Stabilization of Nonlinear Systems via Potential-based Realization

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Abstract: This paper considers the problem of representing a sufficiently smooth nonlinear system as a structured potential-driven system and to exploit the obtained structure for the design of nonlinear state feedback stabilizing controllers. The problem has been studied in recent years for systems modeled as structured potential-driven systems, for example gradient systems, generalized Hamiltonian systems and systems given in Brayton–Moser form. To recover the advantages of those representations for the stabilization of general nonlinear systems, the present note proposes a geometric decomposition technique to re-express a given vector field into a desired potential-driven form. The decomposition method is based on the Hodge decomposition theorem, where a one-form associated to the given vector field is decomposed into its exact, co-exact, and harmonic parts.

Keywords: Nonlinear systems, Hodge decomposition, Feedback stabilization.

1. INTRODUCTION

Analysis and control design based on potential-based representations, such as gradient systems (Cortés et al., 2005), generalized Hamiltonian systems (Ortega et al., 2002; Cheng et al., 2002), and systems given by Brayton–Moser equations (Jeltsema and Scherpen, 2009; García-Canseco et al., 2010; Favache et al., 2011), are now central to nonlinear control theory. Under mild assumptions, stability analysis and design of stabilizing feedback control is greatly simplified for systems given or re-expressed as potential-driven systems, see for example the contributions given in (Ortega et al., 2002, 2003; García-Canseco et al., 2010). For mechanical and electro-mechanical systems, such representations can usually be derived from first principles. However, for applications where the concept of free energy is ill-defined, for example in irreversible non-equilibrium thermodynamic systems (Favache and Dochain, 2010), where such representations are not available *a priori*, these approaches are often limited. In some applications, the problem of re-expressing a system of balance laws as a potential-based representation, using exact (Astolfi and Ortega, 2009) or approximate matching conditions (Ramírez et al., 2009), proved difficult to be solved.

In the present note, we consider the problem of feedback stabilization design for nonlinear control affine systems of the form

$$\dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \tag{1}$$

where $f \in \mathcal{C}^k$, with $k \geq 2$. We assume that some desired equilibrium of x^* is at least locally reachable (Coron, 2007).

The approach proposed here refines the technique proposed originally in (Hudon et al., 2008), which was based on the application of a homotopy operator (see Section 3) on a differential one-form associated to the system (1), to compute a generating potential for a given system. The approximate approach from (Hudon et al., 2008) is based on the Poincaré lemma, which has been used recently to compute vector potentials in (Yap, 2009) and in the context of Brayton–Moser representation in (Favache et al., 2011). In (Guay et al., 2012), a refined approach to this early construction was proposed by further characterizing the structure of the dynamics. This was achieved by identifying the co-exact and the anti-coexact parts of the dynamics using a dual operator to the homotopy operator (see Section 3).

The proposed approach is related to the representation of smooth nonlinear dynamics as the sum of a gradient system and $(n-1)$ Hamiltonian systems, as presented for example in (Roels, 1974) and more recently in (Steeb and Scholes, 2005; Presnov, 2008) using Hodge theory for systems of low dimensionsal systems. In the present note, we propose a decomposition of a class of nonlinear systems and exploit this decomposition to analyze the stability of arbitrary vector fields and the stabilization of nonlinear control affine systems.

The paper is organized as follows. Mathematical preliminaries for the proposed construction are presented in Section 2. The homotopy operator, which is used to identify the gradient part of the dynamics, and the dual homotopy operator, which allows us to invert the co-differential operator, are introduced in Section 3. A decomposition of a class of nonlinear systems is proposed in Section 4. The application of the decomposition to solve stabilization problems is depicted in Section 5. Conclusions and areas for further research are given in Section 6.

2. MATHEMATICAL PRELIMINARIES

The decomposition problem considered is solved using exterior calculus, reviewed extensively in (Edelen, 2005). The derivation of a differential one-form associated to the drift vector field $f(x)$ relies on the canonical Riemannian metric in $\mathbb{R}^n$, given as $g = dx_1 \otimes dx_1 + \ldots + dx_n \otimes dx_n$, with its associated volume form in $\Lambda^n(\mathbb{R}^n)$, expressed as $\mu = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$. For a given drift vector field $f(x) = \sum_{i=1}^n f_i(x)\frac{\partial}{\partial x_i}$, we seek to find a structured potential-based representation. The central element to be exploited in the sequel is the divergence of the vector field $f(x)$, computed following Lee (2006). A $(n-1)$ differential form j is first obtained by taking the interior product of the volume μ with respect to the drift vector field $f(x)$, *i.e.*,

$$j = \left(\sum_{i=1}^n f_i(x)\frac{\partial}{\partial x_i}\right) \lrcorner \mu$$
$$= \sum_{i=1}^n (-1)^{(i-1)} f_i(x) dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n, \qquad (2)$$

where $\widehat{dx_i}$ denotes a removed element such that j is a $(n-1)$ form. Taking the exterior derivative of j, and by the property of the wedge product that $dx_i \wedge dx_i = 0$, we obtain

$$dj = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(x) dx_1 \wedge \ldots \wedge dx_n = \mathrm{div} f(x)\mu. \qquad (3)$$

The proposed construction consists in computing a differential one-form $\omega \in \Lambda^1(\mathbb{R}^n)$ that encodes the divergence of the drift vector field $f(x)$. Such a one-form is obtained by applying the Hodge star operator $\star : \Lambda^k(\mathbb{R}^n) \to \Lambda^{n-k}(\mathbb{R}^n)$ to the $(n-1)$ form j (Morita, 2001, Chapter 4),

$$\omega = \star j = \star(f(x) \lrcorner \mu) = (-1)^{n-1} \sum_{i=1}^n f_i(x) dx_i. \qquad (4)$$

Following (Morita, 2001), we also define the co-differential operator, $\delta : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k-1}(\mathbb{R}^n)$, as

$$\delta = (-1)^{n(k+1)+1} \star d \star. \qquad (5)$$

3. HOMOTOPY AND DUAL HOMOTOPY OPERATORS

This section presents the homotopy operator that inverts locally the exterior derivative d, following (Edelen, 2005), and proposes the construction of a dual homotopy operator that inverts locally the co-differential operator δ, defined by 5.

3.1 Homotopy Operator

The homotopy decomposition considered in the present paper is based on (Edelen, 2005, Chapter 5). A more general construction of such an operator is given in (Lee, 2006). The homotopy operator $\mathbb{H}$ is a linear operator on elements of $\Lambda^k(\mathbb{R}^n)$ that satisfies the identity

$$\omega = d(\mathbb{H}\omega) + \mathbb{H}d\omega, \qquad (6)$$

for a given differential form $\omega \in \Lambda^k(\mathbb{R}^n)$. The first step in the construction of a homotopy operator is to define a star-shaped domain on $\mathbb{R}^n$. An open subset S of $\mathbb{R}^n$ is said to be star-shaped with respect to a point $\boldsymbol{p}^* = (x_1^*, \ldots, x_n^*) \in S$ if the following conditions hold:

- S is contained in a coordinate neighborhood U of $\boldsymbol{p}^*$;
- The coordinate functions of U assign coordinates $(x_1^*, \ldots, x_n^*)$ to $\boldsymbol{p}^*$;
- If p is any point in S with coordinates $(x_1, \ldots, x_n)$ assigned by functions of U, then the set of points $(x^* + \lambda(x - x^*))$ belongs to S, for all $\lambda \in [0, 1]$.

A star-shaped region S has a natural associated vector field $\mathfrak{X}$, defined in local coordinates by

$$\mathfrak{X}(x) = \sum_{i=1}^n (x_i - x_i^*)\frac{\partial}{\partial x_i}, \quad \forall x \in S.$$

For a differential form ω of degree k on a star-shaped region S centered at the origin, the homotopy operator is defined, in coordinates, as

$$(\mathbb{H}\omega)(x) = \int_0^1 \mathfrak{X}(x) \lrcorner \omega(x^* + \lambda(x - x^*))\lambda^{k-1} d\lambda,$$

where $\omega(x^* + \lambda(x - x^*))$ denotes the differential form evaluated on the star-shaped domain in the local coordinates defined above. The important properties of the homotopy operator used in the present context are the following:

1. $\mathbb{H}$ maps $\Lambda^k(S)$ into $\Lambda^{k-1}(S)$ for $k \geq 1$ and maps $\Lambda^0(S)$ identically to zero;
2. $d\mathbb{H} + \mathbb{H}d =$ identity for $k \geq 1$ and $(\mathbb{H}df)(x) = f(x) - f(x_0)$ for $k = 0$;
3. $(\mathbb{H}\mathbb{H}\omega)(x_i) = 0, \quad (\mathbb{H}\omega)(x_i^*) = 0$;
4. $\mathfrak{X}\lrcorner\mathbb{H} = 0, \quad \mathbb{H}\mathfrak{X}\lrcorner = 0$.

The first part of the right hand side of (6), $d(\mathbb{H}\omega)$, is obviously a closed form, since by definition of the exterior derivative, $d \circ d(\mathbb{H}\omega) = 0$. By the first property of the homotopy operator, for $\omega \in \Lambda^k(S)$, we have $(\mathbb{H}\omega) \in \Lambda^{k-1}(S)$, hence $d(\mathbb{H}\omega)$ is also exact on S. We denote the exact part of ω by $\omega_e = d(\mathbb{H}\omega)$ and the anti-exact part by $\omega_a = \mathbb{H}d\omega$, computed as $\omega_a = \omega - \omega_e$.

3.2 Dual Homotopy Operator

In a manner that is similar to the definition of the homotopy operator, one can define a dual homotopy operator based on the notion of co-exact forms (Guay et al., 2012). The dual homotopy operator $\mathbb{S}$ is a linear operator on elements of $\Lambda^k(\mathbb{R}^n)$ that satisfies the identity

$$\omega = \delta(\mathbb{S}\omega) + \mathbb{S}\delta\omega, \qquad (7)$$

for a given differential form $\omega \in \Lambda^k(\mathbb{R}^n)$.

Proposition 1. (Guay et al., 2012) The dual homotopy operator $\mathbb{S}$ can be written in terms of the homotopy operator $\mathbb{H}$ as follows:

$$\mathbb{S} = (-1)^{n(k+1)+1} \star \mathbb{H} \star. \qquad (8)$$

The following theorem summarizes some useful properties of the dual homotopy operator, following directly from the definition of the dual homotopy operator and the properties of the homotopy operator $\mathbb{H}$.

Theorem 2. (Guay et al., 2012) The dual homotopy operator has the following properties:

1. $\mathbb{S}$ maps $\Lambda^k(S)$ into $\Lambda^{k+1}(S)$ for $k \geq 0$ and maps $\Lambda^n(S)$ to zero.
2. $\delta\mathbb{S} + \mathbb{S}\delta =$ identity for $0 \leq k < n-1$ and $(\mathbb{S}\delta\omega)(x) = f(x) - f(x_0)$ for $\omega \in \Lambda^{n-1}(S)$ and $f(x) \in \Lambda^0(S)$.
3. $(\mathbb{S}\mathbb{S}\omega)(x_i) = 0, \quad (\mathbb{S}\omega)(x_i^0) = 0$.
4. $\mathfrak{W} \wedge \mathbb{S} = 0, \quad \mathbb{S}\mathfrak{W}\wedge = 0$.

4. DECOMPOSITION AND POTENTIAL-BASED REPRESENTATIONS

The objective of this section is to show how the homotopy operator $\mathbb{H}$ and the dual homotopy operator $\mathbb{S}$ can be used to represent a smooth nonlinear system of the form (1) as a structured potential-driven system, using a suitably chosen one-form $\omega(x)$ computed in Section 2.

4.1 Potential-based representations

The approach proposed in this paper is focused on the use of homotopy and the dual homotopy operators to extract specific structures of the vector field $X = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$. In order to do so, we first define a non vanishing closed two-form $\Omega = \sum_{1\leq i<j\leq n} dx_i \wedge dx_j$ on $\mathbb{R}^n$. The exterior derivative of the one-form $\omega(x) = \star j$ from (4) is given by

$$d\omega = \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} -\frac{\partial f_i}{\partial x_j} dx_j \wedge dx_i. \tag{9}$$

Computing the co-differential of $\omega(x)$, we obtain

$$\star\omega = \sum_{i=1}^{n} (-1)^{i+1} f_i \alpha_i, \tag{10}$$

where

$$\alpha_i = dx_1 \wedge \ldots \wedge dx_{i-1} \wedge \widehat{dx_i} \wedge dx_{i+1} \wedge \ldots \wedge dx_n$$

where the notation $\widehat{dx_i}$ indicates that this one-form is removed. The exterior derivative is given by

$$d(\star\omega) = \sum_{i=1} (-1)^{i+1} \frac{\partial f_i}{\partial x_i} dx_i \wedge \alpha_i = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \mu. \tag{11}$$

Finally, taking the Hodge star, we obtain the function

$$\delta\omega = \star d \star \omega = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}. \tag{12}$$

4.2 Decomposition of nonlinear systems

The decomposition, based on the homotopy and the dual-homotopy operator, can be used to separate the gradient part from the co-exact part. A residual form resulting from these decompositions is shown to possess special properties that provide some valuable information about the dynamics of the system.

As above, consider the one-form $\omega = \star j$. There are two possible ways to decomposition this system using the homotopy operators $\mathbb{H}$ and $\mathbb{S}$. Using $\mathbb{H}$, one generates the decomposition:

$$\omega = d\mathbb{H}\omega + \mathbb{H}d\omega.$$

The anti-exact part can be further decomposed into a coexact and anti-coexact part using $\mathbb{S}$ as follows:

$$\omega = d\mathbb{H}\omega + \delta\mathbb{S}\mathbb{H}d\omega + \mathbb{S}\delta\mathbb{H}d\omega$$

The first term of this decomposition yields a gradient element of the dynamics. The second term yields the anti-symmetric part of the dynamics. The third component is defined as

$$\gamma_{\mathbb{H}} = \mathbb{S}\delta\mathbb{H}d\omega. \tag{13}$$

By the properties of the dual homotopy operator, it follows that this form is such that:

$$\mathfrak{W} \wedge \gamma_{\mathbb{H}} = 0,$$

where $\mathfrak{W}$ is the one-form defined on the star-shaped domain used to define $\mathbb{S}$, *i.e.* $\mathfrak{W}(x)$ is the one-form dual to the vector field $\mathfrak{X}(x)$. As a result, one can write $\gamma_{\mathbb{H}}$ as

$$\gamma_{\mathbb{H}} = Q(x)\mathfrak{W},$$

where $Q(x)$ is a smooth function. We can summarize this result as follows:

Proposition 3. Consider a smooth nonlinear dynamical system $\dot{x} = f(x)$ with corresponding one-form, $\omega = \star j$. The one-form ω can be decomposed as follows:

$$\omega = d\mathbb{H}\omega + \delta\mathbb{S}\mathbb{H}d\omega + \gamma_{\mathbb{H}} \tag{14}$$

where the one-form $\gamma_{\mathbb{H}} = Q(x)\mathfrak{W}$.

This decomposition yields a decomposition of the one-form (and hence the nonlinear system) as the sum of a gradient form, an anti-symmetric form and a form co-linearly dependent to the one-form used in the definition of the homotopy operators.

Let us consider the decomposition in Proposition 3. By duality, the decomposition of the one-form gives rise to a dynamical system of the form:

$$\dot{x} = \nabla_x P + \sum_{i\neq j}^{n} J_{ij} \nabla_x H_{ij}^T + Q(x)x$$

The first term on the right hand side is the gradient part of the flow. The second term provides the anti-symmetric component of the dynamics. The third term takes the form of the gradient of $\|x\|^2$. This term can be absorbed in some way into the gradient part of the dynamics. Note that if $Q(x) \leq 0$ this part of the vector fields yields flows that are everywhere normal to the $(n-1)$ dimensional sphere that collapse to the origin since $\|x\|^2$ would constitute a Lyapunov function for this part of the dynamics.

5. STABILITY AND STABILIZATION

In this section, we consider the application of the decomposition to solve stabilization problems for control affine nonlinear control system (1).

5.1 Stability

In this study, we identified a normal form that s generated by functions $P(x)$ and $H_{ij}(x)$ $(i \neq j)$. These functions can be used to assist in the stabilization of control affine nonlinear systems of the form

$$\dot{x} = \nabla_x P^T + \sum_{i\neq j} J_{ij} \nabla_x H_{ij}^T + Q(x)x + g(x)u. \tag{15}$$

We make the following assumption.

Assumption 4. Assume that the functions P and H_{ij} are such that for a neighborhood $\mathcal{D}$ of the equilibrium x_e:

(1) $\nabla_x P(x_e) = \nabla_x H_{ij}(x_e) = 0$,
(2) $\nabla_x^2 P(x) \le -\alpha I$,

for all $x \in \mathcal{D}$ and for all i, j and positive constant α.

We obtain the following stability result.

Theorem 1. Let the nonlinear system

$$\dot{x} = f(x) \tag{16}$$

generate a decomposition with potentials that meet Assumption 4. Then the origin is a local exponentially stable equilibrium of the system.

Proof. Let us note that the interesting property of the decomposition is that the antiexact part ω_a that gives rise to the last two terms is such that

$$\mathfrak{X} \lrcorner \omega_a \equiv 0. \tag{17}$$

Let us assume that one chooses the vector field such that $\mathfrak{X} = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$. Then the dynamics can be written as

$$\dot{x} = \nabla_x P^T + U(x), \tag{18}$$

where $U(x) = \sum_{i \neq j} J_{ij} \nabla_x H_{ij}^T + Q(x)x$. It follows by construction that $x^T U(x) \equiv 0$. Furthermore, it always possible by assumption that the gradient of $P(x)$ can be written as $\nabla_x P^T = \Theta(x)x$ where

$$\Theta(x) = \int_0^1 \nabla^2 P(\lambda x) d\lambda. \tag{19}$$

Next consider the simple Lyapunov function, $V = \frac{1}{2}x^T x$. Its derivative with respect to time yields

$$\dot{V} = x^T \nabla_x P^T + x^T U(x) = x^T \nabla_x P^T. \tag{20}$$

Based on the discussion above, it follows that the second term is identically zero. Moreover, it follows that, by assumption, one can write

$$\dot{V} = -x^T \Theta(x) x, \tag{21}$$

and for all $x \in D$ we have

$$\dot{V} \le -\alpha \|x\|^2. \tag{22}$$

As a result, local exponentially stability of the system over $\mathcal{D}$ is achieved, as required.

This construction is remarkably simple and easy to perform using the homotopy operator. One of the interesting aspects of the state-dependent matrix $Q(x)$ is that it inherits a lot of its structure from the Jacobian of $f(x)$. To see this, consider the homotopy operator using the standard canonical vector field, and rewrite it in matrix form as

$$\mathbb{H}\omega = \int_0^1 \mathfrak{X} \lrcorner \omega(\lambda x) d\lambda = \int_0^1 x^T f(\lambda x) d\lambda \tag{23}$$

Again, by assumption, one can write $f(x) = \Psi(x)x$ where $\Psi(x) = \int_0^1 \frac{\partial f(\beta x)}{\partial x} d\beta$. As a result, one gets

$$\mathbb{H}\omega = \int_0^1 \lambda x^T \Psi(\lambda x) x d\lambda. \tag{24}$$

Consequently, one can see that if the matrix $\Psi(x)$ contains zero on its diagonal then so will the matrix $\Theta(x)$. Consider any diagonalizing transformation $z = \phi(x)$ such that $\phi(x)$ is a diffeomorphism and such that the matrix $\bar{\Psi}(z)$ has nonzero diagonal elements. Then the corresponding potential $\bar{P}(z)$ has a diagonal Hessian matrix $\bar{\Theta}(z)$.

The interesting aspect of this construction is that, as opposed to other standard techniques, for example Port-Hamiltonian systems (Ortega et al., 2002), the potential $P(x)$ is not used as a Lyapunov function. The Lyapunov function is provide by the coordinates that generate $P(x)$. From this point of view, one should assume the existence of at least one coordinate representation that yields a suitable potential. That is, a potential that is locally convex. Since stability is preserved by diffeomorphisms, it follows that there exist many other coordinate representations yielding suitable potentials.

The question that remains is whether one can formulate a converse to Theorem 1. That is, we need to check whether the convexity of the potential is also necessary for stability. Let us invoke the standard converse Lyapunov theorem for the dynamics $\dot{x} = f(x)$.

Assuming that the dynamical system is asymptotically stable, it follows that there exists a positive definite function V such that, for $\mathcal{K}$ functions α_1, α_2, α_3 and α_4:

- $$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|) \tag{25}$$
- $$\|\nabla_x V\| \le \alpha_3(\|x\|) \tag{26}$$
- $$\dot{V} \le -\alpha_4(\|x\|) \tag{27}$$

for all $x \in \Omega \subset \mathbb{R}^n$ where $\Omega = \{x \in \mathcal{D} \subset \mathbb{R}^n \,|\, V(x) \le c, c > 0\}$ is a level set of the function V contained inside the region of attraction $\mathcal{D}$ of the dynamical system.

Theorem 2. Assume that the homotopy based decomposition is such that the hessian of $P(x)$ has nonzero diagonal elements at the origin. Let the dynamical system be asymptotically stable at the origin with region of attraction $\mathcal{D}$. Then the potential $P(x)$ is locally convex.

Proof. Assume that the decomposition of the vector field is such that the potential $P(x)$ is not locally convex in a neighborhood $\mathcal{N}$ of the origin. If one considers the function $W = \frac{1}{2}x^T x$, then it follows that

$$\dot{W} = x^T f(x) = x^T \nabla_x P = x^T \Theta(x) x \le \alpha \|x\|^2,$$

where $\alpha > 0$ since the function $P(x)$ is not locally convex on $\mathcal{N}$. Therefore, any initial conditions of the system starting in $\mathcal{N}$ is such that $\|x\|$ increases. This would mean that, by condition (25), the function V would increase in a neighborhood of the origin, a contradiction of (27).

Theorems 1 and 2 establish that the convexity of the potential arising from the homotopy decomposition is necessary and sufficient for the local stability of the nonlinear system. This is a surprising result with considerable potential application in nonlinear controller design. The only requirement is that the diagonal elements of the Hessian matrix, and, hence, $\Theta(x)$, has nonzero diagonal elements. This property can be easily related to the need that the Jacobian matrix of the nonlinear system, and, therefore, $\Psi(x)$, has nonzero diagonal elements.

Let us consider the following example to illustrate the basic idea of this construction.

Example Consider the two-dimensional system:

$$\dot{x}_1 = -x_1, \; \dot{x}_2 = x_1^2 - x_2.$$

This system is globally asymptotically stable at the origin. Additionally, it is such that the Jacobian matrix has nonzero diagonal elements at the origin. By Theorems 1 and 2, it follows that the potential $P(x)$ is locally convex. To demonstrate that this is the case, Let us construct the locally convex potential using the homotopy operator.

The one-form associated with these dynamics about the equilibrium $(0, 0)$ is given by

$$\omega(x) = -x_1 dx_1 + (x_1^2 - x_2)dx_2.$$

Using the homotopy operator, one obtains the potential

$$P(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - \frac{1}{3}x_1^2 x_2.$$

As expected, the potential $P(x)$ is locally convex around the origin. Figure 1 shows the level sets of the potential $P(x)$ along with a vector field plot of the nonlinear system. However, one can clearly sees that the potential is only locally convex and hence only local stability of the origin can be claimed. This means that the Hessian of $P(x)$, and hence $\Theta(x)$, is only locally positive definite. Again the level sets of $P(x)$ demonstrate that this is not the case. And one can only claim local stabilty of this system. Additionally, it is clear from the construction that this is due to the term $-\frac{1}{3}x_1^2 x_2$ in $P(x)$ that yields an off-diagonal Hessian term.

Next consider the state space transformation

$$z_1 = x_1, \, z_2 = x_2 + x_1^2.$$

This is a diagonalizing transformation for the system since the Jacobian of $\tilde{f}(z)$ is now diagonal. In fact, the dynamics are given by

$$\dot{z}_1 = -z_1, \, \dot{z}_2 = -z_2.$$

The corresponding potential is simply

$$\tilde{P}(z) = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2$$

or

$$P(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1^2)^2,$$

which is a global Lyapunov function for this system.

The key point here is that the computed potential is not necessarily used as a Lyapunov function. The extent to which the potential recovers the region of attraction of the origin of the system is largely due to the coordinate transformation considered. Note that, in the absence of a coordinate transformation, the initial potential $P(x)$ still provides a measure of a local region of attraction that can be used to prove the stability of the system.

5.2 Stabilization

For the purpose of stabilization, we consider the damping feedback proposed in (Hudon and Guay, 2009), reviewed extensively in (Malisoff and Mazenc, 2009). For a potential $P(x)$ computed by the homotopy operator, we consider a damping feedback of the form:

$$u = -\kappa g(x)^T \nabla_x P, \quad \kappa > 0. \tag{28}$$

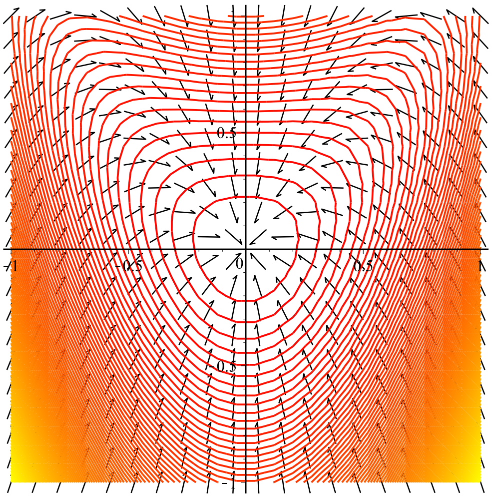

Fig. 1. Level sets of the potential function P for Example 5.1. The arrows show the orientation of the vector field.

Let us again consider the Lyapunov function $V = \frac{1}{2}x^T x$. Assume that the drift vector field of system (1) has been decomposed using the approach proposed. Then the closed-loop system

$$\dot{x} = f(x) - \kappa g(x)g(x)^T \nabla_x P \tag{29}$$

is such that

$$\dot{V} = x^T \left(I - \kappa g(x)g(x)^T\right) Q(x)x.$$

Thus the purpose of the damping feedback in this context is to overcome the lack of local convexity of the potential $P(x)$ and choose κ such that the matrix $\left(I - \kappa g(x)g(x)^T\right)\Theta(x)$ is negative definite in a neighborhood of the origin. The approach proposed here takes full advantage of the homotopy based approach while highlighting the impact of the choice of coordinates on the stabilization problem. The following result is a simple consequence of Theorems 1.

Proposition 1. Consider the nonlinear system (1). Let $P(x)$ be the potential generated by the homotopy decomposition such that, $P(0) = 0$ and the Hessian matrix $\Theta(x)$ has nonzero diagonal elements in a neighborhood of the origin. If there exists a κ such that the matrix $\left(I - \kappa g(x)g(x)^T\right)\Theta(x)$ is negative definite in a neighborhood of the origin, then the closed-loop control system (29) has an locally asymptotically stable equilibrium at the origin.

Example Let us consider a standard backstepping controller design problem, with a nonlinear system in strict-feedback form given by

$$\dot{x}_1 = x_2, \; \dot{x}_2 = x_3, \; \dot{x}_3 = x_1^2 + x_3 + u.$$

We first diagonalize the system using a backstepping-like coordinate transformation

$$z_1 = x_1,\ z_2 = x_1 + x_2,\ z_3 = 2z_2 + x_3.$$

The transformed system takes the form

$$\dot{z}_1 = z_2 - z_1,\ \dot{z}_2 = z_3 - z_2 - z_1,\ \dot{z}_3 = z_1^2 + z_3 - 2z_2 + u,$$

and the corresponding potential is given by

$$\tilde{P}(z) = -\frac{1}{3}z_3 z_1^2 + \frac{1}{2}z_1^2 + \frac{1}{2}z_3 z_2 - \frac{1}{2}z_3^2 + \frac{1}{2}z_2^2.$$

The Hessian of $\tilde{P}(z)$ has clearly nonzero diagonal elements in a neighborhood of the origin. Consider the damping feedback with $\kappa = 3$, *i.e.*,

$$u(z) = -3[0,\ 0,\ 1]\nabla_z \bar{P}(z)^T = -z_1^2 + \frac{3}{2}z_2 - 3z_3.$$

The closed-loop system becomes

$$\dot{z}_1 = z_2 - z_1,\ \dot{z}_2 = z_3 - z_2 - z_1,\ \dot{z}_3 = -\frac{1}{2}z_2 - z_3,$$

which is globally exponentially stable.

6. CONCLUSION

The problem of representing a sufficiently smooth control affine system as a potential-based realization is addressed. Using a homotopy decomposition and a dual homotopy decomposition of a differential one-form that encodes the divergence of the given vector field, constructive conditions for the representation of nonlinear systems as a potential-based realization are developed. Based on this decomposition, we studied the problems of stability and stabilization of nonlinear control affine systems. The problems are shown to be straightforward in the context of potential based realizations.

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