Schrödinger cat states or entangled states Poyatos et al. [1996], Sarlette et al. [2011], Krauter et al. [2011], in the next Section we present a rather classical application of such a technique and stabilize the ground state (lowest energy state) of a two-level quantum system.

3. COOLING OF A SINGLE QUBIT

A quantum bit of information (qubit) is usually encoded in a two-level system. A crucial task in most experiments is the ability to initialize this qubit in its ground state. If the qubit interacts with a cold environment, simply waiting for a sufficiently long time ensures that the qubit will decay to its ground state. However, recent progress in extending the decay times of superconducting qubits makes it necessary to look for fast and efficient ways to actively prepare the ground state, thus avoiding the long passive initializations by waiting for equilibration with the cold bath. Furthermore, very often for superconducting qubits, the thermal environment is hot on the scale of the transition frequency and therefore, even by equilibration, the qubit ends up with some population in the excited state. A fast active cooling of the qubit can also help reduce this thermal population.

One can consider a measurement-based feedback scheme to stabilize the ground state Risté et al. [2012]. In this aim, one performs a fast, single-shot and high-fidelity projective measurement of the qubit state. Whenever we find the qubit in its excited state, we apply a π-pulse rotating the qubit state back to the ground state. While this feedback scheme seems very straight-forward, it necessitates a high-fidelity and fast measurement which, in general, would be possible using a quantum-limited amplifier Roch et al. [2012]. Here, instead, we propose a reservoir engineering scheme by coupling the qubit to a highly dissipative reservoir (here a cavity mode), which efficiently stabilizes the qubit in its ground state, thus cooling it. Before going to the details of the cooling scheme, we recall some

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notations allowing us to present the mathematical model behind the system we consider.

The building block of most circuit QED experiments consists of a single superconducting qubit dispersively coupled to a single cavity mode, modeled by a quantum harmonic oscillator. The state space for a single qubit is defined on a two dimensional complex Hilbert space $\mathcal{H}_S = \text{span}\{|g\rangle, |e\rangle\}$, where $|g\rangle$ stands for the ground and $|e\rangle$ the excited state. Also, the state space for the quantum harmonic oscillator is defined on the infinite dimensional Hilbert space denoted by $\mathcal{H}_R$ (also called Fock space) with inner-product $\langle . \mid . \rangle_{\mathcal{H}_R}$ and norm $\| \cdot \|_{\mathcal{H}_R}$. Also the set $\{|n\rangle \ : \ n \in \mathbb{Z}_0^+ = \{0, 1, \cdots\}\}$ denotes the canonical basis of the Fock space $\mathcal{H}_R$. Physically, the state $|n\rangle$ is a Fock state representing a quantum state with precisely n photons.

For a qubit we define the Pauli operators
$$\sigma_x = |g\rangle\langle e| + |e\rangle\langle g|,$$
$$\sigma_y = i(|g\rangle\langle e| - |e\rangle\langle g|), \quad \sigma_z = |e\rangle\langle e| - |g\rangle\langle g|.$$
Also, for the harmonic oscillator, we define the annihilation, $\mathbf{a}$, and creation, $\mathbf{a}^\dagger$, operators as well as the photon number operator $\mathbf{N} = \mathbf{a}^\dagger\mathbf{a}$:
$$\mathbf{a}|n\rangle = \sqrt{n}|n-1\rangle, \mathbf{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \mathbf{N}|n\rangle = n|n\rangle,$$
for $n \in \mathbb{Z}_0^+$. Their domains are given by
$$\mathcal{D}(\mathbf{a}) = \mathcal{D}(\mathbf{a}^\dagger) = \left\{ \sum_{n=0}^\infty c_n|n\rangle \mid (c_n)_{n=0}^\infty \in \hbar^1(\mathbb{C}) \right\},$$
$$\mathcal{D}(\mathbf{N}) = \left\{ \sum_{n=0}^\infty c_n|n\rangle \mid (c_n)_{n=0}^\infty \in \hbar^2(\mathbb{C}) \right\},$$
where
$$\hbar^k(\mathbb{C}) = \left\{ (c_n)_{n=0}^\infty \in l^2(\mathbb{C}) \mid \sum_{n=0}^\infty n^k|c_n|^2 < \infty \right\}.$$
The qubit of frequency ω_q has a Hamiltonian $\mathbf{H}_S = \frac{\omega_q}{2}\sigma_z$, and the cavity of frequency ω_c has a Hamiltonian $\mathbf{H}_R = \omega_c\mathbf{a}^\dagger\mathbf{a}$. The interaction Hamiltonian for dispersive coupling is given by $\mathbf{H}_{\text{int}} = -\frac{\chi}{2}\sigma_z \otimes \mathbf{a}^\dagger\mathbf{a}$. Hence, the total qubit-cavity Hamiltonian is well approximated by Haroche and Raimond [2006]
$$\mathbf{H}_0 = \frac{\omega_q}{2}\sigma_z \otimes \mathbf{I}_R + \omega_c\mathbf{I}_S \otimes \mathbf{a}^\dagger\mathbf{a} - \frac{\chi}{2}\sigma_z \otimes \mathbf{a}^\dagger\mathbf{a},$$
defined on the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_R$, where $\mathbf{I}_S$ and $\mathbf{I}_R$ denote the identity operators of the Hilbert spaces $\mathcal{H}_S$ and $\mathcal{H}_R$. A way to understand this Hamiltonian is that the bare qubit and cavity frequencies are given by ω_q and ω_c. Whenever the two are coupled, the cavity frequency gets shifted by $\pm\chi/2$ based on the qubit state being $|e\rangle$ or $|g\rangle$. In the same way the qubit's frequency gets shifted by a multiple of χ depending on the number of photons in the cavity.

We effectively engineer the interaction Hamiltonian between the qubit and the cavity by applying two electric driving fields close to the resonance frequencies of the qubit and cavity transitions. The total Hamiltonian is then given by
$$\mathbf{H}(t) = \mathbf{H}_0 +$$
$$(u_q(t)\sigma_+ + u_q^*(t)\sigma_-) \otimes \mathbf{I}_R + \mathbf{I}_S \otimes (u_c(t)\mathbf{a}^\dagger + u_c^*(t)\mathbf{a}), \quad (3)$$
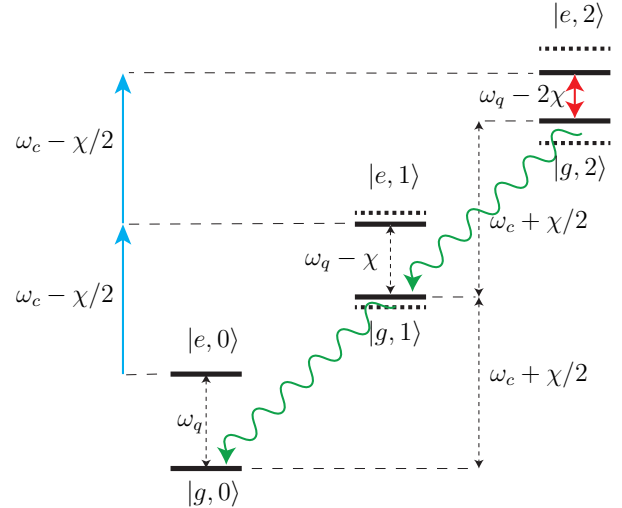

Fig. 2. Diagram for cavity-assisted qubit cooling: 1) (blue arrows) a drive u_c at resonance with the harmonic oscillator's frequency when the qubit is in the excited state (frequency $\omega_c - \chi/2$), drives the cavity to a coherent state with an average photon number of $\bar{n}$ (here $\bar{n} = 2$); 2) (red arrow) a drive u_q at resonance with the qubit's frequency when the harmonic oscillator is in the Fock state $|\bar{n}\rangle$ (frequency $\omega_q - \bar{n}\chi$) induces oscillations between $|e\rangle \otimes |\bar{n}\rangle$ and $|g\rangle \otimes |\bar{n}\rangle$; 3) (green arrows) the population in $|g\rangle \otimes |\bar{n}\rangle$ that does not see the drive u_c gradually decays at a rate κ towards $|g\rangle \otimes |0\rangle$.

where $u_q \in \mathbb{C}$ and $u_c \in \mathbb{C}$ are the two complex control amplitudes (representing the two drive's phase and power). Here $\sigma_- = |g\rangle\langle e|$ and $\sigma_+ = |e\rangle\langle g|$ are, respectively, the qubit's lowering and raising operators. Neglecting the decoherence of the qubit, which we assume to happen at a much slower rate than that of the harmonic oscillator, the system's evolution is modeled by the Lindblad type master equation
$$\frac{d}{dt}\rho = -i[H(t), \rho] + \kappa\mathcal{L}[\mathbf{I}_S \otimes \mathbf{a}]\rho. \quad (4)$$
Here the state space is given by the set of trace-class operators ρ, defined on the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_R$ which are semi-definite positive, Hermitian and of unit trace. We also note that this system is a bilinear control system and that the well-posedness of the equation for smooth bounded control fields u_c and u_q can be derived from Davies [1977].

Throughout this paper we assume the harmonic oscillator's decay rate κ to be much smaller than the coupling strength χ. This regime, which is becoming a very frequent regime in the recent circuit QED experiments is called the strong dispersive regime. While the coupling strength χ represents the separation between the two spectral lines (in a spectroscopy experiment) for the harmonic oscillator's frequency as a function of the qubit state, the decay rate κ provides the linewidth of each of these spectral lines. The assumption $\chi \gg \kappa$ corresponds then to the case of well-resolved spectral lines where one can selectively control the harmonic oscillator's state conditioned on the qubit state. The idea of the cooling scheme is then as follows (see also Fig. 2):
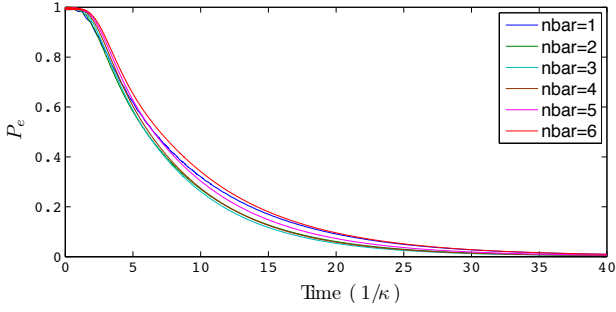
Fig. 3. The population of the excited state $\mathrm{Tr}\left((|e\rangle\langle e|\otimes \mathbf{I}_R)\rho\right)$ for various choices of $\bar{n}$. Following a numerical optimization, we change the qubit drive's amplitude $|u_q|$ according to the choice of $\bar{n}$: $|u_q| = \bar{n}\kappa/2$. These choices ensure near-optimal convergence rates which turn out to be well-approximated by $\kappa/7$ and are independent of $\bar{n}$.

(1) By applying a drive $u_c(t) = |u_c|e^{-i(\omega_c-\chi/2)t}$ at resonance with the harmonic oscillator's frequency when the qubit is in the excited state, and of appropriate constant amplitude $|u_c|$, we drive the harmonic oscillator with a rate of κ to an average number of photons of $\bar{n}$ conditioned on the qubit state being $|e\rangle$. This drive does not affect the harmonic oscillator's state when the qubit is in the ground state;

(2) We also apply a drive $u_q(t) = |u_q|e^{-i(\omega_q-\bar{n}\chi)t}$ at resonance with the qubit's frequency when the harmonic oscillator is in the Fock state $|\bar{n}\rangle$. In this way, we induce oscillations between the state $|e\rangle\otimes|\bar{n}\rangle$ and $|g\rangle\otimes|\bar{n}\rangle$. The frequency of these oscillations is given by $|u_q|$;

(3) During the oscillations between $|e\rangle\otimes|\bar{n}\rangle$ and $|g\rangle\otimes|\bar{n}\rangle$, the population in $|g\rangle\otimes|\bar{n}\rangle$ that does not see the drive $u_c(t)$ gradually decays at a rate κ towards $|g\rangle\otimes|0\rangle$.

In three steps, we have therefore pumped the population from the state $|e\rangle\otimes|0\rangle$ to the state $|g\rangle\otimes|0\rangle$. Note that the pumping rate in this scheme should be given by an aggregate of the decay rate κ and the oscillation rate $|u_q|$. By assuming both these rates much stronger than the qubit's heating rate (that we have neglected in the dynamics (4)), we can cool the qubit much faster than it absorbs thermal photons and therefore we would decrease its temperature Geerlings et al. [2013]. Here, however, neglecting the qubit's heating dynamics, we do not discuss this temperature reduction. We treat this protocol solely as a stabilization scheme, studying its convergence rate through numerical simulations and proving analytically its convergence. Before passing to the details of a mathematical proof of convergence, let us illustrate the performance of the protocol by some numerical simulations.

The simulations of Fig. 3 illustrate the convergence of the scheme. Here, we have simulated the Lindblad master equation (4) considering the parameters $\chi = 10\kappa$ (the simulations are done in the rotating frame of the Hamiltonian $\frac{\omega_q}{2}\sigma_z \otimes \mathbf{I}_R + \omega_c \mathbf{I}_S \otimes \mathbf{a}^\dagger\mathbf{a}$): each curve corresponds to a particular choice of $\bar{n}$; a numerical study shows that for a particular choice of $\bar{n}$, the optimal choice of the qubit's drive amplitude $|u_q|$ (ensuring a near-optimal convergence rate) is near $\bar{n}\kappa/2$. Indeed, as it appears in the simulations of Fig. 3 this near-optimal convergence rate

is well-approximated by $\kappa/7$ and is independent of $\bar{n}$. An analytical study of this optimal convergence rate remains to be done.

4. CONVERGENCE PROOF

While all the results of this section can be proven in the infinite dimensional framework of the dynamics (4), for simplicity sakes we consider only a finite dimensional approximation. Indeed, we truncate the Fock space to the space $\mathcal{H}_R^{N_{\max}}$ spanned by $(|n\rangle)_{0\leq n\leq N_{\max}}$, where $N_{\max}$ is a maximum number of photons much larger than $\bar{n}$. Therefore, ρ is a $2(N_{\max}+1) \times 2(N_{\max}+1)$ Hermitian non-negative matrix defined on $\mathcal{H}_S \otimes \mathcal{H}_R^{N_{\max}}$ with unit trace. Also the annihilation operator $\mathbf{a}$ is an upper-triangular matrix with $(\sqrt{n})_{1\leq n\leq N_{\max}}$ as upper diagonal, the remaining elements being 0.

We prove the following theorem on the convergence of the engineered system towards a small neighborhood of the ground state $\Pi_{g0} = (|g\rangle\otimes|0\rangle)(\langle g|\otimes\langle 0|)$.

Theorem 1. Consider the system (4) where the drives $u_c(t)$ and $u_q(t)$ are simply given by $u_c(t) = \frac{\kappa\sqrt{\bar{n}}}{2}e^{-i(\omega_c-\chi/2)t}$ and $u_q(t) = |u_q|e^{-i(\omega_q-\bar{n}\chi)t}$ and $\bar{n} \in \{1,2\cdots\}$. Furthermore, assume that $\kappa\frac{\sqrt{\bar{n}}}{2}$, $|u_q|$ are much smaller than χ and take $\epsilon = \max\left(\frac{\sqrt{\bar{n}}}{2}\kappa, |u_q|\right)/\chi$. Then, for small enough ϵ, the dynamics (4) admits a unique globally asymptotically stable periodic orbit $\gamma_\epsilon = \Pi_{g0} + O(\epsilon)$.

Proof. Let us start by re-writing the dynamics in the rotating frame of the free Hamiltonian H_0. By defining $\tilde{\rho} = e^{itH_0}\rho e^{-itH_0}$, and applying the operator identities (derived from Campbell-Baker-Hausdorff formula):

$$\tilde{\mathbf{a}} = e^{itH_0}(\mathbf{I}_S \otimes \mathbf{a})\, e^{-itH_0} = e^{-it\omega_c}(\cos(\tfrac{t\chi}{2})\mathbf{I}_S + i\sin(\tfrac{t\chi}{2})\sigma_z) \otimes \mathbf{a},$$

$$\tilde{\sigma}_- = e^{itH_0}(\sigma_- \otimes \mathbf{I}_R)e^{-itH_0} = \sigma_- \otimes \exp\left(-it(\omega_q - \chi\mathbf{a}^\dagger\mathbf{a})\right),$$

we have

$$\frac{d}{dt}\tilde{\rho} = -i[u_q(t)\tilde{\sigma}_+ + u_q^*(t)\tilde{\sigma}_-, \tilde{\rho}]$$
$$- i[u_c(t)\tilde{\mathbf{a}}^\dagger + u_c^*(t)\tilde{\mathbf{a}}, \tilde{\rho}] + \kappa\mathcal{L}[\tilde{\mathbf{a}}]\tilde{\rho}.$$

Averaging the terms of frequency χ in the right hand side of this equation, we find the averaged dynamics

$$\frac{d}{dt}\rho_{\mathrm{av}} = -i|u_q|[\sigma_x \otimes \Pi_{\bar{n}}, \rho_{\mathrm{av}}] - i\frac{\kappa\sqrt{\bar{n}}}{2}[\Pi_e \otimes (\mathbf{a}+\mathbf{a}^\dagger), \rho_{\mathrm{av}}]$$
$$+ \kappa\mathcal{L}[\Pi_g \otimes \mathbf{a}]\rho_{\mathrm{av}} + \kappa\mathcal{L}[\Pi_e \otimes \mathbf{a}]\rho_{\mathrm{av}}, \quad (5)$$

where $\Pi_g = |g\rangle\langle g|$, $\Pi_e = |e\rangle\langle e|$ and $\Pi_{\bar{n}} = |\bar{n}\rangle\langle \bar{n}|$. Applying the averaging theorem Guckenheimer and Holmes [1983], we only need to prove that the averaged dynamics (5) is globally asymptotically stable at Π_{g0}. Note, in particular, that the diagram of Fig. 2 actually represents this averaged dynamics: qubit oscillating when the cavity is in the Fock state $|\bar{n}\rangle\langle\bar{n}|$ and cavity driven when the qubit is in the excited state $|e\rangle\langle e|$.

In the aim of proving the asymptotic convergence of ρ_{av} towards Π_{g0}, we define three projection operators on the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_R^{N_{\max}}$:

$$\Pi_{g,>\bar{n}} = \Pi_g \otimes \sum_{m>\bar{n}} \Pi_m, \qquad \Pi_{g,<\bar{n}} = \Pi_g \otimes \sum_{m<\bar{n}} \Pi_m,$$

$$\Pi_\perp = \Pi_g \otimes \Pi_{\bar{n}} + \Pi_e \otimes \mathbf{I}_R.$$

Noting that $\Pi_{g,>\bar{n}} + \Pi_{g,<\bar{n}} + \Pi_\perp = \mathbf{I}_S \otimes \mathbf{I}_R$, we have
$$\rho_{\mathrm{av}}(t) = (\Pi_{g,>\bar{n}} + \Pi_{g,<\bar{n}} + \Pi_\perp)\rho_{\mathrm{av}}(t)(\Pi_{g,>\bar{n}} + \Pi_{g,<\bar{n}} + \Pi_\perp).$$
As the matrix ρ_{av} is a non-negative Hermitian matrix one can define its square root $\sqrt{\rho_{\mathrm{av}}}$ which is also a non-negative Hermitian matrix. The idea will be to prove that $\Pi_{g,>\bar{n}}\sqrt{\rho_{\mathrm{av}}}$ and $\Pi_\perp \sqrt{\rho_{\mathrm{av}}}$ converge to the zero matrix, and then to conclude by proving that $\Pi_{g,<\bar{n}}\rho_{\mathrm{av}}(t)\Pi_{g,<\bar{n}}$ converges to Π_{g0}.

We start by proving that $\Pi_{g,>\bar{n}}\sqrt{\rho_{\mathrm{av}}}$ converges to zero. Noting that $(\Pi_{g,>\bar{n}}\sqrt{\rho_{\mathrm{av}}})(\Pi_{g,>\bar{n}}\sqrt{\rho_{\mathrm{av}}})^* = \Pi_{g,>\bar{n}}\rho_{\mathrm{av}}\Pi_{g,>\bar{n}}$, we only need to show that $\mathrm{Tr}\,(\Pi_{g,>\bar{n}}\rho_{\mathrm{av}}\Pi_{g,>\bar{n}})$ converges to zero. With this aim, we prove the stronger statement that

$$\mathrm{Tr}\left(\mathbf{a}^\dagger \mathbf{a}\Pi_{g,>\bar{n}}\rho_{\mathrm{av}}(t)\Pi_{g,>\bar{n}}\right) \to 0 \qquad \text{as } t \to \infty. \quad (6)$$

Indeed, since $\mathbf{a}^\dagger\mathbf{a}\Pi_n = n\Pi_n$, the statement (6) clearly implies that $\mathrm{Tr}\,(\Pi_{g,>\bar{n}}\rho_{\mathrm{av}}\Pi_{g,>\bar{n}})$ also converges to 0. Inserting the equation (5), and applying simple relations such as $\Pi_{g,>\bar{n}}\Pi_e = \Pi_{g,>\bar{n}}\Pi_{\bar{n}} = 0$, we have

$$\frac{d}{dt}\mathbf{a}^\dagger\mathbf{a}\Pi_{g,>\bar{n}}\rho_{\mathrm{av}}(t)\Pi_{g,>\bar{n}} = \kappa\Pi_{g,>\bar{n}}\mathcal{L}[\mathbf{I}_S \otimes \mathbf{a}]\rho_{\mathrm{av}}(t)\Pi_{g,>\bar{n}}.$$

Taking the trace of this equation and applying the identity $\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(BA)$, and the fact that the operators $\mathbf{a}^\dagger\mathbf{a}$ and $\Pi_{g,>\bar{n}}$ commute, we have

$$\frac{d}{dt}\mathrm{Tr}\left(\mathbf{a}^\dagger\mathbf{a}\Pi_{g,>\bar{n}}\rho_{\mathrm{av}}(t)\Pi_{g,>\bar{n}}\right) = \kappa\Big(\mathrm{Tr}\left((\mathbf{a}^\dagger)^2\mathbf{a}\Pi_{g,>\bar{n}}\mathbf{a}\rho_{\mathrm{av}}(t)\Pi_{g,>\bar{n}}\right)$$
$$- \mathrm{Tr}\left((\mathbf{a}^\dagger\mathbf{a})^2\Pi_{g,>\bar{n}}\rho_{\mathrm{av}}(t)\Pi_{g,>\bar{n}}\right)\Big).$$

We apply now the operator identity

$$(\mathbf{a}^\dagger)^2\mathbf{a}\Pi_{g,>\bar{n}}\mathbf{a} - (\mathbf{a}^\dagger\mathbf{a})^2\Pi_{g,>\bar{n}} =$$
$$-\mathbf{a}^\dagger\mathbf{a}\Pi_{g,>\bar{n}} - \bar{n}(\bar{n}+1)\Pi_g \otimes \Pi_{\bar{n}+1}.$$

Therefore, we have

$$\frac{d}{dt}\mathrm{Tr}\left(\mathbf{a}^\dagger\mathbf{a}\Pi_{g,>\bar{n}}\rho_{\mathrm{av}}(t)\Pi_{g,>\bar{n}}\right) = -\kappa\mathrm{Tr}\left(\mathbf{a}^\dagger\mathbf{a}\Pi_{g,>\bar{n}}\rho_{\mathrm{av}}(t)\Pi_{g,>\bar{n}}\right)$$
$$- \kappa\bar{n}(\bar{n}+1)\mathrm{Tr}\left(\Pi_g \otimes \Pi_{\bar{n}+1}\rho_{\mathrm{av}}(t)\Pi_g \otimes \Pi_{\bar{n}+1}\right),$$

which clearly proves the statement (6) and even ensures an exponential convergence rate of κ.

Assuming $\Pi_{g,>\bar{n}}\sqrt{\rho_{\mathrm{av}}} = 0$, let us now prove that $\Pi_\perp\sqrt{\rho_{\mathrm{av}}}$ converges to zero, or equivalently $\mathrm{Tr}\,(\Pi_\perp\rho_{\mathrm{av}}\Pi_\perp)$ converges to zero. Defining $\rho_\perp = \Pi_\perp\rho_{\mathrm{av}}\Pi_\perp$, and applying the dynamics (5), we have

$$\frac{d}{dt}\rho_\perp = -i|u_q|[\sigma_x \otimes \Pi_{\bar{n}}, \rho_\perp] - i\frac{\kappa\sqrt{\bar{n}}}{2}[\Pi_e \otimes (\mathbf{a}+\mathbf{a}^\dagger), \rho_\perp]$$
$$- \bar{n}\kappa\Pi_g \otimes \Pi_{\bar{n}}\rho_\perp\Pi_g \otimes \Pi_{\bar{n}} + \kappa\mathcal{L}[\Pi_e \otimes \mathbf{a}]\rho_\perp, \quad (7)$$

Taking the trace of this equation, we have

$$\frac{d}{dt}\mathrm{Tr}\,(\rho_\perp) = -\bar{n}\kappa\mathrm{Tr}\,(\Pi_g \otimes \Pi_{\bar{n}}\rho_\perp(t)\Pi_g \otimes \Pi_{\bar{n}}) \le 0.$$

Therefore, by LaSalle's invariance principle, the Ω-limit set for the dynamics (7) is given by the largest invariant set included in the set $\{\rho_\perp \mid \rho_\perp = \rho_\perp^\dagger, \rho_\perp \ge 0, \mathrm{Tr}\,(\Pi_g \otimes \Pi_{\bar{n}}\rho_\perp(t)\Pi_g \otimes \Pi_{\bar{n}}) = 0\}$. Noting that by invariance, one should also have $\Pi_e \otimes \Pi_{\bar{n}}\rho_\perp\Pi_e \otimes \Pi_{\bar{n}} = 0$, one has the following dynamics for $\rho_e = \Pi_e\rho_\perp\Pi_e$ in the invariant set:

$$\frac{d}{dt}\rho_e = -i\frac{\kappa\sqrt{\bar{n}}}{2}[\mathbf{I}_S \otimes (\mathbf{a}+\mathbf{a}^\dagger), \rho_e] + \kappa\mathcal{L}[\mathbf{I}_S \otimes \mathbf{a}]\rho_e, \quad (8)$$

This is the dynamics for a driven damped harmonic oscillator whose solution converges to the state $\mathrm{Tr}\,(\rho_e(0))\,|\alpha\rangle\langle\alpha|$, where $|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$ with $\alpha = i\sqrt{\bar{n}}$ (for the sake of completeness, we provide a proof of this result in the Appendix). Since $\Pi_{\bar{n}}\rho_e\Pi_{\bar{n}}$ is necessarily zero within the invariant set, we have $\mathrm{Tr}\,(\rho_e) = 0$ within the invariant set which finishes the proof of the statement $\Pi_\perp\sqrt{\rho_{\mathrm{av}}(t)} \to 0$ as $t \to \infty$.

We can therefore restrict ourselves to the dynamics of $\rho_{g,<\bar{n}} = \Pi_{g,<\bar{n}}\rho_{\mathrm{av}}\Pi_{g,<\bar{n}}$. By (5), we have

$$\frac{d}{dt}\rho_{g,<\bar{n}} = \kappa\mathcal{L}[\mathbf{I}_S \otimes \mathbf{a}]\rho_{g,<\bar{n}}.$$

Once again applying the result of the Appendix $\rho_{g,<\bar{n}}$ converges at a rate κ to $|g\rangle\langle g| \otimes |0\rangle\langle 0|$.

$\square$

5. CONCLUSION AND FURTHER DIRECTIONS

Inspired by quantum reservoir engineering techniques, we have proposed a cooling scheme for reseting and stabilizing a single qubit around its ground state. The proposed protocol has been, very recently, successfully experimented Geerlings et al. [2013] and provides a very promising method for rapid cooling of most super-conducting qubits. Furthermore, we have recently extended this scheme to the case of preparing and protecting (against decoherence) a maximally entangled state between two qubits. This extension will be presented in a forthcoming paper.

Appendix A. DRIVEN DAMPED QUANTUM HARMONIC OSCILLATOR

Lemma 2. Consider the dynamics

$$\frac{d}{dt}\rho(t) = -i\epsilon_c[\mathbf{a}+\mathbf{a}^\dagger, \rho(t)] + \kappa\mathcal{L}[\mathbf{a}]\rho(t) \quad (A.1)$$

on the Hilbert space $\mathcal{H}_R$ of the cavity mode. The solution $\rho(t)$ converges at a rate κ to the coherent state $|\alpha\rangle\langle\alpha|$, where $|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$ with $\alpha = 2i\epsilon_c/\kappa$.

Proof. We consider the Lyapunov equation
$$V(\rho) = \mathrm{Tr}\left(\mathbf{a}^\dagger\mathbf{a}D_\alpha^\dagger\rho D_\alpha\right),$$
where $D_\alpha = \exp(\alpha\mathbf{a}^\dagger - \alpha^*\mathbf{a}) : \mathcal{H}_R \to \mathcal{H}_R$ is called the displacement operator. In particular, one has $D_\alpha|0\rangle = |\alpha\rangle$, or equivalently $D_\alpha^\dagger|\alpha\rangle = |0\rangle$ (see e.g. [Haroche and Raimond, 2006, page 118]). Noting that $\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(BA)$, we can write $V(\rho) = \mathrm{Tr}\left(\mathbf{a}D_\alpha^\dagger\rho D_\alpha\mathbf{a}^\dagger\right)$, which together with the positivity of the density matrix ρ implies that $V(\rho) \ge 0$. Also, defining the displaced density matrix $\xi = D_\alpha^\dagger\rho D_\alpha$ (which is still a well-defined density matrix), the Lyapunov function $V(\rho) = \mathrm{Tr}\left(\mathbf{a}^\dagger\mathbf{a}\xi\right)$ indicates the average number of photons for the density matrix ξ. In particular, $V(\rho) = 0$ implies that $\xi = |0\rangle\langle 0|$ and therefore $\rho = D_\alpha|0\rangle\langle 0|D_\alpha^\dagger = |\alpha\rangle\langle\alpha|$.

By inserting the dynamics (A.1), and applying the identity $D_\alpha^\dagger\mathbf{a}D_\alpha = \mathbf{a} - \alpha\mathbf{I}_R$, we have $\frac{d}{dt}\xi = \kappa\mathcal{L}[\mathbf{a}]\xi$. This, together with the commutation relation $[\mathbf{a}, \mathbf{a}^\dagger\mathbf{a}] = \mathbf{a}$, imply

$$\frac{d}{dt}V(\rho) = -\kappa V(\rho),$$

and proves the convergence at a rate κ of ρ towards $|\alpha\rangle\langle\alpha|$.

$\square$

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