Practical and Robust Synchronization of Systems with Additive Linear Uncertainties ⋆

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Abstract: We investigate the synchronization of systems with additive uncertainties. In doing so, we establish a setup of diffusively coupled nonlinear systems that are perturbed by unknown linear functions, each. By assuming bounded solutions of the nominal uncoupled systems, we derive sufficient conditions for boundedness of the solutions of the coupled systems with uncertainties. Next, using the QUAD condition, we derive conditions for the synchronization error to remain bounded. Subsequently, we investigate the impact of the coupling strength on this bound and find that the bound can be made arbitrarily small for sufficiently large gains, thus establishing criteria for practical synchronization. Finally, we consider classes of uncertainties which consist of matrices whose maximal singular value is smaller than a specific value and show practical synchronization for all uncertainties belonging to that class. Therefore, we establish conditions for robust synchronization with respect to such a class. Our theoretical results are validated with a numerical example composed of perturbed Van der Pol oscillators.

1. INTRODUCTION

Synchronization is a widely studied phenomenon that has been investigated both theoretically, e.g. by Mirollo and Strogatz [1990], Pecora and Carroll [1998] and experimentally, e.g. by Oud et al. [2006], Carroll and Pecora [1991]. Physical sytems however never match their mathematically modeled counterparts perfectly. The fundamental theory explaining synchronization does thus not quite explain the effects occurring in real systems. These discrepancies motivate the study of synchronization of uncertain systems, and, in particular, robust synchronization.

On the one hand, researchers have investigated robust synchronization in master-slave systems; Synchronization of systems with parameter uncertainties has been studied using sliding-mode and variable structure control by Etemadi et al. [2006], and using adaptive control by Wang et al. [2008]. A synchronization problem of systems under perturbations was solved using variable structure control in Yau [2004] and estimation of switching gains in Yau and Lin [2005]. Sytems with additive nonlinear uncertainties can be synchronized employing observers, as it has been shown by Pogromsky and Nijmeijer [1998].

On the other hand, researchers have sought to investigate robust synchronization in networks; $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance of linear systems under perturbations has been investigated with suitable distributed controllers through Li et al. [2011] and $\mathscr{H}_\infty$ consensus of sytems under perturbations and coupling uncertainties including time-delays has been studied using reduced order systems through Lin et al. [2008]. Das and Lewis [2010] propose adaptive control protocols at every node to solve a synchronization problem of nonlinear heterogeneous systems under perturbations. Networks of both, continuous and discrete-time linear systems with additive linear uncertainties have been shown to synchronize when applying a distributed control law designed with LMI and $\mathscr{H}_\infty$ techniques, cf. Li et al. [2011]. $\mathscr{H}_2$ performace of the consensus of single-integrators under perturbations has been studied using the agreement protocol by Zelazo and Mesbahi [2009b]. Synthesis of heterogeneous linear systems under perturbations for $\mathscr{H}_2$-robustness has been presented using semi-definite programming for fixed topology and using Kruskal's algorithm for fixed dynamics in Zelazo and Mesbahi [2009a].

Moreover, robust synchronization is closely related to synchronization of heterogeneous networks. An internal model principle for synchronization has been studied by Wieland and Allgöwer [2009] for nonlinear systems and by Wieland et al. [2011], Seyboth et al. [2012], among others, for linear systems. Furthermore, we will excessively use the notion of QUAD vector fields, studied by DeLellis et al. [2009, 2011].

In the present paper, we study diffusively coupled homogeneous nonlinear systems under additive linear (heterogeneous) uncertainties. The uncertainty setup for a single system i is depicted in Fig. 1.

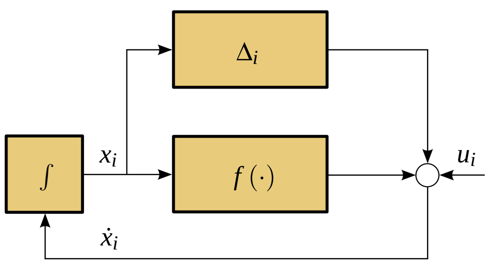

Fig. 1. Uncertainty setup for a single system i: The nonlinear function $f(\cdot)$ is assumed identical for all systems, but the additive uncertainty Δ_i is heterogeneous.

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Prior research has focused on designing control laws, such as pinning control, observers, state feedback, or coupling graphs to achieve exact synchronization. We choose to leave the dynamics of the single nodes, as well as the network architecture, untouched. We do not want to impose additional control laws on the network or to single systems but only analyze the effect of the diffusive coupling strength, herein modelled through a coupling gain k, on the synchronization behavior of the systems, and, in particular, on the synchronization error. Therefore, compared to the literature presented above, we will not be able to achieve exact synchronization. Instead, we will establish bounds for synchronization errors depending on both, the uncertainties and the coupling gains.

Structure of the paper. The remainder of the paper is structured as follows; In Section 2, we state the problem investigated herein, including system and uncertainty setup, couplings, and assumptions. Section 3 contains our main results, where Subsection 3.1 contains sufficient conditions for the boundedness of the coupled systems, and Subsection 3.2 shows how we can guarantee boundedness of the synchronization error. In Subsection 3.3, the notion of practical synchronizability is introduced and Subsection 3.4 extends the previous results to robust synchronization by considering an entire class of uncertainties. Section 4 validates our theoretical results by a numerical example composed of perturbed Van der Pol oscillators with uncertainties and Section 5 concludes the paper.

Notation. Variables are formatted italic, operators upright, and sets blackboard bold. $\mathbb{R}$ is the field of real numbers, by $\mathbb{R}^n$ we denote the space of n-tuples of real numbers, and by $\mathbb{R}^{n\times m}$ the space of matrices composed of m n-tuples. With $\otimes$ and $\oplus$ we denote the direct product and direct sum, respectively and $\operatorname{diag}(x_i)$ is the direct sum of $x_1 \oplus \cdots \oplus x_N$, where N can be concluded from the context. By I_n and $\mathbf{1}_n$ we mean the identity of $\mathbb{R}^{n\times n}$ and the vector of ones in $\mathbb{R}^n$, respectively. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function mapping $x_i \mapsto f(x_i)$, then, with a slight abuse of notation, by writing $(\mathbf{1}_k \otimes f)(\cdot)$, we mean a function $(\mathbf{1}_k \otimes f) : \mathbb{R}^{kn} \to \mathbb{R}^{km}$, which is the stack of k copies of f and maps $\left[x_1^\top \cdots x_k^\top\right]^\top \mapsto \left[f^\top(x_1) \cdots f^\top(x_k)\right]^\top$. For norms (and also for induced norms), we write $\|\cdot\|$. A transpose is expressed by superindexing $\top$ and an overdot abbreviates $\frac{\mathrm{d}}{\mathrm{d}t}$, where t is the time. Furthermore, $\max(\cdot)$ is the maximum, $\sup(\cdot)$ the supremum, $\operatorname{spec}(\cdot)$ the spectrum, and $\operatorname{rank}(\cdot)$ the rank. A $[0,\infty) \to [0,\infty)$ function is said to be class $\mathscr{K}$, if it is zero at zero, strictly increasing, and continuous, and a $[0,\infty)^2 \to [0,\infty)$ function is class $\mathscr{KL}$, if it is class $\mathscr{K}$ in the first argument and decreasing to zero in the second argument.

2. PROBLEM STATEMENT

We consider N dynamical systems

$$\dot{x}_i = f(x_i) + \Delta_i x_i + u_i, \tag{1}$$

where i is from the index set $\{1 \cdots N\}$, $x_i \in \mathbb{R}^n$ is the state of system i, $\Delta_i \in \mathbb{R}^{n\times n}$ is its uncertainty, $u_i \in \mathbb{R}^n$ its input, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is identical for all i. In particular, the maximal singular value of Δ_i shall be given through $\sigma_i^2 = \operatorname{maxspec}\left(\Delta_i^\top \Delta_i\right)$ and $f(x_i)$ is assumed to be $\mathrm{QUAD}(P,\omega)$, i.e. $(a-b)^\top (f(a) - f(b)) - (a-b)^\top P (a-b) \leq -\omega (a-b)^\top (a-b)$, with P some diagonal matrix and ω some finite scalar, as introduced by DeLellis et al. [2011]. We will often also utilize the estimate $P - \omega I_n \leq q_i I_n$, where I_n is the identity of $\mathbb{R}^{n\times n}$. Furthermore, we assume that every solution to the auxiliary system $\dot{\xi}_i = f(\xi_i)$, $\xi_i \in \mathbb{R}^n$, which is the uncoupled nominal system, where $\xi_i(0) = x_i(0)$ is assumed for the sake of interpretability, is bounded by a closed ball of radius $\frac{L}{c}$ plus its initial condition. That is, there exist finite positive scalars L, c, such that

$$\|\xi_i(t)\|^2 \leq \|\xi_i(0)\|^2 + \frac{L}{c}. \tag{2}$$

Boundedness according to (2) can be shown using a quadratic function $\frac{1}{2}\xi_i^\top \xi_i$ if there exist finite positive scalars L, c, such that

$$\xi_i^\top f(\xi_i) \leq -c\xi_i^\top \xi_i + L, \tag{3}$$

cf. Raffoul [2003]. For the average of all solutions under consideration, and thus the solution we would potentially synchronize to, we have

$$s(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t), \tag{4}$$

and we describe the deviation of system i from $s(t)$ as the synchronization error $e_i(t) = x_i(t) - s(t)$.

We approach the synchronization of the above systems by placing them on the vertices of a directed, weighted graph $\mathscr{G}$, encoded through its negative Laplacian $A = [a_{ij}]$. Herein, an element a_{ij} determines, whether or not the state of system j is used as input for system i and how much it is scaled in between, yielding

$$u_i = \sum_{j=1}^{N} k a_{ij} x_j, \tag{5}$$

where $k \in \mathbb{R}$ is a gain. We assume that A has an eigenvector of ones corresponding to the eigenvalue zero and all other eigenvalues negative. As a notational convention, we write $A \in \mathbb{W}$, where $\mathbb{W}$ is defined as follows: $\mathbb{W} = \bigcup_{m=2}^{\infty} \mathbb{W}^m$, $\mathbb{W}^m = \{W \in \mathbb{R}^{m\times m} \; | W\mathbf{1}_m = 0,\; \operatorname{rank}(W) = m-1,\; \max(\operatorname{spec} W \setminus \{0\}) < 0\}$, $\mathbf{1}_m = [1 \cdots 1]^\top \in \mathbb{R}^m$, and we denote the largest nonzero eigenvalue of A by $\lambda = \max(\operatorname{spec} A \setminus \{0\})$. This is the interpretation of having diffusive couplings between the systems (1). Note also, that every negative Laplacian of a directed, connected graph of appropriate dimension satisfies these assumptions.

Employing the coupling, we can write

$$\dot{x} = (\mathbf{1}_N \otimes f)(x) + (\operatorname{diag}(\Delta_i) + A \otimes I_n)x \tag{6}$$

as a shorthand notation for (1) under (5), where $\otimes$ is the Kronecker product, $(\mathbf{1}_N \otimes f) : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is the stacked vector of $f(x_i)$ and $\operatorname{diag}(\Delta_i) = \Delta_1 \oplus \cdots \oplus \Delta_N$, where $\oplus$ is the direct sum. Furthermore, differentiating (4), we get

$$\begin{aligned}
\dot{s} &= \frac{1}{N}\sum_{i=1}^{N}\left(f(x_i) + \Delta_i x_i + \sum_{j=1}^{N} k a_{ij} x_j\right) \\
&= \overline{\Delta} s + \frac{1}{N}\sum_{i=1}^{N}\left(f(x_i) + \Delta_i e_i + \sum_{j=1}^{N} k a_{ij} x_j\right) \\
&= \overline{\Delta} s + \frac{1}{N}\left(\mathbf{1}_N^\top \otimes I_n\right)(kA \otimes I_n)x + \frac{1}{N}\sum_{i=1}^{N}\left(f(x_i) + \Delta_i e_i\right) \\
&= \overline{\Delta} s + \frac{1}{N}\left(\mathbf{1}_N^\top \otimes I_n\right)(kA \otimes I_n)e + \frac{1}{N}\sum_{i=1}^{N}\left(f(x_i) + \Delta_i e_i\right),
\end{aligned} \tag{7}$$

where $N\overline{\Delta} = \sum_{i=1}^{N} \Delta_i$.

3. MAIN RESULTS

The starting point for our analyses are networks of coupled nominal systems which synchronize to a common trajectory.

This is guaranteed through the assumption of having QUAD vector fields under couplings of the assumed form, cf. DeLellis et al. [2011]. In this section, we first present sufficient conditions for boundedness of the coupled systems in presence of uncertainties and then, as a second step, derive upper bounds for the synchronization error depending on the uncertainties.

3.1 Boundedness of the Averaged Solution

The bound on the synchronization errors established later in this paper will naturally depend on the supremum norm of $s(t)$. However, $x(t)$ (and thus $s(t)$) can be destabilized by suitable Δ_i. In such a case, a bound depending on the supremum norm of $s(t)$ would not be meaningful. As we want to exclude such cases, we thus head to establish bounds on s, yet reasoning with assumptions regarding $f(\cdot)$.

The function $f(\xi_i)$ under considerations shall be stacked in form $(\mathbf{1}_N \otimes f)(\xi)$ and, utilizing the above, the auxiliary system $\dot{\xi} = (\mathbf{1}_N \otimes f)(\xi)$ is introduced, where $\xi \in \mathbb{R}^{Nn}$, and, for sake of interpretability, $\xi(0) = x(0)$ is assumed. Taking (3), we get

$$\xi^\top \dot{\xi} \leq -c\xi^\top \xi + LN, \tag{8}$$

that is, $\xi(t)$ is bounded by

$$\|\xi(t)\|^2 \leq \|\xi(0)\|^2 + \frac{LN}{c}, \tag{9}$$

see Raffoul [2003]. We now argue that, under certain conditions, this bound on ξ induces a bound on s.

Theorem 1. Assume that there exist finite positive scalars L, c such that $\xi_i^\top \dot{\xi}_i \leq -c\xi_i^\top \xi_i + L$. Then, if

$$\operatorname{diag}(\Delta_i) + kA \otimes I_n - cI_{Nn} < 0, \tag{10}$$

there exist finite positive scalars L', c', such that $s(t)$ is bounded by

$$\|s(t)\|^2 \leq \|x(0)\|^2 + \frac{L'}{c'}. \tag{11}$$

Proof. Consider the Lyapunov function candidate

$$U = \frac{1}{2} x^\top x \tag{12}$$

and its corresponding directional derivative

$$\dot{U} = \sum_{i=1}^{N} x_i^\top \dot{x}_i. \tag{13}$$

Substituting (1) together with (5), we get

$$\dot{U} = \sum_{i=1}^{N} x_i^\top \left(f(x_i) + \Delta_i x_i + \sum_{j=1}^{N} k a_{ij} x_j \right). \tag{14}$$

Consequently, utilizing the properties of the direct product and the vector of ones,

$$\dot{U} = x^\top (\operatorname{diag}(\Delta_i) + kA \otimes I_n) x + x^\top (\mathbf{1}_N \otimes f)(x) \tag{15}$$

follows. Furthermore, utilizing (8), we get

$$\dot{U} \leq x^\top (\operatorname{diag}(\Delta_i) + kA \otimes I_n) x - cx^\top x + LN. \tag{16}$$

Using (10), we can find a positive finite scalar c', such that

$$\operatorname{diag}(\Delta_i) + kA \otimes I_n - cI_{Nn} \leq -c' I_{Nn}, \tag{17}$$

and additionally define some finite positive scalar L' such that

$$LN \leq L'. \tag{18}$$

Taking this into account, we can as well write

$$x^\top (\operatorname{diag}(\Delta_i) + kA \otimes I_n - cI_{Nn}) x + LN \leq -c' x^\top x + L', \tag{19}$$

instead of (17) and (18). Then, comparing (19) to (12) and (16), we see that the former is just $\dot{U} \leq -2c'U + L'$ and thus (Raffoul [2003]) implies $\|x(t)\|^2 \leq \|x(0)\|^2 + \frac{L'}{c'}$, which concludes the proof as $\|s\| \leq \|x\|$. ■

We are now in the position to discuss the geometrical interpretation of our novel bounds L', c'. First, let us see that L' is larger than L, linearly growing with N, and that larger L' lets our ball-like bound on s grow. Second, let us see that c' is linearly growing with c, that it is equal to c for the uncoupled nominal case, that it can be chosen larger if the couplings or the uncertainties become more negative and that larger c' lets our ball-like bound on s shrink.

3.2 Synchronization with a Bounded Error

In general, we would want $e_i(t)$ to go to zero for all i. Notably, this is not possible without additional assumptions if $\Delta_i \neq \overline{\Delta}$ for any i Wieland and Allgöwer [2009]. Instead, we want to state conditions for e_i to remain small, dependent on our choice of k.

Definition 1. N systems (1) under coupling (5) are said to synchronize with a bounded error, if

$$\|e(t)\| \leq \beta(\|e(0)\|, t) + \varepsilon \sup_{0 \leq \tau \leq t} \|s(\tau)\|,$$

and $\sup_{0 \leq \tau \leq t} \|s(\tau)\|$ is finite, where β is some class $\mathcal{KL}$ function and ε a finite positive scalar.

Theorem 2. Consider N systems (1) under coupling (5). Suppose that $A \in \mathbb{W}$, $f(\cdot)$ is QUAD(P, ω) with $P - \omega I_n \leq qI_n$, $\lambda < 0$, k such that $q + \max_i \sigma_i + k\lambda < 0$, there exist finite positive scalars L, c, such that $\xi_i^\top \dot{\xi}_i \leq -c\xi_i^\top \xi_i + L$ and $\operatorname{diag}(\Delta_i) + kA \otimes I_n - cI_{Nn} \leq 0$. Then the systems (1) under coupling (5) synchronize with a bounded error where ε is given through

$$\varepsilon = \frac{\left\|\operatorname{diag}\left(\Delta_i - \overline{\Delta}\right)\right\|}{-(q + \max_i \sigma_i + k\lambda)} \sqrt{N}.$$

Proof. We consider the Lyapunov function candidate

$$V = \frac{1}{2} \sum_{i=1}^{N} e_i^\top e_i \tag{20}$$

and its corresponding directional derivative

$$\dot{V} = \sum_{i=1}^{N} e_i^\top \dot{e}_i = \sum_{i=1}^{N} e_i^\top (\dot{x}_i - \dot{s}). \tag{21}$$

Now, substituting (1) together with (5) and (7), it follows that

$$\dot{V} = \sum_{i=1}^{N} e_i^\top \Big(f(x_i) + \Delta_i x_i + \sum_{j=1}^{N} (k a_{ij} x_j) - \overline{\Delta} s - \\ - \frac{1}{N} \sum_{j=1}^{N} \left(f(x_j) + \Delta_j e_j \right) - \frac{1}{N} \left(\mathbf{1}_N^\top \otimes I_n \right) (kA \otimes I_n) e \Big). \tag{22}$$

Consequently, by substituting

$$\frac{1}{N} \sum_{j=1}^{N} \Delta_j e_j = \frac{1}{N} \left(\mathbf{1}_N^\top \otimes I_n \right) \operatorname{diag}(\Delta_j) e,$$

subtracting $\sum_{i=1}^{N} e_i^\top \sum_{j=1}^{N} k a_{ij} s$, which is just zero since $A \in \mathbb{W}$, adding $\sum_{i=1}^{N} e_i^\top \frac{1}{N} \sum_{j=1}^{N} f(x_j)$ and subtracting $\sum_{i=1}^{N} e_i^\top f(s)$, which are both zero since $\sum_{i=1}^{N} e_i = 0$, we get

$$\dot{V} = \sum_{i=1}^{N} e_i^\top \Big(f(x_i) - f(s) + \Delta_i x_i + \sum_{j=1}^{N} (k a_{ij} x_j) - \sum_{j=1}^{N} (k a_{ij} s) - \\ - \overline{\Delta} s - \frac{1}{N} \left(\mathbf{1}_N^\top \otimes I_n \right) \operatorname{diag}(\Delta_j) e - \frac{1}{N} \left(\mathbf{1}_N^\top \otimes I_n \right) (kA \otimes I_n) e \Big). \tag{23}$$

Furthermore, utilizing $e_i = x_i - s$ and the QUAD inequality together with the estimate q, we get the upper bound

$$\dot{V} \le \sum_{i=1}^{N} e_i^\top \left(q e_i + \Delta_i e_i + \Delta_i s + \sum_{j=1}^{N} k a_{ij} e_j - \overline{\Delta} s \right), \qquad (24)$$

where the term $\sum_{i=1}^{N} e_i^\top \frac{1}{N} \left(\mathbf{1}_N^\top \otimes I_n\right) \left(\operatorname{diag}(\Delta_j) + (kA \otimes I_n)\right) e$ has been dropped for being zero (again reasoning that $\sum_{i=1}^{N} e_i = 0$). Collecting the products of e_i and s,

$$\dot{V} \le \sum_{i=1}^{N} e_i^\top \left((qI_n + \Delta_i) e_i + \sum_{j=1}^{N} k a_{ij} e_j + \left(\Delta_i - \overline{\Delta}\right) s \right) \qquad (25)$$

follows straightforward. We now remove the sums to be able to henceforth use the compact notation

$$\begin{aligned} \dot{V} \le & e^\top \left(qI_{Nn} + \operatorname{diag}(\Delta_i) + kA \otimes I_n\right) e \\ & + e^\top \operatorname{diag}\left(\Delta_i - \overline{\Delta}\right) (\mathbf{1}_N \otimes s). \end{aligned} \qquad (26)$$

We substitute $qI_{Nn} + \operatorname{diag}(\Delta_i) + kA \otimes I_n \le q + \max_i \sigma_i + k\lambda$ and $e^\top \operatorname{diag}\left(\Delta_i - \overline{\Delta}\right) (\mathbf{1}_N \otimes s) \le \|e\| \left\|\operatorname{diag}\left(\Delta_i - \overline{\Delta}\right)\right\| \|\mathbf{1}_N \otimes s\|$ to get

$$\dot{V} \le e^\top \left(q + \max_i \sigma_i + k\lambda \right) e + \|e\| \left\|\operatorname{diag}\left(\Delta_i - \overline{\Delta}\right)\right\| \|\mathbf{1}_N \otimes s\|. \qquad (27)$$

Subsequently, choosing k such that $q + \max_i \sigma_i + k\lambda < 0$, which is always possible, abbreviating $\max_i \sigma_i$ by simply writing σ and introducing a scalar θ, $0 < \theta < 1$, it follows, that

$$\begin{aligned} \dot{V} \le & e^\top e (q + \sigma + k\lambda)(1 - \theta) + \\ & e^\top e (q + \sigma + k\lambda)\theta + \|e\| \left\|\operatorname{diag}\left(\Delta_i - \overline{\Delta}\right)\right\| \|\mathbf{1}_N \otimes s\|, \end{aligned} \qquad (28)$$

which leads to $\dot{V} < 0$ for $\|e\| \ge \frac{\left\|\operatorname{diag}\left(\Delta_i - \overline{\Delta}\right)\right\|}{-\theta(q+\sigma+k\lambda)} \|\mathbf{1}_N \otimes s\|$ and hence to

$$\|e(t)\| \le \beta(\|e(0)\|, t) + \frac{\left\|\operatorname{diag}\left(\Delta_i - \overline{\Delta}\right)\right\|}{-(q + \sigma + k\lambda)} \sqrt{N} \sup_{0 \le \tau \le t} \|s(\tau)\| \qquad (29)$$

where $\sup_{0 \le \tau \le t} \|s(\tau)\|$ is finite by means of Theorem 1, which thus concludes the proof. ∎

Remark. It is possible to assume $\overline{\Delta} = 0$, since otherwise $f(x_i)$ can be redefined to $f(x_i) + \overline{\Delta} x_i$ (if the latter is still QUAD). Taking this into account, the ε given above reduces to $\varepsilon = \frac{\|\operatorname{diag}(\Delta_i)\|}{-(q+\sigma+k\lambda)} \le \frac{\sigma}{-(q+\sigma+k\lambda)}$.

Given the above theorem, we can now interpret its consequences. The more the uncertainties differ, (or, considering the remark, the larger the uncertainties become,) the larger the impact of s on e becomes. On the other hand, the larger k becomes, the smaller the impact of s on e becomes. This interpretation shall be utilized later to establish the notion of practical synchronizability. First, however, let us consider the influence that q has. Naturally, linear vector fields are QUAD. Thus, let us now take a little detour and consider the case where $f(\cdot)$ is a linear map F. As we take the step from (23) to (24), we could see that we would instead have $\dot{V} = \sum_{i=1}^{N} \left(F e_i + \Delta_i e_i + \Delta_i s + \sum_{j=1}^{N} k a_{ij} e_j - \overline{\Delta} s \right)$ and can hence formulate a corollary regarding the linear case.

Corollary. Considering N *linear* systems $\dot{x}_i = F x_i + \Delta_i x_i + u_i$, $F \in \mathbb{R}^{n \times n}$ under coupling (5). Suppose that $A \in \mathbb{W}$, and k such that $\|I_N \otimes F\| + \max_i \sigma_i + k\lambda < 0$, then the systems (1) under coupling (5) synchronize with a bounded error where ε is given through $\varepsilon = \frac{\sigma}{-(\|I_N \otimes F\| + \sigma + k\lambda)}$.

3.3 Practical Synchronization

Now having done this, let us get back to the interpretation of the influence of k on ε. Reconsider that we have assumed that k is chosen such that $q + \sigma + k\lambda < 0$. Given this, we can see that we can make ε arbitrarily small by choosing k sufficiently large. However, to establish the notion of practical synchronizability, where a hard bound on e can be designed arbitrarily, we also have to know some upper bound for the supremum of s. We have already established such a bound (resulting from the assumptions of Theorem 1) through

$$\begin{aligned} & \sup_{0 \le \tau \le t} \|s(\tau)\|^2 \ge \|s(t)\|^2, \\ & \|s(t)\|^2 \le \|x(0)\|^2 + \frac{L'}{c'} \\ \Rightarrow & \sup_{0 \le \tau \le t} \|s(\tau)\|^2 \le \|x(0)\|^2 + \frac{L'}{c'}, \end{aligned} \qquad (30)$$

which follows from the Weierstraß extreme value theorem.

Definition 2. N systems (1) under coupling (5) are said to be practically synchronizable, if, for any positive finite choice of ε', there exists a gain k, such that

$$\|e(t)\| \le \beta(\|e(0)\|, t) + \varepsilon',$$

where β is some class $\mathscr{KL}$ function.

Theorem 3. Consider N systems (1) under coupling (5). Suppose that $A \in \mathbb{W}$, $f(\cdot)$ is QUAD(P, ω) with $P - \omega I_n \le qI_n$, $\lambda < 0$, k such that $q + \max_i \sigma_i + k\lambda < 0$, there exist finite positive scalars L, c, such that $\xi_i^\top \dot{\xi}_i \le -c \xi_i^\top \xi_i + L$ and $\operatorname{diag}(\Delta_i) + kA \otimes I_n - cI_{Nn} \le 0$. Then the systems (1) under coupling (5) are practically synchronizable for any gain

$$k \ge -\frac{\sigma \sqrt{N \|x(0)\|^2 + \frac{NL'}{c'}} + (q + \sigma)\varepsilon'}{\varepsilon' \lambda}.$$

Proof. By the above assumptions, we can reconsider Theorem 2 and hence conclude

$$\|e(t)\| \le \beta(\|e(0)\|, t) + \frac{\sigma}{-(q + \sigma + k\lambda)} \sqrt{N} \sup_{0 \le \tau \le t} \|s(\tau)\|. \qquad (31)$$

In addition, taking the consequence of Theorem 1, we have

$$\|s(t)\|^2 \le \|x(0)\|^2 + \frac{L'}{c'}, \qquad (32)$$

and thus, taking (30), also

$$\begin{aligned} \|e(t)\| \le & \beta(\|e(0)\|, t) + \\ & + \frac{\sigma}{-(q + \sigma + k\lambda)} \sqrt{N \|x(0)\|^2 + \frac{NL'}{c'}}. \end{aligned} \qquad (33)$$

Taking any arbitrary value for ε', we can always choose k, such that

$$k \ge -\frac{\sigma \sqrt{N \|x(0)\|^2 + \frac{NL'}{c'}} + (q + \sigma)\varepsilon'}{\varepsilon' \lambda}. \qquad (34)$$

Now, resubstituting (34) into (33), we have $\|e(t)\| \le \beta\left(\left\|e(0)\right\|, t\right) + \varepsilon'$, which hence concludes the proof. ∎

The above theorem states that, loosely speaking, we can always choose k large enough to make the error converge into an arbitrarily small ball.

3.4 Robust Synchronization

So far, we have interpreted the influence of k, λ, and q on our error bound. It remains to discuss the influence of σ

and how this interpretation is related to the notion of robust synchronization. Therefore, first, let us suppose that all Δ_i are from some compact set $\mathbb{D} \subset \mathbb{R}^{n\times n}$ containing all possible uncertainties. Suppose that d is the set of all maximal singular values of all elements of $\mathbb{D}$ given through

$$d = \left\{\sigma \,\middle|\, \Delta \in \mathbb{D}, \sigma^2 = \text{maxspec}\left(\Delta^\top \Delta\right)\right\}. \quad (35)$$

Then, certainly, $\sigma \leq \max d = d'$, and thus

$$\frac{\sigma}{-(q+\sigma+k\lambda)} \leq \frac{d'}{-(q+d'+k\lambda)}. \quad (36)$$

Definition 3. N systems (1) under coupling (5) are said to be robustly synchronizable with respect to $\mathbb{D}$, if, for any $\Delta_1, \cdots, \Delta_N \in \mathbb{D}$ and positive finite choice of ε', there exists a gain k, such that

$$\|e(t)\| \leq \beta(\|e(0)\|, t) + \varepsilon',$$

where β is some class $\mathscr{KL}$ function.

Theorem 4. Consider N systems (1) with $\Delta_1, \cdots, \Delta_N \in \mathbb{D}$ and $\mathbb{D} \subset \mathbb{R}^{n\times n}$ compact under coupling (5). Suppose that $A \in \mathbb{W}$, $f(\cdot)$ is QUAD(P,ω) with $P - \omega I_n \leq qI_n$, $\lambda < 0$, k such that $q + d' + k\lambda < 0$, there exist finite positive scalars L, c, such that $\xi_i^\top \dot{\xi}_i \leq -c\xi_i^\top \xi_i + L$ and $d' + kA \otimes I_n - cI_{Nn} \leq 0$. Then the systems (1) under coupling (5) are robustly synchronizable for any gain

$$k \geq -\frac{d'\sqrt{\|Nx(0)\|^2 + \frac{NL'}{c'}} + (q+d')\varepsilon'}{\varepsilon'\lambda}.$$

Proof. We structure our proof into three steps. First, let us show how the average solution $s(t)$ in the above setup is still bounded. Second, how systems (1) under coupling (5) are still synchronizing with a bounded error, and third, how systems (1) under coupling (5) are still practically synchronizable.

Let us thus consider the assumption that $d' + kA \otimes I_n - cI_{Nn} \leq 0$. Notably, $d' + kA \otimes I_n - cI_{Nn} \geq \sigma + kA \otimes I_n - cI_{Nn} \geq \text{diag}(\Delta_i) + kA \otimes I_n - cI_{Nn}$. If the former is less or equal to zero, then so is the latter. we can find a positive finite scalar c', such that $d' + kA \otimes I_n - cI_{Nn} \leq -c' I_{Nn}$, and additionally define some finite positive scalar L' such that $LN \leq L'$ and hence conclude that s is bounded by $\|s(t)\|^2 \leq \|x(0)\|^2 + \frac{L'}{c'}$.

Consequently, applying Theorem 2, we get

$$\|e(t)\| \leq \beta(\|e(0)\|, t) + \frac{\left\|\text{diag}\left(\Delta_i - \overline{\Delta}\right)\right\|}{-(q+\sigma+k\lambda)} \sqrt{N} \sup_{0\leq\tau\leq t} \|s(\tau)\|. \quad (37)$$

Substituting our bound for s and (36), we have

$$\|e(t)\| \leq \beta(\|e(0)\|, t) + \frac{d'}{-(q+d'+k\lambda)} \sqrt{\|Nx(0)\|^2 + \frac{NL'}{c'}}. \quad (38)$$

Last, taking any arbitrary value for ε', we can always choose k, such that

$$k \geq -\frac{d'\sqrt{\|Nx(0)\|^2 + \frac{NL'}{c'}} + (q+d')\varepsilon'}{\varepsilon'\lambda}, \quad (39)$$

and substitute k back to get $\|e(t)\| \leq \beta(\|e(0)\|, t) + \varepsilon'$, which thus concludes the proof. ■

4. NUMERICAL EXAMPLE

We want to illustrate the above deductions by an exemplary setup consisting of 6 Van der Pol oscillators with uncertainties under diffusive coupling at different gains.

The ith Van der Pol oscillator in two dimensional form is given through

$$\dot{x}_{i1} = \frac{1}{\alpha} x_{i2} \quad + \quad \Delta_{i1} x_{i1} \quad + \quad u_{i1}, \quad (40)$$

$$\dot{x}_{i2} = \underbrace{\alpha\left(x_{i2} - \frac{1}{3}x_{i2}^3 - x_{i1}\right)}_{\text{Van der Pol equation}} + \underbrace{\Delta_{i2} x_{i2}}_{\text{uncertainties}} + \underbrace{u_{i2}}_{\text{couplings}}, \quad (41)$$

where we abbreviate $x_i^\top = [x_{i1}\ x_{i2}]$ and $u_i^\top = [u_{i1}\ u_{i2}]$, respectively. Also, we choose α to be equal to 5. In addition, the diffusive all-to-all coupling law $A = -6I_6 + \mathbf{1}_6^\top \otimes \mathbf{1}_6$ is applied, so that $\lambda = -6$. Notably, DeLellis et al. [2011] have shown that this setup achieves exact synchronization (which is synchronization with a bounded error and $\varepsilon = 0$) for $k > \frac{q}{\lambda}$ if we set $\Delta_1 = \Delta_2 = 0$, which is also a consequence of Theorem 2. In particular, the Van der Pol oscillator has been shown to be semi-contracting by Wang and Slotine [2005] and thus QUAD(P,ω) with $P = I_2\omega$, allowing for q to be any value greater than or equal to zero (DeLellis et al. [2011]).

Here, we consider uncertainties and investigate their impact on ε. The uncertainties under consideration are distributed randomly between 0 and -0.5. The Van der Pol oscillator loses its periodic orbit for larger uncertainties. In this example, we have $\left\|\text{diag}\left(\Delta_i - \overline{\Delta}\right)\right\| = 0.2089$ and $\sigma = 0.4769$. According to Theorem 2, k has to be chosen such that $0.4769 - 6k < 0$, which is true for $k > 0.0795$. Therefore, we would have $\varepsilon = \frac{0.2089\sqrt{6}}{-0.4769+6k}$. This setting is simulated in MATLAB using `ode45` at three different gains 0.1, 1, and 10. The respective supremum norms of s are found experimentally (thus solving the problem first, before knowing the bounds). The solutions for x_{i1} and x_{i2} are plotted for all i at the different coupling gains in Figure 2 together with the errors $\|e\|$ and their upper bounds given through Theorem 2.

The plots show, that the errors hold the previously established bounds well. Also, it turns out, that the bounds become quite small for comparatively small gains. The solutions agree with our interpretations as for the smaller gains, the solutions remain close to each other and then approach each other with arbitrary precision as the gain is increased.

5. CONCLUSION

We have presented a synchronization problem for homogeneous nonlinear systems, that each are bounded and QUAD. These systems were subject to linear additive uncertainties and equipped with diffusive couplings of adjustable strength. We have referred to some literature stating that such networks would synchronize when assuming zero uncertainties. Consequently, we have derived conditions for the uncertain network to remain bounded, arguing with the boundedness conditions for the single nominal systems. Using this, we could derive sufficient conditions for the synchronization error to remain bounded, and, investigating the latter further, we analyzed first the influence of our gain on this bound, yielding arbitrarily small bounds for sufficiently large gains (and thus practical synchronization), and second the influence of our uncertainties on this bound, yielding bounds for a whole class of uncertainties (and thus robust synchronization). We have validated our results with an exemplary setup composed of perturbed Van der Pol oscillators.

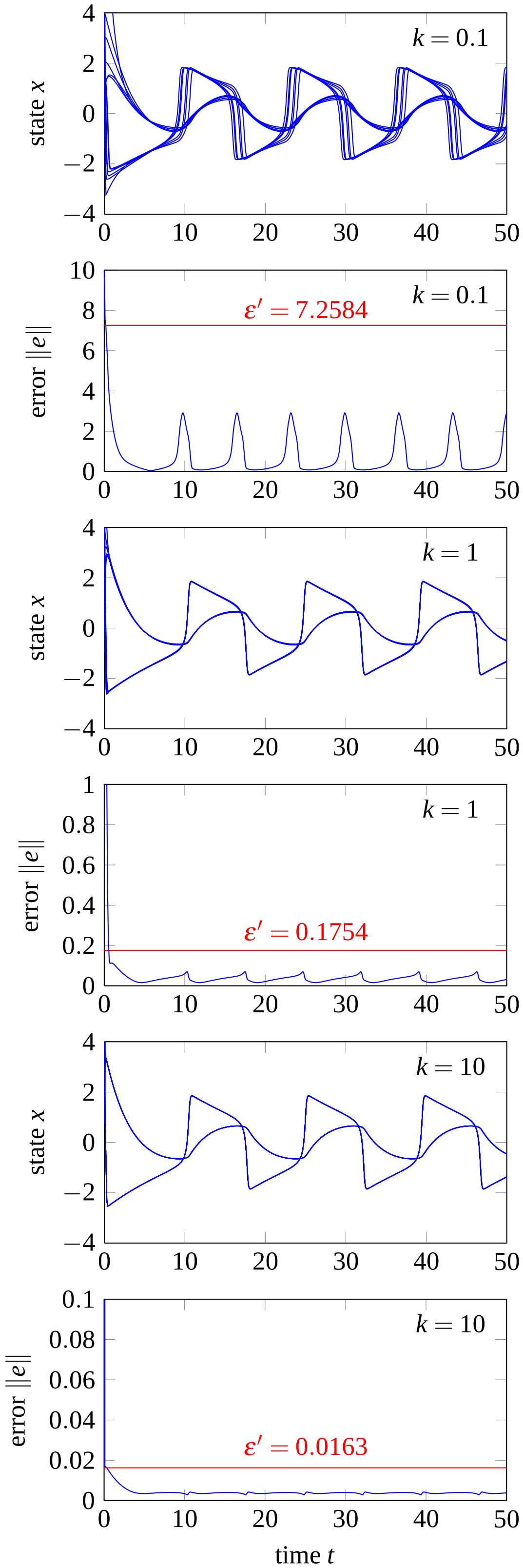

Fig. 2. Solutions for states x and errors e: The solutions and errors are plotted at three different gains $k = 0.1, 1, 10$, as indexed in the respective plot. The upper bounds ε' for $\|e\|$ are plotted red (——). As expected, the errors satisfy the computed bounds and decrease with increasing k.

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