Approximation of reachability sets for nonlinear unicycle control system using the comparison principle ⋆

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Abstract: In this paper the application of the comparison principle for Hamilton-Jacobi equations to a particular nonlinear control system is discussed. Two classes of approximations of the reachability sets for this system are constructed. Numerical examples of the reachability set approximations and the solutions to control synthesis problem are given to illustrate the proposed approach.

1. INTRODUCTION

The computation of *reachability sets* which consist of all system states reachable with some of the available controls is a key problem in the mathematical theory of controlled processes (Krasovski [1964, 1971], Kurzhanski et al. [2002], Chernousko [1994], Lygeros et al. [1999], Krogh and Stursberg [2003], Patsko et al. [2003]). In particular, due to them it is possible to solve the *problem of control synthesis*.

It is known that the reachability problem may be reduced to the investigation of appropriate problems of dynamic optimization. Here we introduce the value function $V(t, x)$ as satisfying in some generalized sense a corresponding Hamilton-Jacobi equation, such that the reachable set $X[t] = X(t, t_0, X^0)$ from initial states $x(t_0) = x^0 \in X^0$ is a level set of $V(t, x)$ (Kurzhanski et al. [2001]). Hence the reachability problem may be solved by finding the value function. For general nonlinear systems this may be done by some numerical methods for solving the Hamilton-Jacobi equation (e.g. Sethian [1999], Osher and Fedkiw [2002]). However, these methods are not applicable to the problems of high dimension as the amount of computations needed to preserve the same accuracy grows exponentially with dimension. On the other hand, for linear systems many methods of reachability set approximation were developed. They represent reachability sets as unions or intersections of simpler standard-shape domains, such for example, as ellipsoids (Kurzhanski et al. [2002]) or parallelotopes (e.g. Kostousova [1998]). These methods have the possibility to parallelization as the estimates in those methods are computed independently. This property allows to solve control synthesis problems of quite high dimension (see control synthesis example for 500-dimensional linear system in Daryin and Kurzhanski [2013]). In fact, these standard-shape estimates could be obtained via comparison principle (Kurzhanski [2006]). The comparison principle method could be thought as general approach of deriving such estimates. However, for nonlinear systems the problem of deriving such estimates is much more complicated as we have to consider the properties of specific classes of nonlinear systems to obtain suitable families of estimates that produce good approximations for reachability sets.

Here we apply comparison principle to construct reachability sets for a concrete five-dimensional nonlinear control system. It is so called dynamic unicycle system. The properties of three-dimensional simplified analogue of this system, so called kinematic unicycle system, has been studied in Murray et al. [1994], A. De Luca et al. [1998]. Its reachability sets were constructed in Patsko et al. [2003].

In this paper we obtain two families of estimates to the value function of reachability problem for this system: one family of nonsmooth estimates (both lower and upper) and one family of more rough quadratic estimates that corresponds to external ellipsoidal estimates of the reachability set. Then we use these two families of estimates to construct internal and external approximations of the reachability and to solve the problem of control synthesis. For the first family we represent the estimate as a function $f_1(x) + f_2(\varphi)$ with separated variables x being four-dimensional and φ being one-dimensional. Using the comparison principle we obtain ordinary differential equations for the parameters of f_1. To find f_2 we numerically solve one-dimensional Hamilton-Jacobi equation that is much simpler than the original HJB equation in the five-dimensional phase space. For the second family we use the comparison principle to obtain quadratic lower estimates of the value function. In fact, using the latter estimates we can construct not only the reachability set approximations but also the reachability tube approximations as those estimates for different values of t are computed recursively.

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It should be noted than the given scheme allows to obtain similar families of ellipsoidal estimates for other nonlinear systems. However, some part of the estimates (depending on the nonlinear control system under investigation) may eventually degenerate or become too rough. It should mentioned also that the given approach may be developed further to be used in the problems of team control for groups of unicycles, but this is beyond the scope of this paper and may be one of the topics for future research.

2. COMPARISON PRINCIPLE

In this section let us state the comparison principle in the form that will be used in the following. This is a slight generalization of the comparison principle given in Kurzhanski [2006]. Consider two Hamilton-Jacobi equations with the Hamiltonians $H(t,x,p)$ and $\mathcal{H}(t,x,p)$ respectively. We suppose that conditions of the uniqueness theorem for these equations are satisfied (e.g. Crandall et al. [1984], Subbotin [1995], Bardi and Capuzzo Dolcetta [1997], Fleming and Soner [2006]). Let $\tilde{w}(t,x)$ and $\hat{w}(t,x)$ denote the respective viscosity solutions of these two equations with the same initial condition $w(t_0,x) = h(x)$. In the following $\bar{x} = \bar{x}(t)$ and $\bar{p} = \bar{p}(t)$ will denote the components of characteristic system

$$\begin{aligned} \dot{x} &= H_p(t,x,p), \quad x(t_0) = y, \\ \dot{p} &= -H_x(t,x,p), \; p(t_0) = h_x(y). \end{aligned}$$

Suppose we have either

$$H(t,x,\tilde{p}) \le \mathcal{H}(t,x,\tilde{p}), \quad \forall\,(\tilde{q},\tilde{p}) \in D^-\tilde{w}(t,x) \tag{1}$$

or

$$H(t,x,\hat{p}) \le \mathcal{H}(t,x,\hat{p}), \quad \forall\,(\hat{q},\hat{p}) \in D^+\hat{w}(t,x), \tag{2}$$

where D^- and D^+ denote sub- and superdifferetials respectively (e.g. Clarke et al. [1998]). Then the inequality

$$\hat{w}(t,x) \le \tilde{w}(t,x) \tag{3}$$

holds for all $t \ge t_0$. If besides that the functions $\tilde{w}(t,x)$ and $\hat{w}(t,x)$ are twice continuously differentiable at every point of the curve $\bar{x}(t)$, the functions $H(t,x,p)$ and $\mathcal{H}(t,x,p)$ are continuously differentiable in (x,p), and the following equality holds

$$H\left(t,\bar{x}(t),\bar{p}(t)\right) = \mathcal{H}\left(t,\bar{x}(t),\bar{p}(t)\right)$$

then we have

$$\hat{w}\left(t,\overline{x}(t)\right) = \tilde{w}\left(t,\overline{x}(t)\right), \quad \forall t \in [t_0,t_1].$$

Remark 1. Instead of conditions (1) or (2) we will actually use a simpler condition $H(t,x,p) \le \mathcal{H}(t,x,p)$ for all (t,x,p) as in Kurzhanski [2006].

Remark 2. If the equality (2) is satisfied then $\bar{x}(t)$ and $\bar{p}(t)$ are also components of characteristic system for the equation with the Hamiltonian $\mathcal{H}(t,x,p)$.

Next, consider a control system

$$\dot{x} = f(t,x,u) \tag{4}$$

with the set of initial states $\mathcal{X}^0$. The set of all admissible controls is

$$\mathcal{U}(t) = L^\infty([t_0,t];U)$$

where U is a convex compact. The reachability set for this system is a set

$$\mathcal{X}[t] = \left\{ x \;\middle|\; \exists u(\cdot) \in \mathcal{U}(t) : \; x(t;t_0,x^0,u(\cdot)) = x, \; x^0 \in \mathcal{X}^0 \right\}.$$

Consider the functional

$$J(u(\cdot)) = h(x(t_0)), \quad x(t) = x, \; u(\cdot) \in \mathcal{U}(t)$$

such that $\mathcal{X}^0 = \{x | \; h(x) \le 0\}$. Here $x(\tau) = x(\tau;t,x,u(\cdot))$. Then the value function

$$V(t,x) = \min_{u(\cdot)\in\mathcal{U}(t)} \left\{ J\left(u(\cdot)\right) \mid x(t) = x \right\}$$

solves the Cauchy problem for the Hamilton-Jacobi-Bellman equation

$$V_t + H(t,x,V_x) = 0, \; V(t_0,x) = h(x). \tag{5}$$

There is connection between the value function and the reachability set (Kurzhanski et al. [2001]):

$$\mathcal{X}[t] = \{x \mid V(t,x) \le 0\}.$$

Applying comparison principle with $\tilde{w}(t,x) = V(t,x)$ we obtain a function $w^+(t,x)$ such that

$$w^+(t,x) \le V(t,x)$$

and hence

$$\mathcal{X}[t] \subseteq \mathcal{X}^+[t] = \left\{ x \;\middle|\; w^+(t,x) \le 0 \right\}.$$

Similarly applying comparison principle with $\hat{w}(t,x) = V(t,x)$ we obtain a function $w^-(t,x)$ such that

$$V(t,x) \le w^-(t,x)$$

and hence

$$\mathcal{X}^-[t] = \left\{ x \;\middle|\; w^-(t,x) \le 0 \right\} \subseteq \mathcal{X}[t].$$

3. DYNAMIC UNICYCLE SYSTEM

In this section we apply the comparison principle above to the dynamic unicycle control system whose three-dimensional simplified kinematic version was discussed in Murray et al. [1994], A. De Luca et al. [1998], Roublev [2010]:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= v\cos\varphi, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= v\sin\varphi, \\ \dot{\varphi} &= \alpha u \end{aligned} \tag{6}$$

with u being a control parameter, and v, α being positive constants. Here the compact set U is $[-1,1]$. The initial set is of the form

$$\mathcal{X}^0 = \{(x,\varphi) \mid h_1(x) + h_2(\varphi) \le 0\} \tag{7}$$

where $h_1(x) = \left(x - x^0, X^0(x - x^0)\right)$ and the function $h_2(\varphi)$ is continuous. Here $X^0 = (X^0)^T > 0$.

This control system is connected to the dynamic model for a car-like robot. Namely, x_1 and x_3 are the cartesian coordinates of the rear wheel, x_2 and x_4 are the corresponding velocities, v is the absolute value of velocity, φ is the steering angle, and α is the maximum absolute value of the angular velocity.

Here for the value function we have the HJB equation (5) with the Hamiltonian

$$H(x,\varphi,p) = p_1x_2 + p_3x_4 + vp_2\cos\varphi + vp_4\sin\varphi + \alpha\,|p_5|\,.$$

We propose to look for lower and upper estimates of the value function in the following simple form:

$$w(t,x,\varphi) = w^1(t,x) + w^2(t,\varphi).$$

Then the Hamiltonian $\mathcal{H}(t,x,\varphi,p)$ corresponding to that estimates should have the variables (x,p_x) and (φ,p_φ) separated in the same way (here $p=[p_x^T,p_\varphi]^T$). To obtain lower estimates of the value function we estimate the Hamiltonian $H(x,\varphi,p)$ from the above ensuring that the equality holds when $x=\bar{x}(t)$, $\varphi=\bar{\varphi}(t)$ and $p=\bar{p}(t)$. For the functions w^1 and w^2 we obtained the following equations:

$$\begin{aligned} & w_t^1 + w_{x_1}^1 x_2 + w_{x_3}^1 x_4 + v w_{x_2}^1 \cos\bar{\varphi} + v w_{x_4}^1 \sin\bar{\varphi} + \\ & + \frac{1}{2} v \left(w_{x_2}^1 - \bar{p}_2\right)^2 + \frac{1}{2} v \left(w_{x_4}^1 - \bar{p}_4\right)^2 = 0 \end{aligned} \tag{8}$$

and

$$\begin{aligned} & w_t^2 + v\bar{p}_2 (\cos\varphi - \cos\bar{\varphi}) + v\bar{p}_4 (\sin\varphi - \sin\bar{\varphi}) + \\ & + \frac{1}{2} v (\cos\varphi - \cos\bar{\varphi})^2 + \frac{1}{2} v (\sin\varphi - \sin\bar{\varphi})^2 + \alpha \left|w_\varphi^2\right| = 0 \end{aligned} \tag{9}$$

with the initial conditions

$$w^1(t_0,x) = h_1(x), \quad w^2(t_0,\varphi) = h_2(\varphi).$$

We look for the solution to the first equation in the following form:

$$w^1(t,x) = (x - x^*(t), K(t)\,(x - x^*(t))) + \mu(t).$$

It is easy to obtain the equations for the parameters $P(t)$, $x^*(t)$, $\mu(t)$:

$$\dot{K} = -\left(KA + A^T K + 4KBK\right),$$
$$\dot{x}^* = Ax^* + r(t),\ \dot{\mu} = -\frac{1}{2} v \left(\bar{p}_2^2 + \bar{p}_4^2\right),$$

where $[e_1,\ldots,e_4]$ is the identity matrix and

$$A = e_1 e_2^T + e_3 e_4^T,\ B = \frac{1}{2} v (e_2 e_2^T + e_4 e_4^T).$$

The equation (9) is one-dimensional HJB-type equation. We propose two possible ways to deal with it. First, it could be solved by using one of the possible numerical schemes (e.g. Subbotin [1995]). The solution to this equation is generally continuous but nonsmooth function so that the function $w(t,x,\varphi)$ will be also nonsmooth. Second, the solution of (9) could be approximated by applying comparison principle again. In both cases we will have the following estimate

$$\mathcal{X}[t] \subseteq \mathcal{X}^+[t] = \{(x,\varphi) \mid w(t,x,\varphi) \le 0\}.$$

Next we will obtain ellipsoidal estimates for this system by applying comparison principle to (9). To do so we estimate the trigonometric functions in the second equation as follows

$$\cos\phi \le a(\phi - \bar{\phi})^2 + 2b(\phi - \bar{\phi}) + c,$$

where

$$a = \begin{cases} -\sin\bar{\phi}/\big(2(\bar{\phi} - \pi)\big), & \bar{\phi} \ne \pi \\ \dfrac{1}{2}, & \bar{\phi} = \pi \end{cases},$$

$$b = a(\bar{\phi} - \pi), \quad c = \cos\bar{\phi}, \quad \bar{\phi} \in [0, 2\pi].$$

This inequality turns to equality for $\phi = \bar{\phi}$. Trigonometric functions $-\cos\phi$, $\sin\phi$, $-\sin\phi$ may be estimated from the above in the same way. Applying these inequalities we obtain the equation that is quadratic in (φ,p_5). Then the whole equation for $w(t,x,\varphi)$ becomes quadratic in (x,φ,p) so that its solution will be a quadratic function provided $h_2(\varphi)$ is a quadratic function:

$$h_2(\varphi) = \gamma(\varphi - \varphi^0)^2 - 1.$$

In that case the level sets of $w(t,x,\varphi)$ are ellipsoids. The equation of theirs shape matrices has the form

$$\dot{\mathcal{K}}(t) = \mathcal{A}\mathcal{K}(t) + \mathcal{K}(t)\mathcal{A}' + \mathcal{K}(t)\mathcal{C}(t)\mathcal{K}(t) + 4\mathcal{B}(t) \tag{10}$$

where $\mathcal{B}(t)$ and $\mathcal{C}(t)$ are positive definite matrices that depend on the chosen characteristic $(\bar{x}(t),\bar{p}(t))$:

$$\mathcal{A} = e_1 e_2^T + e_3 e_4^T,\ \mathcal{B} = \frac{1}{2} v (e_2 e_2^T + e_4 e_4^T) + \frac{1}{2|\bar{p}_5|} \alpha e_5 e_5^T,$$

$$\mathcal{C} = v(\lambda_1 a_1 + \lambda_2 a_2) e_5 e_5^T,\ \lambda_1 = \bar{p}_2 - \cos\bar{\varphi},\ \lambda_2 = \bar{p}_4 - \sin\bar{\varphi}.$$

Here $[e_1,\ldots,e_5]$ is the identity matrix. For a particular choice of $(\bar{x}(t),\bar{p}(t))$ equation (10) may not have a solution on the interval of consideration. However, if the solution exists it defines an ellipsoid that touches the reachability set. This in particular implies that $\bar{x}(t)$ lies on the convex hull of the reachability set $\mathcal{X}[t]$.

For the upper estimates of the value function we obtained very similar equations to those for the lower estimates:

$$\begin{aligned} & w_t^1 + w_{x_1}^1 x_2 + w_{x_3}^1 x_4 + v w_{x_2}^1 \cos\bar{\varphi} + v w_{x_4}^1 \sin\bar{\varphi} - \\ & - \frac{1}{2} v \left(w_{x_2}^1 - \bar{p}_2\right)^2 - \frac{1}{2} v \left(w_{x_4}^1 - \bar{p}_4\right)^2 = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} & w_t^2 + v\bar{p}_2 (\cos\varphi - \cos\bar{\varphi}) + v\bar{p}_4 (\sin\varphi - \sin\bar{\varphi}) - \\ & \frac{1}{2} v (\cos\varphi - \cos\bar{\varphi})^2 - \frac{1}{2} v (\sin\varphi - \sin\bar{\varphi})^2 + \alpha \left|w_\varphi^2\right| = 0 \end{aligned} \tag{12}$$

where $w(t,x,\varphi) = w^1(t,x) + w^2(t,\varphi)$ as previously. These equations could be solved in the same way to obtain nonsmooth upper estimates of the value function. This gives us the inclusion

$$\mathcal{X}^-[t] = \{(x,\varphi) \mid w(t,x,\varphi) \le 0\} \subseteq \mathcal{X}[t].$$

For both lower and upper nonsmooth estimates the equality

$$V(t,\bar{x}(t),\bar{\varphi}(t)) = w(t,\bar{x}(t),\bar{\varphi}(t))$$

holds when $V(t,x,\varphi)$ and $w(t,x,\varphi)$ are twice continuously differentiable on the curve $\bar{x}(t)$.

Let us proceed with the problem of control synthesis. We consider the same dynamic unicycle system (6) but instead of the initial condition (7) we have the terminal condition

$$x(t_1) \in \mathcal{M} = \{(x,\varphi) \mid h_1(x) + h_2(\varphi) \le 0\}.$$

Similarly to the reachability set one could define the backward reachability set for this system:

$$\mathcal{W}[t] = \left\{x \mid \exists u(\cdot) \in \mathcal{U}(t):\ x(t;t_1,x^1,u(\cdot)) = x,\ x^1 \in \mathcal{M}\right\}.$$

Here $\mathcal{U}(t) = L^\infty([t,t_1];U)$. The estimates for the backward reachability sets are constructed in the same way as for the forward rechability sets. One may formally substitute w_t^i, w_x^i for $-w_t^i$ and $-w_x^i$ respectively into the equations (11), (12) to obtain $w^-(t,x,\varphi)$ whose zero level set $\mathcal{W}^-[t]$ is an internal estimate of $\mathcal{W}[t]$. Knowing that estimate one may construct control strategy $u(t,x)$ that ensures the inclusion $x(t_1) \in \mathcal{M}$ as soon as $x(t) \in \mathcal{W}^-[t]$ using extremal aiming methods (e.g. Krasovski and Subbotin [1988]). For that purpose define the sets:

$$W^0(t,x) = \operatorname{Arg}\min_{w \in \mathcal{W}^-[t]} \|x - w\|,$$

$$S^0(t,x) = \left\{s^0 = w^0 - x:\ w^0 \in W^0(t,x)\right\}.$$

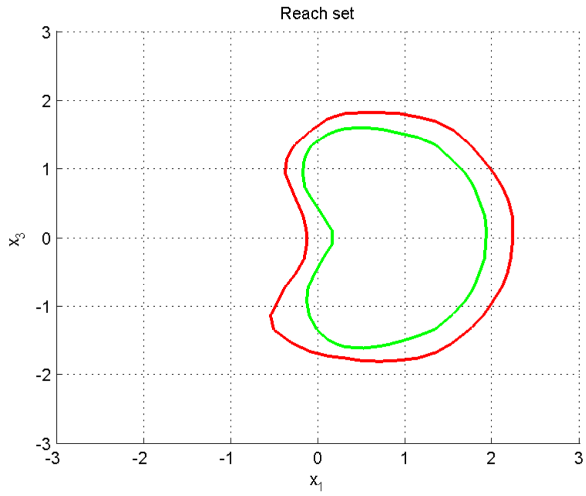

Fig. 1. The projection of internal and external approximations of the reachability set at $t_1 = 1.5$

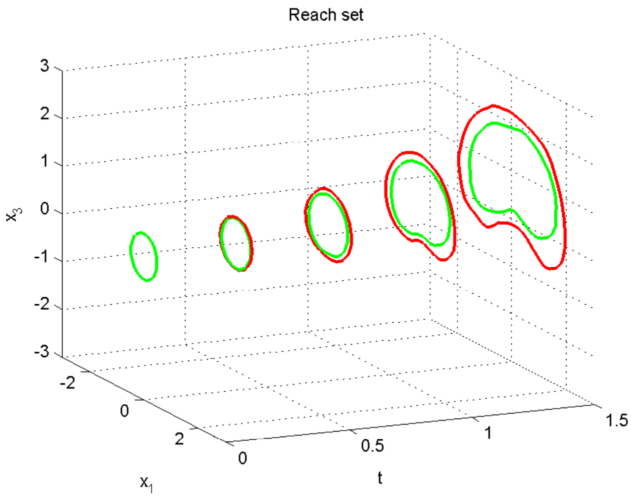

Fig. 2. The evolution of the projections of internal and external approximations of the reachability set

Then the control strategy is defined by

$$u(t,x) \in \operatorname{Arg}\max_{u \in U}[u s_5^0(t,x)].$$

where $s^0 \in S^0(t,x)$.

4. NUMERICAL EXAMPLES

In the first example (see Fig. 1 and 2) we took 10 internal and 10 external estimates to construct internal and external approximations of the reachability set. In the second example (see Fig. 3 and 4) we intersected 10 ellipsoidal tubes to obtain external approximation of the reachability tube. In the control synthesis example we took a starting point $x(0) = x^0 = [11.25, 2.68, -27.49, -5.02, -2.55]^T$ and the internal estimate $\mathcal{W}^-[t]$ which contains x^0 at $t = 0$. Then we synthesize the control $u^*(t)$ corresponding to that estimate and the point x^0 (see Fig. 5 and 6).

5. CONCLUSION

Although the comparison principle approach by itself does produce a general algorithm of reachability set approxi-

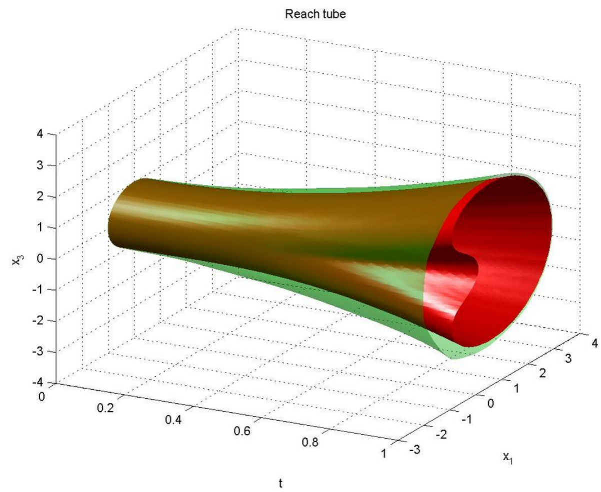

Fig. 3. The reachability tube and its ellipsoidal approximation

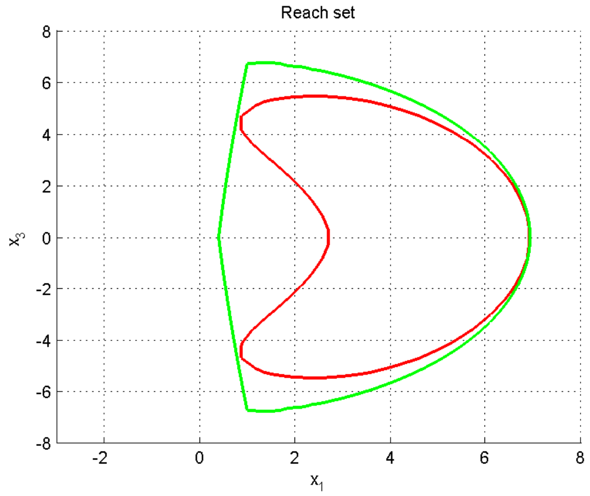

Fig. 4. The reachability set and its ellipsoidal approximation at $t_1 = 1$.

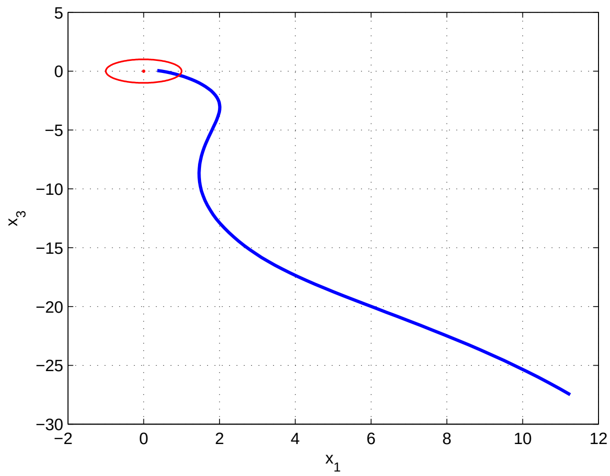

Fig. 5. The projection of the trajectory in the control synthesis example

mation, it provides a way of dealing with various specific classes of nonlinear systems while preserving certain advantages that ellipsoidal or parallelotope approaches have in linear control theory. However, in nonlinear problems there are new difficulties to deal with as the form of the value function estimate and hence the form of reachability set estimate are to be chosen accordingly to the system under consideration.

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