# Solving Partially Hyper-Sensitive Optimal Control Problems Using Manifold Structure

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**Abstract:** Hyper-sensitivity to unknown boundary conditions plagues indirect methods of solving optimal control problems as a Hamiltonian boundary-value problem for both state and costate. Yet the hyper-sensitivity may imply manifold structure in the Hamiltonian flow, knowledge of which would yield insight regarding the optimal solutions and suggest a solution approximation strategy that circumvents the hyper-sensitivity. Finite-time Lyapunov exponents and vectors provide a means of diagnosing hyper-sensitivity and determining the associated manifold structure. A solution approximation approach is described that requires determining the unknown boundary conditions, such that the solution end points lie on certain invariant manifolds, and matching of forward and backward segments. The approach is applied to the optimal control of a nonlinear spring-mass-damper system. The approximate solution is shown to be accurate by comparison with a solution obtained by a collocation method.

Keywords: Optimal control, Lyapunov stability, system order reduction, manifold structure.

## 1. INTRODUCTION

The first-order necessary conditions for the solution to an optimal control problem comprise a Hamiltonian boundary-value problem (HBVP). An optimal control problem is called hyper-sensitive if the final time is large relative to some of the contraction and expansion rates of the associated Hamiltonian system. The solution to a hyper-sensitive problem can be qualitatively described in three segment as "take-off", "cruise" and "landing" analogous to optimal flight of an aircraft between distant locations (Kokotovic et al. [1986]). The "cruise" segment is primarily determined by the cost function and the state dynamics, whereas the "take-off" and "landing" segments are determined by the boundary conditions and the goal of connecting these to the "cruise" segment.

As the final time increases so does the duration of the cruise segment which shadows a trajectory on a reduced-order slow invariant manifold. When the final time is large, the sensitivity of the final state to the unknown initial conditions makes the HBVP ill-conditioned. The ill-conditioning can be removed by approximating the solution by a composite solution: a trajectory on a center-stable manifold that satisfies the initial boundary conditions is matched with a trajectory on a center-unstable manifold that satisfies the final boundary conditions.

The key to implementing this approach is a means of determining the unknown boundary conditions such that the solution end points lie on the appropriate invariant manifolds to sufficient accuracy. If the differential equations are in singularly perturbed normal form, then appropriate equilibrium-based manifold structure can be used. However, since the singularly perturbed normal form is often not available and a general approach to converting a system to this form does not exist, a method that does not require this normal form is desired. We describe how finite-time Lyapunov exponents and vectors can be used for this purpose and the issues involved.

Relevant previous research concerning optimal control is discussed and cited in [Rao and Mease, 2000, Topcu and Mease, 2006, Aykutlug and Mease, 2009]. And, though the focus in Guckenheimer and Kuehn [2009] is not optimal control, the proposed solution approach keys off the same geometric structure as our method, yet it is different in that it relies on the singularly perturbed form. The manifold structure of two-timescale ("slow-fast") systems is addressed in [Fenichel, 1979, Jones, 1994]. The use of finite-time Lyapunov exponents and vectors is guided by the asymptotic theory of partially hyperbolic sets, given in [Hasselblatt and Pesin, 2006]. Finite-time Lyapunov exponents and vectors have been studied and applied in several areas; atmospheric sciences, e.g., [Kalnay, 2003, Hartmann et al., 1996, Danforth and Yorke, 2006, Danforth and Kalnay, 2008], oceanic circulation and fluid mechanics, e.g., [Wolfe and Samelson, 2007, Haller, 2011, Lekien et al., 2007]; for further references in the literature see for example [Mease et al., 2003, 2012].

## 2. OPTIMAL CONTROL PROBLEM AND ASSOCIATED HAMILTONIAN BOUNDARY-VALUE PROBLEM

We consider the Lagrangian optimal control problem: determine the control function $u$ on the time interval $[0, t_f]$ that minimizes the cost function

$$\mathrm{J} = \int_0^{t_f} L(x(t), u(t))dt \tag{1}$$

$$\text{subject to} \quad \dot{x} = f(x, u)$$
$$x(0) = x_0,\ x(t_f) = x_f$$

where we assume that the vector field, $f(x, u)$, and $L(x, u)$ are smooth in both $x$ and $u$, and that $t_f$ is given. The state vector

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$x(t) \in \mathbb{R}^n$ and the control $u(t) \in \mathbb{R}^m$. The first-order necessary conditions for optimality lead to the Hamiltonian boundary-value problem (HBVP)

$$\begin{aligned} \dot{x} &= \frac{\partial H^*}{\partial \lambda} \\ \dot{\lambda} &= -\frac{\partial H^*}{\partial x} \\ x(0) = x_0&, \text{ and } x(t_f) = x_f, \end{aligned} \tag{2}$$

where $\lambda(t) \in \mathbb{R}^n$ is the costate vector and $H^* = L(x, u^*) + \lambda^T f(x, u^*)$ is the Hamiltonian evaluated at the optimal control $u^*(x(t), \lambda(t)) = \arg\min H(x(t), \lambda(t), u(t))$. We assume $u^*$ is a smooth function of $x$ and $\lambda$. The augmented state $p = (x, \lambda)$ is a point in the phase space $\mathbb{R}^{2n}$. In terms of $p$, we write the Hamiltonian dynamics in (2) as

$$\dot{p} = h(p) \tag{3}$$

where $h(p) = (\partial H/\partial \lambda, -\partial H/\partial x)^T$ is the Hamiltonian vector field. The linearized dynamics

$$\dot{v} = Dh(p)v. \tag{4}$$

are analyzed to characterize the timescales and associated phase space geometry for the flow of (3).

## 3. HYPER-SENSITIVITY

If numerical solution of an OCP proves difficult and reducing the final time alleviates the difficulty, hyper-sensitivity should be investigated. By observing how the solution evolves as $t_f$ is varied, the relevant phase space region can be identified. Using finite-time Lyapunov analysis, as described in the next section, on a grid of phase points in this region, the spectrum of exponential rates can be determined. If the spectrum uniformly separates into fast-stable, slow, and fast-unstable subsets, and the 'fast' rates are indeed fast relative to the time interval of interest, then hyper-sensitivity is confirmed. To describe the general case, let $n$ be the dimension of the state dynamics; then it follows that $2n$ is the dimension of the associated Hamiltonian system. The spectrum also reveals the equal dimensions, $n^s$ and $n^u$, of the fast-stable and fast-unstable behavior, respectively. If $n^s + n^u = 2n$, then the OCP is completely hyper-sensitive. If $n^s + n^u < 2n$, then the OCP is partially hyper-sensitive.

## 4. FINITE-TIME LYAPUNOV ANALYSIS

In Mease et al. [2003, 2012], finite-time Lyapunov analysis (FTLA) was applied to autononmous nonlinear dynamical systems to define and diagnose two-timescale behavior and compute points on a slow manifold, if one exists. The approach is to decompose the tangent bundle into subbundles on the basis of the characteristic exponential rates for the associated linear flow, and then to translate the tangent bundle structure into manifold structure in the base space.

In FTLA, the characteristic exponential rates and associated directions are given, respectively, by finite-time Lyapunov exponents (FTLEs) and finite-time Lyapunov vectors (FTLVs). This approach has been guided by the asymptotic theory of partially hyperbolic invariant sets, e.g., Barreira and Pesin [2002]. The finite-time tangent bundle decomposition can be viewed as an approximation of the asymptotic Oseledets' decomposition Barreira and Pesin [2002]. It has been established in Mease et al. [2012] that under certain conditions the finite-time decomposition approaches the (suitably defined) asymptotic decomposition exponentially fast, the rate being given by the size of the gaps in the spectrum of the FTLEs.

In Mease et al. [2003, 2012] it is shown how finite-time Lyapunov analysis (FTLA) can be used to diagnose multiple timescale behavior in dynamical system models. In the present context, the goal of FTLA is to determine if the nonlinear Hamiltonian system, $\dot{p} = h(p)$, has, at each point $p$ in a neighborhood $P \subset \mathbb{R}^{2n}$ of the solution of interest, a tangent space splitting $T_p\mathbb{R}^{2n}(p) = E^s(p) \oplus E^c(p) \oplus E^u(p)$, where all the vectors in the fast-stable subspace $E^s(p)$ contract exponentially fast in forward time and all the vectors in the fast-unstable subspace $E^u(p)$ contract exponentially fast in backward time, and all the vectors in the slow (i.e., center) subspace $E^c(p)$ change more slowly, under the linearized Hamiltonian flow. Note that in the finite-time setting, we use the terms stable and unstable for simplicity, even though fast contracting and fast expanding are more appropriate. Also we use the term center for the subspace associated with the FTLEs of small, but not necessarily zero, magnitude. In the asymptotic theory of partially hyperbolic sets Barreira and Pesin [2002], Hasselblatt and Pesin [2006], the splitting is invariant, however, when defined in terms of finite-time Lyapunov exponents and vectors (FTLE/Vs), the splitting only approximates an invariant splitting. Associated with the splitting is a manifold structure as depicted in Fig. 1.

Finite-time Lyapunov exponents (FTLE) characterize the average exponential expansion/contraction rates of a nonlinear system. The solution of (3) for the initial condition $p$ is denoted by $p(t) = \phi(t, p)$, where $\phi(t, \cdot) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the $t$-dependent flow associated with the vector field $h$ and $\phi(0, p) = p$. Let $\Phi$ denote the transition matrix for the linearized dynamics (4), defined such that $\Phi(0, p) = I$, the $2n \times 2n$ identity matrix. A vector $v \in T_p\mathbb{R}^{2n}$, propagated for $T$ units of time along the trajectory $\phi(t, p)$, evolves to the vector $\Phi(T, p)v$ in the tangent space $T_{\phi(T,p)}\mathbb{R}^{2n}$. The forward and backward FTLEs are given by

$$\begin{aligned} \mu^+(T, p, v) &= \frac{1}{T}\ln \sigma^+(T, p, v) = \frac{1}{T}\ln \frac{\|\Phi(T, p)v\|}{\|v\|}, \\ \mu^-(T, p, v) &= \frac{1}{T}\ln \sigma^-(T, p, v) = \frac{1}{T}\ln \frac{\|\Phi(-T, p)v\|}{\|v\|}, \end{aligned} \tag{5}$$

for propagation time $T$. We distinguish variables associated with forward-time propagation and backward-time propagation using the superscripts "$^+$" and "$^-$" respectively. The propagation time $T$, also referred to as the averaging time, is always taken to be positive whether forward or backward. For $v = 0$, define $\mu^+(T, p, 0) = \mu^-(T, p, 0) = -\infty$. A Lyapunov exponent allows the corresponding multiplier to be interpreted as an average exponential rate, i.e., $\sigma(T, p, v) = \exp[\mu(T, p, v)T]$; the average is over the time interval $[0, T]$.

Discrete forward and backward Lyapunov spectra, for each $(T, p)$, can be defined as follows. Define $l_i^+(T, p)$, $i = 1, \ldots, 2n$, to be the orthonormal basis of $T_p\mathbb{R}^{2n}$ with the minimum sum of exponents, i.e., the minimum value of $\Sigma_{i=1}^{2n}\mu_i^+(T, p, l_i^+(T, p))$ over all orthonormal bases, [Dieci and Vleck, 2002]. The forward Lyapunov spectrum is the set of exponents corresponding to the minimizing solution, namely, $\{\mu_i^+(T, p), i = 1, \ldots, 2n\}$. The Lyapunov spectrum is unique, though the minimizing basis is not in general.

One way [Dieci and Vleck, 2002, Mease et al., 2003] to obtain a minimizing basis (FTLVs) and the forward (similarly backward) Lyapunov spectrum (FTLEs) is to compute the singular value decomposition (SVD) of

$$\Phi(T,p) = N^+(T,p)\Sigma^+(T,p)L^+(T,p)^T,$$

where

$$\Sigma^+(T,p) = diag(\sigma_1^+(T,p),\ldots,\sigma_{2n}^+(T,p))$$

contains the singular values, all positive and ordered such that $\sigma_1^+(T,p) \leq \sigma_2^+(T,p) \leq \cdots \leq \sigma_{2n}^+(T,p)$, and to compute the Lyapunov exponents as $\mu_i^+(T,p) = (1/T)\ln\sigma_i^+(T,p)$, $i = 1,\ldots,2n$. The column vectors of the matrix $L^+(T,p)$ are the minimizing orthonormal basis vectors $l_i^+(T,p), i = 1,\ldots,2n$ for $T_p\mathbb{R}^{2n}$, and the column vectors of the orthogonal matrix $N^+(T,p)$ are denoted $n_i^+(T,p), i = 1,\ldots,2n$.

Similarly, the backward exponents can be obtained from the singular value decomposition

$$\Phi(T,p) = N^+(T,p)\Sigma^+(T,p)L^+(T,p)^T,$$

where

$$\Sigma^+(T,p) = diag(\sigma_1^+(T,p),\ldots,\sigma_{2n}^+(T,p))$$

. Assume the ordering on the diagonal of $\Sigma^-(T,p)$ is such that $\sigma_1^-(T,p) \geq \cdots \geq \sigma_{2n}^-(T,p)$. The column vectors of the orthogonal matrix $L^-(T,p)$ are denoted by $l_i^-(T,p)$, $i = 1,\ldots,2n$. For the column vectors of $L^-(T,p)$ and the orthogonal matrix $N^-(T,p)$, we have $l_i^-(T,p) \in T_p\mathbb{R}^{2n}$ whereas $n_i^-(T,p) \in T_{\phi(-T,p)}\mathbb{R}^{2n}$.

The $l_i^+(T,p)$ and the $l_i^-(T,p)$ vectors, for $i = 1,\ldots,2n$, referred to as forward and backward FTLVs, respectively, will be used to define subspaces in $T_p\mathbb{R}^{2n}$ associated with different exponential rates. Methods based on QR decomposition provide alternatives to computing FTLE/Vs Dieci and Vleck [2002].

The forward (backward) Lyapunov spectra are non-degenerate for particular arguments $(T,p)$, if there are $2n$ distinct forward (backward) FTLEs . We assume that for all $T \geq T_o$ and all values of $p$ under consideration, the forward and backward FTLE spectra are non-degenerate. $T_o$ is chosen just large enough to avoid initial transients in the exponents that are not representative of their behavior on most of the time interval of interest.

## 5. SOLUTION APPROXIMATION APPROACH

The solution approximation approach for partially hyper-sensitive optimal control problems follows from the observation that the solution shadows a trajectory on the center (i.e., slow) manifold most of the time, and the trajectory being shadowed is the intersection of center-stable and center-unstable manifolds. Let $\sigma$ denote the trajectory being shadowed on the center manifold $W^c$. The center-stable (center-unstable) manifold containing $\sigma$ is denoted $W^{cs}(\sigma)$ ($W^{cu}(\sigma)$). Both of these manifolds have dimension $n^s + 1 = n^u + 1$, and $W^{cs}(\sigma) \cap W^{cu}(\sigma) = \sigma$. There is an initial boundary-layer in which the solution shadows a trajectory on the center-stable manifold $W^{cs}(\sigma)$ and approaches $\sigma$ in forward time. Similarly there is a final boundary-layer in which the solution shadows a trajectory on the center-unstable manifold $W^{cu}(\sigma)$ and approaches $\sigma$ in backward time. Thus the strategy is to use the trajectories being shadowed to approximate the solution. From the tangent space geometry, conditions can be formulated for computing the unknown initial and final conditions such that the initial and final phases lie on $W^{cs}(\sigma)$ and $W^{cu}(\sigma)$ respectively.

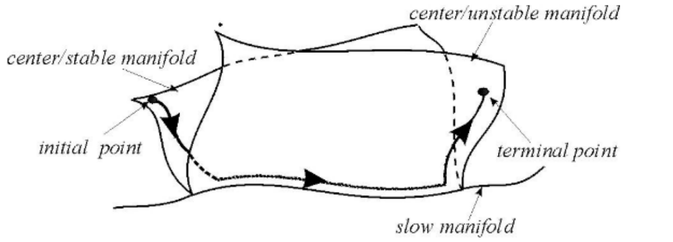


Fig. 1. Geometry of solution to partially hyper-sensitive optimal control problem in Hamiltonian phase space. Note that the slow manifold is even dimensional.

### 5.1 Matching on Slow Manifold

Choosing the unknown initial costates (unknown initial conditions) so that the initial phase point is on a center-stable manifold, suppresses the fast unstable behavior, and leaves $n - n^u$ degrees of freedom to control the trajectory and in particular, what trajectory it will approach on the slow manifold. Similarly, choosing the unknown final costates so that the final phase point is on a center-unstable manifold, suppresses the fast stable behavior for backward time integration, and leaves $n - n^s$ degrees of freedom to control the trajectory and in particular, what trajectory it will approach on the slow manifold. These degrees of freedom are used to match the forward and backward trajectories, within a specified tolerance, and construct a composite (approximate) solution.

More specifically, of the $n$ unknown initial conditions, $n^u$ are specified to satisfy the condition that $p \in W^{cs}$ whereas the remaining $n - n^u$ determine the trajectory on $W^c$. Similarly, of the $n$ unknown final conditions, $n^s$ are specified to satisfy the condition that $p \in W^{cu}$ whereas the remaining $n - n^s$ determine the trajectory on $W^c$. This means that $n - n^u$ initial conditions and $n - n^s$ final conditions are adjusted to achieve matching on $W^c$ at a selected matching time $t_m$.

Thus rather than determining all $n$ unknown conditions either at the initial time or the final time such that the conditions at the other end are satisfied, which would be very difficult due to hyper-sensitivity, we only determine $n - n^u = n - n^s$ at each end to construct a solution by matching in the middle and approximately on $W^c$, determining the other unknown boundary conditions at each end such that the fast-unstable motion is suppressed in the direction of integration. [1]

### 5.2 Computing Boundary Conditions on Invariant Manifolds

Assume the columns of a matrix $B$ are a basis for the tangent space $T_p\mathbb{R}^{2n}$ at each phase point $p$, consistent with the partially hyperbolic Lyapunov spectrum. Further assume the structure $B(p) = [B^s(p)\ B^c(p)\ B^u(p)]$, where $B^s(p) \in \mathbb{R}^{2n\times n^s}$, $B^c(p) \in \mathbb{R}^{2n\times n^c}$, and $B^u(p) \in \mathbb{R}^{2n\times n^u}$ contain the column vectors that span the stable, center and unstable subspaces respectively. At each phase point $p$, the vector $h(p)$ can be expressed as

$$\dot{p} = h(p) = B^s(p)w_s(p) + B^c(p)w_c(p) + B^u(p)w_u(p), \quad (6)$$

where $w_s(p)$, $w_c(p)$ and $w_u(p)$ are determined by $w_s(p) = [B^s(p)]^\dagger h(p)$, $w_c(p) = [B^c(p)]^\dagger h(p)$, $w_u(p) = [B^u(p)]^\dagger h(p)$. $[B^s(p)]^\dagger \in \mathbb{R}^{n^s\times 2n}$, $[B^c(p)]^\dagger \in \mathbb{R}^{n^c\times 2n}$ and $[B^u(p)]^\dagger \in$

[1] This strategy was first proposed by S.-H. Lam, circa 1990.

 

$\mathbb{R}^{n^u \times 2n}$ are composed of the appropriate rows of $B(p)^{-1}$, respectively.

To place the initial phase point on the center-stable manifold, $W^{cs}$, we impose the condition $w_u = 0$. Assume there is a splitting of the tangent space at $p(0) = p_0$, such that $T_{p_0}\mathbb{R}^{2n} = E^{cs}(p_0) \bigoplus (E^{cs}(p_0))^{\perp}$, where $E^{cs}(p_0) = E^{s}(p_0) \bigoplus E^{c}(p_0)$ and $(E^{cs}(p_0))^{\perp}$ is its orthogonal complement. If $p_0$ were on a center-stable manifold, i.e., $w_u(p_0) = 0$, then the following condition would hold

$$< h(p_0), v >= 0,\ \forall v \in (E^{cs}(p_0))^{\perp}. \tag{7}$$

An approximation of the subspace $E^{cs}(p_0)$, denoted $\hat{E}^{cs}(p_0)$, can be obtained by using the finite-time Lyapunov vectors

$$\hat{E}^{cs}(p_0) = span\{l_1^+(T, p_0), \cdots, l_{n^s+n^c}^+(T, p_0)\}$$

and an approximation of the subspace $(E^{cs}(p_0))^{\perp}$ is

$$(\hat{E}^{cs}(p_0))^{\perp} = span\{l_{n^s+n^c+1}^+(T, p_0), \cdots, l_{2n}^+(T, p_0)\}.$$

Similarly, at the final condition $p(t_f) = p_f$, assume there is a splitting of the tangent space $T_{p_f}\mathbb{R}^{2n} = E^{cu}(p_f) \bigoplus (E^{cu}(p_f))^{\perp}$, where $E^{cu}(p_f) = E^{u}(p_f) \bigoplus E^{c}(p_f)$ and $(E^{cu}(p_f))^{\perp}$ is its orthogonal complement. If $p_f$ were on the center-unstable manifold, i.e., if $w_s(p_f) = 0$, then the following condition would hold

$$< h(p_f), v >= 0,\ \forall v \in (E^{cu}(p_f))^{\perp}. \tag{8}$$

Approximations to subspaces $E^{cu}(p_f)$ and $(E^{cu}(p_f))^{\perp}$ can be obtained by using finite-time Lyapunov vectors

$$\hat{E}^{cu}(p_f) = span\{l_{n^s+1}^-(T, p_f), \cdots, l_{2n}^-(T, p_f)\}$$

and

$$(\hat{E}^{cu}(p_f))^{\perp} = span\{l_1^-(T, p_f), \cdots, l_{n^s}^-(T, p_f)\}.$$

Using these finite-time Lyapunov subspaces and the orthogonality conditions (7) and (8), one can choose the unknown boundary conditions to locate the initial and final phase points approximately on the appropriate invariant manifolds. Then the Hamiltonian system can be integrated forward or backward in time to reach the slow invariant manifold. However, because the boundary conditions are only approximately on the invariant manifolds the trajectories will depart. The procedure in the next subsection re-initializes the integration to keep the trajectory close to the invariant manifold.

### *5.3 Re-Initialization Procedure*

In this subsection we describe the re-initialization procedure that repeatedly projects the evolving phase trajectory toward $W^{cs}$. For the initial segment, we are ideally computing a trajectory on $W^{cs}$. However, because the initialization of $p(t_0)$ on $W^{cs}$ is only approximate, the trajectory will start off of $W^{cs}$ and depart farther with time due to the fast-unstable component of the vector field.

At $t = 0$, the fast-unstable component of the vector field that cannot be suppressed is

$$w_u(p_0) = [B^u(p_0)]^{\dagger} \hat{B}^{cs}(T, p_0) h(p_0)$$

where

$$\hat{B}^{cs}(T, p_0) = \hat{B}^{s}(T, p_0)[\hat{B}^{s}(T, p_0)]^{\dagger} + \hat{B}^{c}(T, p_0)[\hat{B}^{c}(T, p_0)]^{\dagger}.$$

When we map this component forward in time, we get

$$w_u(t) = \Phi_u(t, p_0)([B^u(p_0)]^{\dagger} \hat{B}^{cs}(T, p_0) h(p_0)).$$

To force the trajectory to follow $W^{cs}$ more closely, the phase is re-initialized periodically to bring it closer to $W^{cs}$ and reduce $w_u$. Let $[0, t_m]$ be subdivided into intervals $[t_{i-1}, t_i]$, $i = 1, \ldots, k$, where $t_0 = 0$ and $t_k = t_m$. By imposing $\hat{w}_u(t_i) = 0$, for $i = 1, \cdots, n-1$, and updating current phase point $p(t_i)^-$ to $p(t_i)^+$, we project the phase point closer to the invariant manifold at each $t_i$. Figure 2 illustrates the re-initialization procedure. Re-initialization introduces discontinuities that could be reduced by more frequent updates or smoothed by backward-time integration, because the $W^{cs}$ is attracting in backward time.

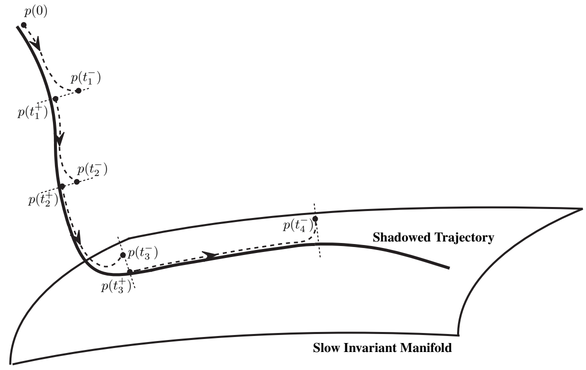


Fig. 2. Illustration of the re-initialization procedure to counteract the departures, due to the fast unstable component of the vector field, and thus shadow more closely the trajectory on $W^{cs}$. $p(t_i^-)$ and $p(t_i^+)$, $i = 1, 2, 3$, denote the phase space points right before and after the $i$-th re-initialization.

## 6. MINIMUM ENERGY CONTROL OF SPRING-MASS-DAMPER SYSTEM

We illustrate the approach for the optimal control problem

$$\begin{aligned} \min \quad J &= \frac{1}{2}\int_0^{t_f} u^2 dt \\ \text{such that} \quad \dot{x_1} &= x_2 \\ m\dot{x_2} &= -k_1 x_1 - k_2 x_1^3 - c x_2 + u \\ x_1(0) &= 2.4, \quad x_2(0) = 0.0, \\ x_1(t_f) &= -2.4, \quad x_2(t_f) = 0.0, \end{aligned} \tag{9}$$

with a specified final time. The first-order necessary conditions lead to the Hamiltonian boundary-value problem

$$\begin{aligned} \dot{x}_1 &= x_2 \\ m\dot{x}_2 &= -k_1 x_1 - k_2 x_1^3 - c x_2 - \tilde{\lambda}_2 \\ \dot{\lambda}_1 &= \tilde{\lambda}_2(k_1 + 3k_2 x_1^2) \\ m\dot{\tilde{\lambda}}_2 &= -\lambda_1 + c\tilde{\lambda}_2 \end{aligned} \tag{10}$$

where the costate associated with $x_2$ is $\lambda_2 = m\tilde{\lambda}_2$; see Kokotovic et al. [1986]. The boundary conditions are those on the state given above.

This system has two-timescale behavior when the mass $m$ is sufficiently small. When the final time is long relative to fast contraction and expansion rates, but not the slow contraction and expansion rates, the optimal control problem is partially hyper-sensitive with $n = 2$, $n^s = n^u = 1$ and $n^c = 2$. For the numerical results, we use $k_1 = 1$, $k_2 = 0.1$, $c = 1.265$, $m = 0.1$ and $t_f = 2.0$. FTLEs at three different phase points on the center-stable manifold are shown in Fig. 3, where each

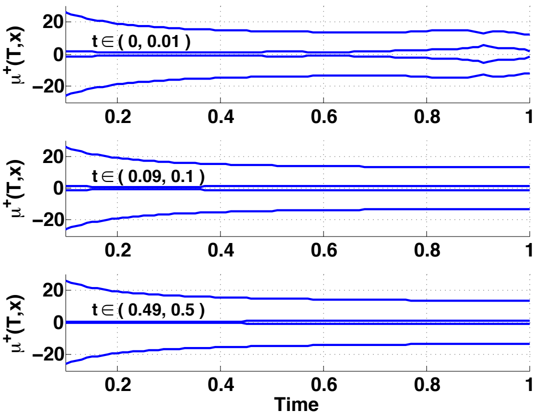


Fig. 3. Finite-time Lyapunov exponents versus averaging time at three phase points, respectively, [2.400; 0.000; 0.786; 0.076], [2.270 ;-2.411 ;0.989 ;0.093], and [1.150; -2.506; 1.961; 0.170], which are obtained by the 2nd iteration of the orthogonality conditions at the start of the time intervals shown on the subplots.

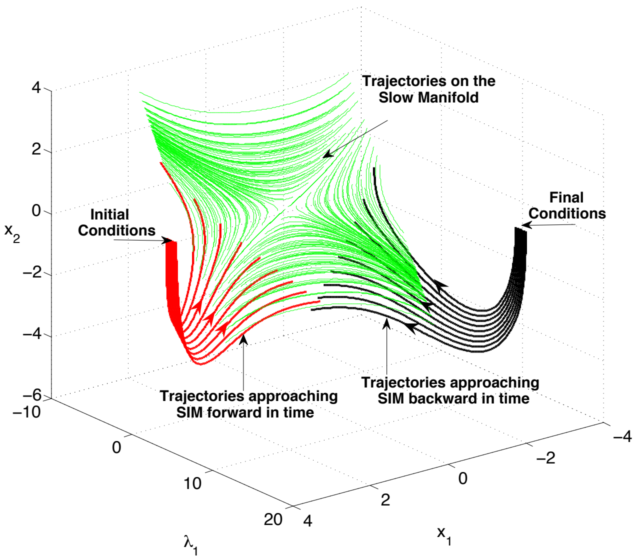


Fig. 4. One-parameter families of forward and backward trajectories to visualize the matching on the slow invariant manifold (SIM).

plot shows the FTLEs versus averaging time $T$. At each re-initialization ($\Delta t = 0.01$), the orthogonality conditions are applied twice to project the phase point closer to the appropriate manifold. Although in each case there is a gap between the slow and fast exponents, this gap becomes more uniform as the phase point gets closer to the center-stable manifold. When the gap between the slow and fast exponents is uniform for longer times, longer averaging times can be used in finite-time Lyapunov analysis, which leads to better approximation of the ideal asymptotic FTLEs/FTLVs. For the numerical results, an averaging time $T \leq 1.0$ was used in all cases.

The approximate solution is constructed by integrating the Hamiltonian dynamics forward in time starting from an initial condition approximately on the center/stable manifold and backward in time starting from a final condition approximately on the center/unstable manifold. This is done by using the re-initialization procedure to project the phase back toward the appropriate manifold, namely, by applying the orthogonality condition

$$\langle h(p(t_i)), l_4^+(T, p(t_i))\rangle = 0, \tag{11}$$

during the integration of the forward segment and

$$\langle h(p(t_i)), l_1^-(T, p(t_i))\rangle = 0. \tag{12}$$

during the integration of the backward segment. Specifically at the initial and final times, conditions (11) and (12) provide two additional boundary conditions; they are used to specify $\lambda_2$ for both forward and backward integration. The two trajectories departing from the initial and final points are matched on the center manifold, using the remaining degree of freedom at each boundary, namely the value of $\lambda_1$. Two families of trajectories can be generated by varying the costates $\lambda_1(0)$ and $\lambda_1(t_f)$.

The matching on the center manifold occurs for unique values of $\lambda_1(0)$ and $\lambda_1(t_f)$ which are computed through a search-based automated technique. Starting from a broad set of values for the initial and final costates, the distance between the end points of each forward and backward trajectory is calculated. Since we are operating with finite-time Lyapunov vectors and $t_f$ is not infinite either, we will not be able to find two trajectories that perfectly match on the center invariant manifold. Therefore, we allow the distance between the forward and backward trajectories to be greater than zero but less than a specified value. In this example using $\lambda_1(0) = 0.786$ and $\lambda_1(t_f) = 10.354$, the forward and backward trajectories are matched with a error less than $0.0015$.

Figure 4 shows the forward and backward families of trajectories for different values of $\lambda_1(0)$ and $\lambda_1(t_f)$ respectively. Also shown is an approximation of the center (i.e., slow) manifold based on the zeroth-order singular perturbation approximation. With $m = 0$, Eqs. (10) are differential-algebraic; the differential equations for $x_1$ and $\lambda_1$ are integrated for different initial conditions $(x_1, \lambda_1)$ and with $x_2$ and $\lambda_2$ determined from the algebraic equations. The existence of forward and backward trajectories that can be matched on the slow manifold to form a composite solution can be visualized.

To assess the accuracy of the solution, it is compared to solutions obtained by two other methods. One method is identical to our method except that the basis for the partially hyperbolic tangent space splitting is constructed from eigenvectors of $Dh(p)$ rather than FTLVs. The second method is to use the general purpose OCP solver GPOPS (Rao et al. [2010]). The three solutions are displayed in Fig. 5 and the three control profiles are displayed in Fig. 6. Re-initialization with $\Delta t = 0.01$ is used for both the FTLA method and the eigenvector method. The FTLA and GPOPS solutions are indistinguishable, whereas the eigenanalysis solution is less accurate. For larger $x_1$, the eigenvector method is not applicable, because the eigenvalues and eigenvectors become complex and do not reveal the two timescales.

## 7. CONCLUSIONS

A method for approximately solving partially hyper-sensitive of optimal control problems has been described. Finite-time Lyapunov exponents and vectors were used to diagnose and characterize the associated geometric structure, leading to a solution approximation by numerically matching forward and backward trajectory segments. The approach was illustrated on an optimal control problem for a nonlinear spring-mass-damper system and shown to produce an accurate solution approximation.

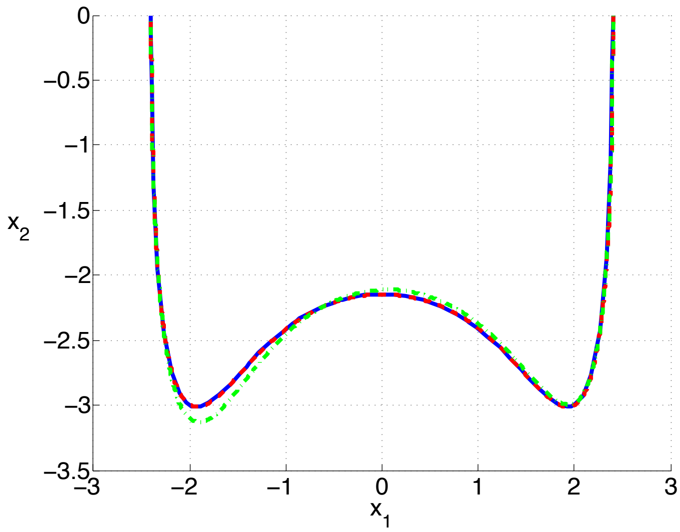


Fig. 5. Solutions obtained by eigenanalysis (green dashed line), FTLA (red dashed-line), GPOPS (blue line).

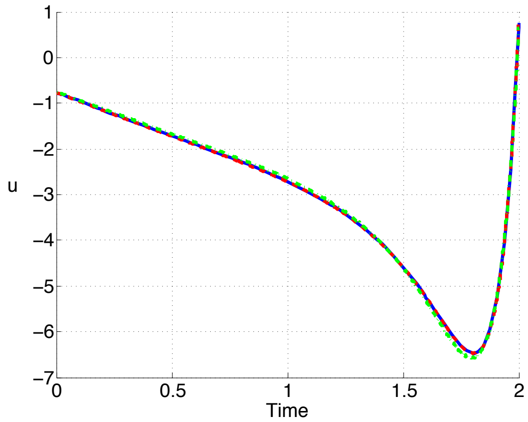


Fig. 6. Control, $u$, obtained by eigenanalysis (green dashed line), FTLA (red dashed-line), GPOPS (blue line).

## REFERENCES


E. Aykutlug and K. D. Mease. Approximate solution of hypersensitive optimal control problems using finite-time Lyapunov analysis. St. Louis, Missouri, USA, 2009. Proceedings of the American Control Conference.

L. Barreira and Y. B. Pesin. *Lyapunov Exponents and Smooth ergodic Theory*. American Mathematical Society, Providence, USA, 2002.

C. M. Danforth and E. Kalnay. Using singular value decomposition to parametrize state-dependent model errors. *Journal of atmospheric sciences*, 65(4):1467 – 1478, 2008.

C. M. Danforth and J. A. Yorke. Making forecasts for chaotic physical processes. *Physical review letters*, 96:144102, 2006.

L. Dieci and E. S. Van Vleck. Lyapunov spectral intervals: theory and computation. *J. Numerical Analysis*, 40(5):516–542, 2002.

N. Fenichel. Geometric singular perturbation theory for ordinary differential equations. *Journal of Differential Equations*, 31:53 – 98, 1979.

J. Guckenheimer and C. Kuehn. Computing slow manifolds of saddle type. *SIAM J. Appl. Dyn. Syst.*, 8(3):854– 879, 2009.

G. Haller. A variational theory of hyperbolic Lagrangian coherent structures. *Physica D*, 240:574 – 598, 2011.

D. L. Hartmann, R. Buizza, and T. N. Palmer. Finite-time instabilities of lower-stratophic flow. *Journal of the Atmospheric Sciences*, 53(15):2129 – 2143, 1996.

B. Hasselblatt and Y. Pesin. *Partially Hyperbolic Dynamical Systems, Handbook of Dynamical Systems v.1B*. Elsevier, Boston, 2006.

C. K. R. T. Jones. Geometric singular perturbation theory. *Montecatini, Terme, R. Johnson Ed., Springer Lecture Notes in Mathematics*, 169:44 – 118, 1994.

E. Kalnay. *Atmospheric Modeling, Data Assimilation and Predictability*. Cambridge University press, Cambridge, UK, 2003.

P. V. Kokotovic, H. K. Khalil, and J. OReilly. *Singular Perturbation Methods in Control: Analysis and Design*. Academic Press, 1986.

F. Lekien, S. C. Shadden, and J. E. Marsden. Lagrangian coherent structures in n-dimensional systems. *Journal of Mathematical Physics*, 48:065404, 2007.

K. D. Mease, S. Bharadwaj, and S. Iravanchy. Timescale analysis for nonlinear dynamical systems. *Journal of Guidance Control and Dynamics*, 26(1):318 – 330, 2003.

K. D. Mease, U. Topcu, E. Aykutlug, and M. Maggia. Characterizing two-timescale nonlinear dynamics using finite-time Lyapunov exponents and vectors. eprint available at arxiv.org (arXiv:0807.0239v2 [math.DS]), 2012.

A. V. Rao and K. D. Mease. Dichotomic basis approach to solving hyper-sensitive optimal control problems. *Automatica*, 35:633 – 642, 2000.

A.V. Rao, D.A. Benson, C.L. Darby, M.A. Patterson, C. Francolin, I. Sanders, and G.T. Huntington. Algorithm 902: Gpops, a matlab software for solving multiple-phase optimal control problems using the gauss pseudospectra method. *ACM Trans. Math. Software*, 37(2), 2010.

U. Topcu and K. D. Mease. Using Lyapunov vectors and dichotomy to solve hyper-sensitive optimal control problems. San Diego, CA, USA, 2006. Proceedings of the $45^{th}$ IEEE Conference on Decision and Control.

C. L. Wolfe and R. M. Samelson. An efficient method for recovering Lyapunov vectors from singular vectors. *Tellus*, 59A:355 – 366, 2007.