Control of Two-wheeled Mobile Robot via Homogeneous Semiconcave Control Lyapunov function

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Abstract: Semiconcave control Lyapunov functions for globally asymptotic stabilizing controllable systems are available. However, a semiconcave control Lyapunov function for nonholonomic systems has not been proposed yet. For a two-wheeled mobile robot, we construct a homogeneous semiconcave control Lyapunov function and a control law with the function. The advantages of the proposed method are confirmed by computer simulation.

1. INTRODUCTION

Stabilization of a nonholonomic system is difficult because the system cannot be stabilized by any static continuous feedback controller. Moreover, there are no smooth control Lyapunov functions available.

For every globally asymptotically stabilizable system, there always exist semiconcave control Lyapunov functions (CLFs)[1]. However, a semiconcave CLF for nonholonomic systems has not been proposed even for a Brockett integrator.

The present paper proposes a semiconcave CLF for a Brockett integrator. Then, we show that the proposed CLF is a homogeneous function with dilation. Moreover, we propose a controller based on the proposed CLF such that the exponential stability is guaranteed.

The chained system is equivalent to the Brockett integrator under a coordinate transformation. We also show a semiconcave CLF for the chained system.

Finally, we apply the proposed CLF and controller for a stabilization problem of a two-wheeled mobile robot. The advantages of the proposed method are confirmed by computer simulation.

This paper is organized as follows. We summarized definitions and basic properties that are used in Section 2. We state the problem discussed in the paper and the main results in Section 3. We demonstrate the effectiveness of the proposed method by computer simulation in Section 4. Section 5 shows the conclusion of this paper.

2. PRELIMINARY

We introduce basic definitions of mathematical terms and their fundamental properties. Throughout the paper, we use a signum function defined as follows:

$$\operatorname{sgn} x = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}. \tag{1}$$

We denote a scalar product by $\langle \cdot, \cdot \rangle$.

2.1 Control System

We consider the following input-affine nonlinear control system

$$\dot{x} = f(x) + g(x)u, \tag{2}$$

where $x \in \mathbb{R}^n$ is a state and $u \in \mathbb{R}^m$ is an input.

Particularly, the following input symmetrically affine system is the center of interest of the paper:

$$\dot{x} = g(x)u. \tag{3}$$

$g_i(x)$ denotes the ith column vector of $g(x)$.

Definition 1. (Carathéodory Solution). [7] Consider the following differential equation:

$$\dot{x} = f(x). \tag{4}$$

A function $x(t)$ is called a Carathéodory solution of (4) on the interval $I \subset [0, +\infty)$ if it is absolutely continuous on every compact subinterval of I and satisfies

$$\dot{x} = f(x(t)) \quad a.e. \quad t \in I. \tag{5}$$

2.2 Homogeneous System[3]

Definition 2. (Dilation) Let $\varepsilon > 0$. The mapping $\Delta_\varepsilon^r x = [\varepsilon^{r_1} x_1, ..., \varepsilon^{r_n} x_n], \forall x \in \mathbb{R}^n \setminus \{0\}$ is said to be a dilation on $\mathbb{R}^n$, where $r = [r_1, r_2, ..., r_n]$ is a constant vector satisfying $0 < r_i < \infty (i = 1, ..., n)$. Note that we often refer to r as a dilation exponent.

Definition 3. (Homogeneous Function) A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be homogeneous of degree $k \in \mathbb{R}$ with respect to the dilation $\Delta_\varepsilon^r x$ if $V(\Delta_\varepsilon^r x) = \varepsilon^k V(x)$.

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Definition 4. (Homogeneous System) System $\dot{x} = f(x) + g(x)u$ is said to be homogeneous of degree $\tau \in \mathbb{R}$ with respect to the dilations $\Delta_\varepsilon^r x$ and $\Delta_\varepsilon^s u$ if $f(\Delta_\varepsilon^r x) + g(\Delta_\varepsilon^r x)\Delta_\varepsilon^s u = \varepsilon^\tau \Delta_\varepsilon^r \{f(x) + g(x)u\}$.

Definition 5. (Homogeneous Norm): The function $\|x\|_{\{r,p\}} = \left(\sum_{i=1}^n |x_i|^{p/r_i}\right)^{1/p}$ $(x \in \mathbb{R}^n)$ is said to be a homogeneous p-norm.

Note that the homogeneous norm is a homogeneous function of degree 1 with respect to dilation exponent r for all $p > 0$.

Lemma 1. We suppose that the system (4) is homogeneous of degree τ and always has a Carathéodory solution for every initial condition, and the origin is asymptotically stable. Then, the following statements are true:

(S1) If $\tau > 0$, there exists a positive constant $d > 0$ such that for any solution $x(t)$ and all $t \geq 0$

$$\|x(t)\|_{\{r,p\}} \leq d\left(1 + \|x(0)\|_{\{r,p\}}^\tau t\right)^{-1/\tau} \|x(0)\|_{\{r,p\}}. \tag{6}$$

(S2) If $\tau = 0$, there exist positive constants $d_1, d_2 > 0$ such that for any solution $x(t)$ and all $t \geq 0$

$$\|x(t)\|_{\{r,p\}} \leq d_1 e^{-d_2 t}\|x(0)\|_{\{r,p\}}. \tag{7}$$

(S3) If $\tau < 0$, the origin is finite-time stable.

For homogeneous functions, the following lemma holds:

Lemma 2. Consider a homogeneous function $V : \mathbb{R}^n \to \mathbb{R}$ of degree $k > 0$ with respect to dilation exponent r. Then if V is positive definite, V is a proper function.

Proof 1. Note that a set $\{x|\ \|x\|_{\{r,2\}} = 1\}$ is compact, and a constant V_1 defined as follows is well defined:

$$V_1 = \min_{x \in \{x|\ \|x\|_{\{r,2\}}=1\}} V(x). \tag{8}$$

Because V is positive definite, $V_1 > 0$. For every $x \in \mathbb{R}^n$, there exists $x_0 \in \{x|\ \|x\|_{\{r,2\}} = 1\}$ and $\varepsilon > 0$ such that $x = \Delta_\varepsilon^r x_0$. Note that the homogeneous norm is a homogeneous function of degree 1; $\|x\|_{\{r,2\}} = \|\Delta_\varepsilon^r x_0\|_{\{r,2\}} = \varepsilon$.

Let L be a positive constant and consider $x \in \mathbb{R}^n$ such that $V(x) \leq L$. Then, the following inequality holds:

$$V(x) = V(\Delta_\varepsilon^r x_0) = \varepsilon^k V(x_0) \geq \varepsilon^k V_1. \tag{9}$$

Therefore,

$$\|x\|_{\{r,2\}} \leq \left(\frac{V(x)}{V_1}\right)^{1/k} \leq \left(\frac{L}{V_1}\right)^{1/k} \tag{10}$$

for every $x \in \{x|V(x) \leq L\}$.

Consequently, the set $\{x \in \mathbb{R}^n|V(x) \leq L\}$ is bounded for every $L > 0$.

Note that V is a continuous function. $V^{-1}([0, L])$ is a closed set. Thus, $\{x \in \mathbb{R}^n|V(x) \leq L\}$ is a bounded closed set. Therefore, V is a proper function.

2.3 Semiconcave Function[4]

Definition 6. (Locally Semiconcave Function) A function $V : X \to \mathbb{R}$ is said to be locally semiconcave with linear modulus if it is continuous and there exists $C \geq 0$ such that $V(x) + V(y) - 2V((x + y)/2) \leq C\|x - y\|^2$ for all $x, y \in X$, where X is an arbitrary convex compact subset of $\mathbb{R}^n$. The constant C above is called a semiconcavity constant for V in X.

Theorem 1. [4] Let $V : X \to \mathbb{R}$ be a locally semiconcave function. Then V can be locally written as the minimum of functions of class C^1. More precisely, for any $X \subset \mathbb{R}^n$ compact, there exist a compact set $\Theta \subset \mathbb{R}^{2n}$ and a continuous function $F : \Theta \times X \to \mathbb{R}$ such that $F(\theta, \cdot)$ is C^1 for any $\theta \in \Theta$, the gradients $D_x F(\theta, \cdot)$ are equicontinuous, and

$$V(x) = \min_{\theta \in \Theta} F(\theta, x), \forall x \in X. \tag{11}$$

Corollary 1. If $V : X \to \mathbb{R}$, with X open convex, is such that $V = V_1 + V_2$, where $V_1 \in C^1(X)$ and V_2 is a locally semiconcave function, V is also a locally semiconcave function.

Proof 2. As V_2 is semiconcave, for every compact set $X \subset \mathbb{R}^n$ V_2 can be written as

$$V_2(x) = \min_{\theta \in \Theta} F(\theta, x), \forall x \in X, \tag{12}$$

where $\Theta \subset \mathbb{R}^{2n}$ is an appropriate compact set. Hence,

$$V(x) = \min_{\theta \in \Theta} [V_1(x) + F(\theta, x)], \forall x \in X. \tag{13}$$

Therefore, V is a locally semiconcave function by Proposition 3.4.1 in [4].

According to Theorem 1, the derivative of F plays an important role in semiconcave function analysis. Hence, we define the disassembled differential defined as follows:

Definition 7. (Disassembled Differential). Suppose that $V : X \to \mathbb{R}$ is a locally semiconcave function. Then, the following set-valued map $\bar{D}V : X \to 2^{T_x X}$ is said to be a disassembled differential of V:

$$\bar{D}V(x) = \left\{ d\bar{V}_\theta(x) \middle| \theta \in \{\theta \in \Theta | V(x) = \bar{V}_\theta(x)\}\right\}. \tag{14}$$

2.4 Locally Semiconcave Control Lyapunov Function

[6]

Definition 8. (Locally Semiconcave Control Lyapunov Function (CLF)) A locally semiconcave control Lyapunov function for system (2) is a locally semiconcave function $V : X \to \mathbb{R}$ such that the following properties hold.

(A1) V is proper; that is, the set $\{x \in X|V(x) \leq L\}$ is compact for every $L > 0$.

(A2) V is positive definite; that is , $V(0) = 0$, and $V(x) > 0$ for all $x \in X \setminus \{0\}$.

(A3) For arbitrary $R_2 > R_1 > 0$, there exist a compact set $\bar{U} \subset U$, a positive real constant Q and a discontinuous mapping $p : X \to T_x X$ such that $p(x) \in \bar{D}V(x)$, and

$$\min\langle p(x), f(x, u)\rangle < -Q,\ \forall x \in \{x|R_1 \leq V(x) \leq R_2\}. \tag{15}$$

3. HOMOGENEOUS CONTROL LYAPUNOV FUNCTION FOR BROCKETT INTEGRATOR

3.1 Brockett integrator[2]

The present paper considers asymptotic stabilization problem of the Brockett integrator defined as follows:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ x_2 u_1 - x_1 u_2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ -x_1 \end{bmatrix} u_2$$
$$= g_1(x)u_1 + g_2(x)u_2, \tag{16}$$

where $x = [x_1, x_2, x_3]^T \in X = \mathbb{R}^3$ is a state and $u = [u_1, u_2]^T \in U = \mathbb{R}^2$ is an input. According to the definition of the homogeneous system, we can confirm that (16) is homogeneous of degree $k = 0$ with respect to the dilation exponent $r = [1, 1, 2]$ and $s = [1, 1]$.

Note that the Brockett integrator is a canonical system of the nonholonomic systems; there does not exist any continuous static feedback controller for asymptotic stabilization at the origin.

On the contrary, every stabilizable system including a nonholonomic system attains a semiconcave CLF; however, a semiconcave CLF for nonholnomic systems was not proposed, even for the Brockett integrator.

In the following main theorem of the paper, we propose a semiconcave homogeneous CLF for a Brockett integrator:

Theorem 2. Consider system (16). Then, the following function is a locally semiconcave homogeneous CLF of degree $\tau = 4$ with respect to dilation exponent $r = [1, 1, 2]$:

$$V(x) = x_1^4 + x_2^4 + \frac{|x_3|^3}{\left(\sqrt{x_1^2 + x_2^2} + \sqrt{|x_3|}\right)^2}. \tag{17}$$

We prove the theorem in the following subsection.

3.2 Proof of Theorem 2

To prove Theorem 2, we use two lemmas.

Lemma 3. The function V defined by (17) is a homogeneous function of degree $k = 4$ with respect to the dilation exponent $r = [1, 1, 2]$.

Proof 3. Let the dilation exponent r be $[1, 1, 2]$, and we can obtain $V(\Delta_\varepsilon^r x) = \varepsilon^4 V(x)$. Therefore, V is a homogeneous function of degree $k = 4$ with respect to r.

Note that (17) is homogeneous with respect to the same dilation exponent as the Brockett integrator (16).

Lemma 4. Function (17) is a locally semiconcave function.

Proof 4. In function (17), the first and the second terms are clearly C^2 functions. According to Corollary 1, function (17) is locally semiconcave if the last term is locally semiconcave.

The last term in (17) can be written as follows:

$$\frac{|x_3|^3}{\left(\sqrt{x_1^2 + x_2^2} + \sqrt{|x_3|}\right)^2} = \min_{\theta \in [0, 2\pi]} \left\{ \frac{|x_3|^3}{\left(x_1 \cos\theta + x_2 \sin\theta + \sqrt{|x_3|}\right)^2} \right\}. \tag{18}$$

Let $F(\theta, x) = |x_3|^3 / \left(x_1 \cos\theta + x_2 \sin\theta + \sqrt{|x_3|}\right)^2$. If $F(\theta, x)$ are C^2 for all $\theta \in [0, 2\pi]$, (17) is locally semiconcave according to Theorem 1.

Indeed F is differentiable in the neighborhood of $[x_1, x_2] = [0, 0]$. The first derivative of F with respect to x can be calculated as follows.

$$\frac{\partial F}{\partial x_1} = \frac{-2|x_3|^3 \cos\theta}{\left(x_1 \cos\theta + x_2 \sin\theta + \sqrt{|x_3|}\right)^3},$$
$$\frac{\partial F}{\partial x_2} = \frac{-2|x_3|^3 \sin\theta}{\left(x_1 \cos\theta + x_2 \sin\theta + \sqrt{|x_3|}\right)^3},$$
$$\frac{\partial F}{\partial x_3} = \frac{\left(3x_1 \cos\theta + 3x_2 \sin\theta + 2\sqrt{|x_3|}\right) x_3^2 \operatorname{sgn} x_3}{\left(x_1 \cos\theta + x_2 \sin\theta + \sqrt{|x_3|}\right)^3}. \tag{19}$$

Note that all of the derivatives are continuously differentiable. Similar to the first derivative, the second derivative F is also continuous. Accordingly, the function F is C^2 for all $\theta \in [0, 2\pi]$.

From the foregoing discussion, it is seen that the function V is a locally semiconcave function.

By using the preceding two lemmas, we can prove Theorem 2.

Proof 5. (Proof of Theorem 2). It is obvious that the function V is positive definite. Then by Lemmas 2 and 3, the function V is proper.

Note that (17) is differentiable except that $[x_1, x_2] = [0, 0]$. Lie derivatives of the functions are obtained as (20), (21), (22), and (23) in the next page. We design a discontinuous function p as follows:

$$p(x) = \begin{cases} \dfrac{\partial V}{\partial x} & ([x_1, x_2] \neq [0, 0]) \\ \dfrac{\partial F}{\partial x}(0, x) & ([x_1, x_2] = [0, 0]) \end{cases}. \tag{24}$$

Then, $\langle p(x), g_1(x)\rangle \neq 0$ and $\langle p(x), g_2(x)\rangle \neq 0$ for all x. Therefore, there exists $u \in U$ such that $\langle p(x), f(x, u)\rangle < 0, \forall x$. According to Lemmas 3 and 4 and Definition 8, V is a locally semiconcave CLF.

Figure 1 illustrates the function V on $x_2 = 0$, Figure 2 the function V on $x_3 = 2.0$, and Figure 3 the function V on $x_3 = 1.0$. By these figures, we can find discontinuity in V on $[x_1, x_2] = [0, 0]$.

3.3 Controller Design

In this paper, we choose the following controller for the Brockett integrator (16).

$$u_1 = -\left|\langle p(x), g_1(x)\rangle\right|^{1/3} \operatorname{sgn}\langle p(x), g_1(x)\rangle,$$
$$u_2 = -\left|\langle p(x), g_2(x)\rangle\right|^{1/3} \operatorname{sgn}\langle p(x), g_2(x)\rangle. \tag{25}$$

Then, the following lemma holds:

Lemma 5. Consider system (16) and controller (25). Then there exists a Carathéodory solution for every $x \in \mathbb{R}^n$.

By Lemma 5, we can apply a standard discussion to prove the asymptotic stability as follows.

$$
\begin{aligned}
L_{g_1}V(x) =& \frac{\partial V}{\partial x_2} + \frac{\partial V}{\partial x_3} x_2 \\
=& 4x_1^3 - \frac{2x_1|x_3|^3}{\left(\sqrt{x_1^2+x_2^2}+\sqrt{|x_3|}\right)^3 \sqrt{x_1^2+x_2^2}} + \frac{x_2\left(3\sqrt{x_1^2+x_2^2}+2\sqrt{|x_3|}\right)x_3^2 \operatorname{sgn} x_3}{\left(\sqrt{x_1^2+x_2^2}+\sqrt{|x_3|}\right)^3}, \quad (20)
\end{aligned}
$$

$$
\begin{aligned}
L_{g_2}V(x) =& \frac{\partial V}{\partial x_2} - \frac{\partial V}{\partial x_3} x_1 \\
=& 4x_2^3 - \frac{2x_2|x_3|^3}{\left(\sqrt{x_1^2+x_2^2}+\sqrt{|x_3|}\right)^3 \sqrt{x_1^2+x_2^2}} - \frac{x_1\left(3\sqrt{x_1^2+x_2^2}+2\sqrt{|x_3|}\right)x_3^2 \operatorname{sgn} x_3}{\left(\sqrt{x_1^2+x_2^2}+\sqrt{|x_3|}\right)^3}. \quad (21)
\end{aligned}
$$

$$
L_{g_1}F(\theta, x) = -\frac{2|x_3|^3 \sin\theta}{\left(x_1\cos\theta + x_2\sin\theta + \sqrt{|x_3|}\right)^3} + \frac{x_2\left(3x_1\cos\theta + 3x_2\sin\theta + 2\sqrt{|x_3|}\right)x_3^2 \operatorname{sgn} x_3}{\left(x_1\cos\theta + x_2\sin\theta + \sqrt{|x_3|}\right)^3}, \quad (22)
$$

$$
L_{g_2}F(\theta, x) = -\frac{2|x_3|^3 \cos\theta}{\left(x_1\cos\theta + x_2\sin\theta + \sqrt{|x_3|}\right)^3} - \frac{x_1\left(3x_1\cos\theta + 3x_2\sin\theta + 2\sqrt{|x_3|}\right)x_3^2 \operatorname{sgn} x_3}{\left(x_1\cos\theta + x_2\sin\theta + \sqrt{|x_3|}\right)^3}. \quad (23)
$$

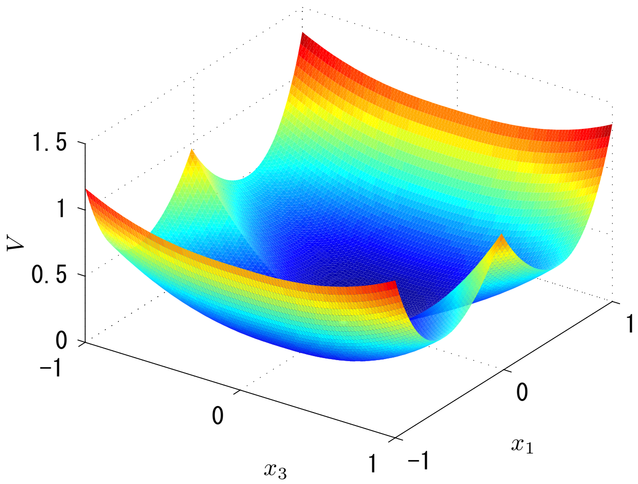

Fig. 1. CLF for a Brockett integrator on $x_2 = 0$

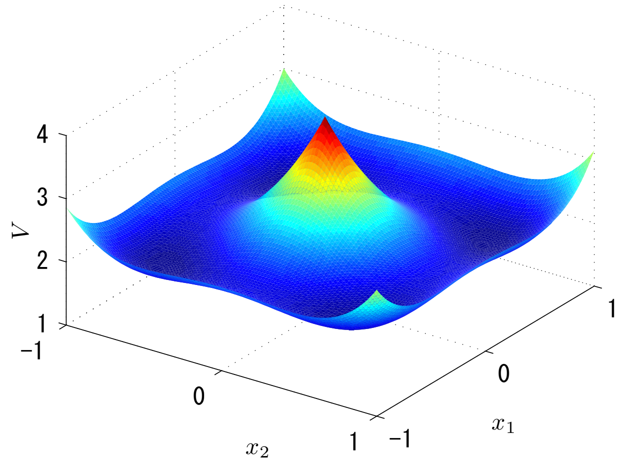

Fig. 2. CLF for a Brockett integrator on $x_3 = 2.0$

Theorem 3. Consider system (16) and controller (25). Then the origin is exponentially stable.

Proof 6.

$$
\dot{V} = -\left|\langle p(x), g_1(x)\rangle\right|^{4/3} - \left|\langle p(x), g_2(x)\rangle\right|^{4/3} < 0. \quad (26)
$$

Moreover, there does not exist a sequence such that $-\left|\langle p(x_i), g_1(x_i)\rangle\right|^{4/3} - \left|\langle p(x_i), g_2(x_i)\rangle\right|^{4/3} \to 0$ if $x_i \to x_n$ except $x_n = 0$. Hence, the origin is globally asymptotically stable.

Furthermore, the closed-loop system is homogeneous of degree 0 with respect to dilation exponent $[1, 1, 2]$. Therefore, the origin is exponentially stable.

3.4 Chained system

A chained system defined as follows is another canonical form of nonholonomic control systems:

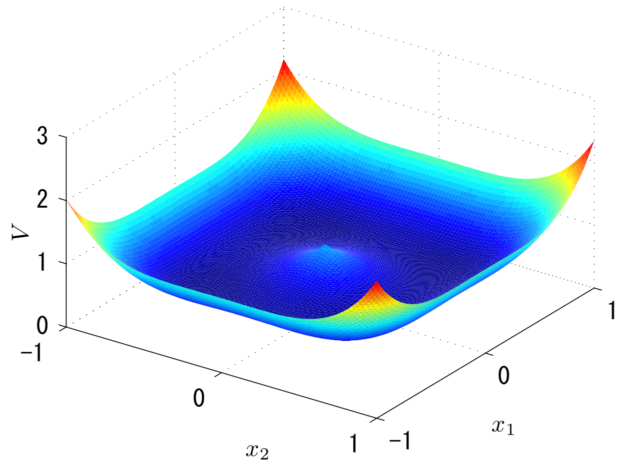

Fig. 3. CLF for a Brockett integrator on $x_3 = 1.0$

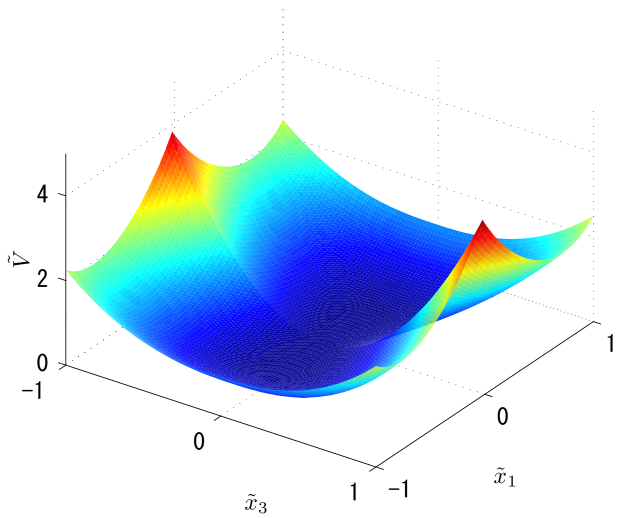

Fig. 4. CLF for a chained system on $\tilde{x}_2 = 0$

$$\dot{\tilde{x}} = \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \tilde{x}_2 u_1 \end{bmatrix}. \quad (27)$$

The chained system (27) is equivalent to the Brockett integrator (16) under the following coordinate transformation :

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ (x_3 + x_1 x_2)/2 \end{bmatrix}. \quad (28)$$

According to (17) and (28), a semiconcave CLF for a chained system is obtained as follows:

$$\tilde{V}(\tilde{x}) = \tilde{x}_1^4 + \tilde{x}_2^4 + \frac{|2\tilde{x}_3 - \tilde{x}_1\tilde{x}_2|^3}{\left(\sqrt{\tilde{x}_1^2 + \tilde{x}_2^2} + \sqrt{|2\tilde{x}_3 - \tilde{x}_1\tilde{x}_2|}\right)^2}. \quad (29)$$

Figure 4 illustrates the function $\tilde{V}$ on $x_2 = 0$.

The function $\tilde{V}$ on the chained system holds the same properties as one of the Brockett integrators; homogeneity, semiconcavity, and CLF are held.

4. APPLICATION TO TWO-WHEELED MOBILE ROBOT

4.1 Controller Design

In this section, we apply the proposed method to position control of a two-wheeled mobile robot. We consider a two-wheeled mobile robot as illustrated in Figure 5. We assume that each wheel on the robot can move with the desired velocity without slipping. $[\hat{x}_1, \hat{x}_2] \in \mathbb{R}^2$ is the Cartesian coordinate of the center of the robot, and $\hat{x}_3 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is the angle between the heading direction and $\hat{x}_1$-axis. $\hat{u}_1$ is the velocity of the right wheel, and $\hat{u}_2$ that of the left wheel. Thus, the state vector of the robot $\hat{x} = [\hat{x}_1, \hat{x}_2, \hat{x}_3]^T \in \hat{X} = \mathbb{R}^2 \times (-\frac{\pi}{2}, \frac{\pi}{2})$, the input vector $\hat{u} = [\hat{u}_1, \hat{u}_2]^T \in \hat{U} = \mathbb{R}^2$. In addition, v, ω and W denote the linear velocity of the robot, the angular velocity, and the distance between the right and the left wheels of the robot, respectively. Then, the following relations hold.

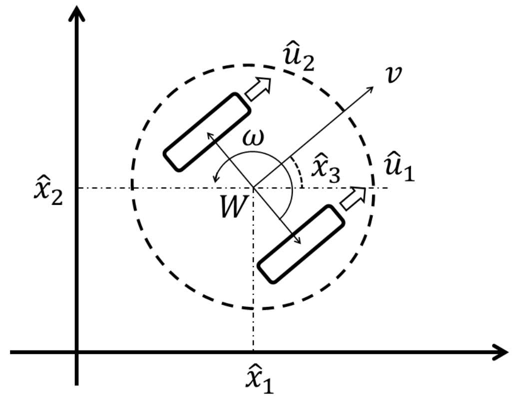

Fig. 5. Model of a two-wheeled robot

$$v = (\hat{u}_1 + \hat{u}_2)/2, \qquad \omega = (\hat{u}_1 - \hat{u}_2)/W. \quad (30)$$

The control system of a two-wheeled robot model with v and ω is obtained as follows:

$$\dot{\hat{x}} = \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} = \begin{bmatrix} v\cos\hat{x}_3 \\ v\sin\hat{x}_3 \\ \omega \end{bmatrix}. \quad (31)$$

By equation (30), this system (31) is equivalent to

$$\dot{\hat{x}} = \begin{bmatrix} \cos\hat{x}_3/2 & \cos\hat{x}_3/2 \\ \sin\hat{x}_3/2 & \sin\hat{x}_3/2 \\ 1/W & -1/W \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix}. \quad (32)$$

System (31) is transformed into a chained system by the following coordinate and input transformations:

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} \hat{x}_1 \\ \tan\hat{x}_3 \\ \hat{x}_2 \end{bmatrix}, \quad (33)$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} (\hat{u}_1 + \hat{u}_2)\cos\hat{x}_3/2 \\ (\hat{u}_1 - \hat{u}_2)\sec^2\hat{x}_3/W \end{bmatrix}. \quad (34)$$

In the previous section, a Brockett integrator is transformed into a chained system. As a result, coordinate transformation (33) and input transformation (34), system (32) is transformed into a Brockett integrator with the following coordinate and input transformation:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \hat{x}_1 \\ \tan\hat{x}_3 \\ 2\hat{x}_2 - \hat{x}_1\tan\hat{x}_3 \end{bmatrix}, \quad (35)$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} (\hat{u}_1 + \hat{u}_2)\cos\hat{x}_3/2 \\ (\hat{u}_1 + \hat{u}_2)\sec\hat{x}_3 \end{bmatrix}. \quad (36)$$

We apply the proposed controller (25) to the Brocket integrator. Then, the original input $\hat{u}$ of the two-wheeled mobile robot is obtained as follows:

$$\hat{u} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} = \begin{bmatrix} \sec x_3 u_1 + W\cos^2 x_3 u_2/2 \\ \sec x_3 u_1 - W\cos^2 x_3 u_2/2 \end{bmatrix}. \quad (37)$$

4.2 Computer Simulation

We show the result of computer simulation in this subsection. The initial value is set at $x = [-1.5, 1.0, \pi/3]^T$ [m,m,rad]. Figure 6 shows the time histories of the state variables and Figure 7 those of the inputs. They confirm

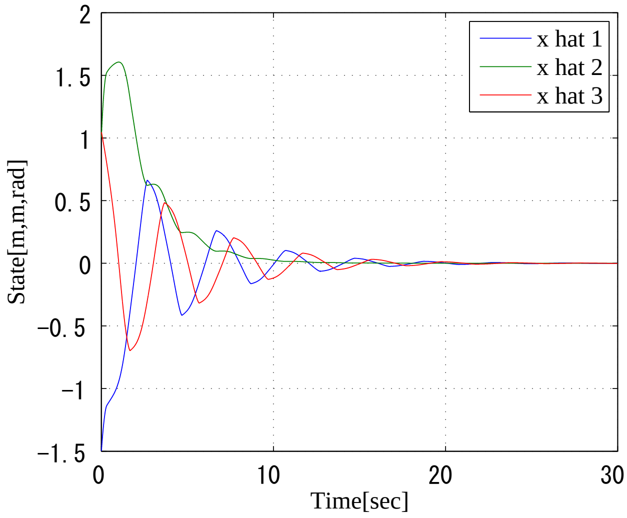

Fig. 6. State of the system with a controller.

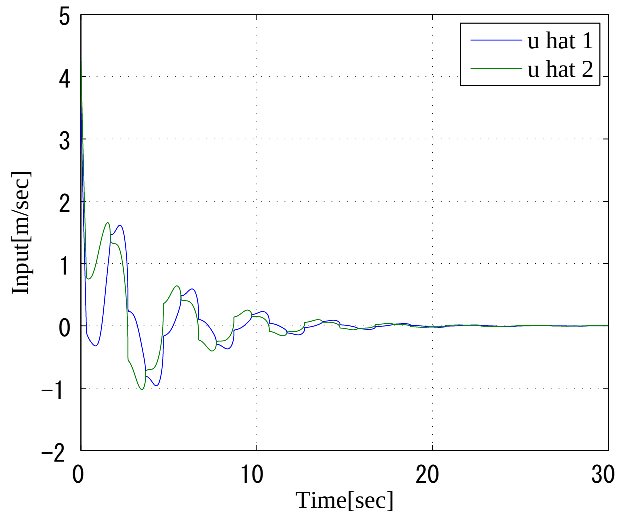

Fig. 7. Input of the system with a controller.

that the state converges to the origin. Although the control law itself is discontinuous, we can find the inputs change smoothly. If initial value $x_3 \notin (-\pi/2, \pi/2)$, $x_3 \to \pi$. This is due to the function tangent of transformation (35).

Figure 8 depicts the trajectory in the $[x_1, x_2] \in \mathbb{R}^2$. The proposed trajectory is nonsmooth and not the best physical solution. Although the best solution is a future task, note that the state and the inputs smoothly change.

5. CONCLUSION

The design of a control law for a nonholonomic system was successful. The control law uses a semiconcave control Lyapunov function. We applied this it to a two-wheeled mobile robot. The advantages of the proposed methods are confirmed by computer simulation.

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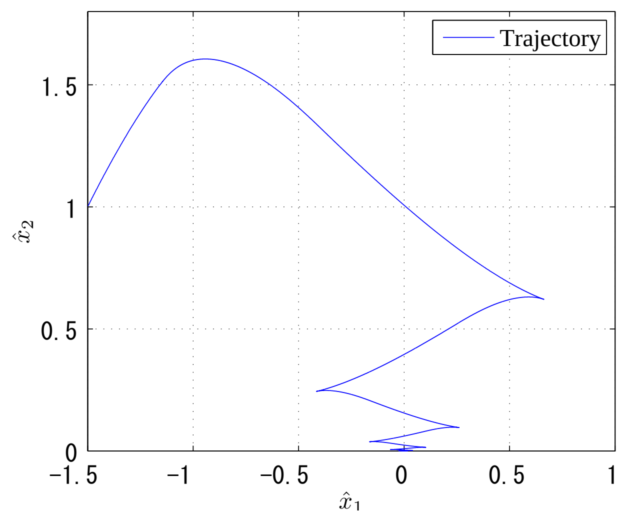

Fig. 8. Trajectory in the $\mathbb{R}^2$ plane from $(-1.5, 1.0, \pi/3)$.

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