Sure almost global vs. global almost sure synchronization on the circle: the virtues of stochastic hybrid feedback

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Abstract: This paper introduces a stochastic, hybrid algorithm for global almost sure synchronization of two agents evolving on the circle. Recent results on stochastic hybrid systems are exploited and a Lyapunov-based proof of global almost sure synchronization is obtained. The same arguments establish that global almost sure practical synchronization is achieved in the presence of sufficiently small disturbances. In contrast, sure almost global algorithms are more seriously affected by adversarial disturbances. Namely, there is a set of initial conditions with nonzero measure from which the agents may not approximately synchronize.

1. INTRODUCTION

We consider global asymptotic synchronization of two agents evolving on the circle. An excellent review article concerning *almost global* synchronization on nonlinear spaces is provided in [Sepulchre, 2011]. See [Scardovi et al., 2007] and [Sarlette and Sepulchre, 2011] for details about algorithms for almost global synchronization of agents evolving on the unit circle. In this paper, we provide a stochastic, hybrid state feedback algorithm that achieves *global almost sure* synchronization for two communicating agents. An alternative stochastic algorithm that achieves global almost sure synchronization of agents on the circle appears in [Sarlette et al., 2008]; it is based on a gossip algorithm given in [Boyd et al., 2006]. The ideas present in the current paper have been extended to a finite number of agents on the circle with all-to-all communication in [Hartman et al., 2013] and [Subbaraman et al., 2013]. In addition to our nominal global almost sure synchronization result, we establish a robustness property with respect to sufficiently small, worst-case perturbations. In contrast, non-stochastic almost globally asymptotically synchronizing algorithms do not confer this robustness property. Our description of the stochastic, hybrid state feedback algorithm and its behavior in the presence of adversaries is based on the theory of stochastic hybrid inclusions as developed in [Teel, 2013]. This class of systems extends to the stochastic case the systems considered in [Goebel and Teel, 2006], [Goebel et al., 2009], [Goebel et al., 2012].

The paper is organized as follows: In Section 2 we review a standard algorithm for almost global synchronization of two agents evolving on the unit circle. We also illustrate how the algorithm degrades from an almost global result in the presence of adversarial disturbances. In Section 3 we develop a stochastic, hybrid algorithm that achieves global almost sure synchronization and state results about how the (practical) synchronization remains global, almost sure synchronization in the presence of sufficiently small adversarial inputs. Section 4 contains a Lyapunov-based proof of our main results.

2. ALMOST GLOBAL ASYMPTOTIC SYNCHRONIZATION: A REVIEW

2.1 The nominal case

Consider two controlled agents $\xi_i \in \mathbb{R}^2$, $i \in \{1,2\}$, evolving on the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ with dynamics

$$\dot{\xi}_i = u_i J \xi_i \qquad \xi_i \in \mathbb{S}^1 \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

where $u_i \in \mathbb{R}$ is the control variable. The synchronization objective is to make $\xi_1^T \xi_2 = 1$. Define the energy function $W : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as

$$W(\xi) := 1 - \xi_1^T \xi_2 \tag{1}$$

and consider the state feedback synchronization algorithm

$$u_i = u_i(\xi) := \kappa \xi_j^T J \xi_i \quad j \neq i \in \{1,2\}\,,\ \kappa > 0. \tag{2}$$

This algorithm has two equilibrium configurations $\xi_i = \pm\xi_j$, since $u_i(\xi) = 0$ if and only if $\xi_i = \pm\xi_j$. Fortunately, the equilibrium $\xi_i = -\xi_j$ is unstable and from every other initial condition the solutions converge to the configuration $\xi_i = \xi_j$. This fact can be established by examining the derivative of W along solutions:

$$\langle \nabla W(\xi), \mathrm{diag}(u_1(\xi)J, u_2(\xi)J)\xi \rangle = -2\kappa \cdot (\xi_2^T J \xi_1)^2.$$

Alternatively, the unstable equilibrium can be removed (as long as generalized solutions, e.g. [Hàjek, 1979], are not considered) by making $u_i(\xi)$ discontinuous when $\xi_1^T \xi_2 = -1$, for example, $\kappa > 0$,

$$u_i = \widehat{u}_i(\xi) = \begin{cases} \kappa & \xi_1^T \xi_2 = -1 \\ \kappa \dfrac{\xi_j^T J \xi_i}{|\xi_j^T J \xi_i|} & \xi_1^T \xi_2 \in (-1,0) \\ \kappa \xi_j^T J \xi_i & \xi_1^T \xi_2 \in [0,1]. \end{cases} \tag{3}$$

However, we will see that such a feedback is still susceptible to adversarial disturbances.

2.2 (Lack of) Robustness to adversaries

Consider the situation where an adversary for agent i reflects the measurement of agent j about the line passing through agent i and the center of the circle when agent j is close to that line. Mathematically, let $\varepsilon > 0$ be arbitrary and, for $j \in \{1,2\}$, let $Q_{j,\varepsilon}$ be a continuous mapping from $\mathbb{S}^1 \times \mathbb{S}^1$ to the set of 2 by 2 matrices with orthonormal columns satisfying

$$\begin{aligned} Q_{j,\varepsilon}(\xi) &= I - 2J\xi_i\xi_i^T J^T \qquad \text{when } |\xi_j^T J \xi_i| \le \varepsilon \\ |Q_{j,\varepsilon}(\xi)\xi_j - \xi_j| &\le 2\varepsilon \qquad \forall \xi \in \mathbb{S}^1 \times \mathbb{S}^1. \end{aligned}$$

Then, in place of (2), consider

$$\begin{aligned} u_1 = u_{1,\varepsilon,d}(\xi) &:= u_1(\xi_1, Q_{2,\varepsilon}(\xi)\xi_2) \\ u_2 = u_{2,\varepsilon,d}(\xi) &:= u_2(Q_{1,\varepsilon}(\xi)\xi_1, \xi_2). \end{aligned}$$

Note that $u_{i,\varepsilon,d}(\xi) = -u_i(\xi)$ for $|\xi_j^T J \xi_i| \le \varepsilon$, from which it follows that the disturbance succeeds at asymptotically stabilizing the previously unstable configuration $\xi_i = -\xi_j$, with basin of attraction having a positive, one-dimensional measure on the unit circle that is proportional to ε.

Similarly, in place of (3), consider

$$\begin{aligned} u_1 = \widehat{u}_{1,\varepsilon,d}(\xi) &:= \widehat{u}_1(\xi_1, Q_{2,\varepsilon}(\xi)\xi_2) \\ u_2 = \widehat{u}_{2,\varepsilon,d}(\xi) &:= \widehat{u}_2(Q_{1,\varepsilon}(\xi)\xi_1, \xi_2). \end{aligned}$$

Now for $0 < |\xi_j^T J \xi| \le \varepsilon$, $\widehat{u}_{i,\varepsilon,d}(\xi) = -\mathrm{sgn}(u_i(\xi))$, from which it again follows that the disturbance succeeds at stabilizing (in finite time) the configuration $\xi_i = -\xi_j$, with basin of attraction having a positive one-dimensional measure on the unit circle that is proportional to ε.

Due to the susceptibility of almost global algorithms to the effect of arbitrarily small adversaries, we turn our attention to a stochastic hybrid algorithm that achieves global almost sure synchronization and robustness.

3. AN ALGORITHM FOR ALMOST SURE GLOBAL ASYMPTOTIC SYNCHRONIZATION

The algorithm presented in this section is inspired by the framework for stochastic hybrid systems with adversaries recently developed in [Teel, 2013]. Our closed-loop system has the form of a stochastic hybrid system written as

$$x \in C \qquad \dot{x} \in F(x) \tag{4a}$$
$$x \in D \qquad x^+ \in G(x, v^+) \tag{4b}$$
$$v \sim \mu(\cdot) \tag{4c}$$

where C is the flow set, F is the flow map, D is the jump set, and G is the jump map. The distribution function μ is derived from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence of independent, identically distributed (i.i.d.) input random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. In particular, with $\mathbf{v}_i : \Omega \to \mathbb{R}^m$, $i \in \mathbb{Z}_{\ge 1}$, denoting the elements of the sequence, and thus having the property that $\mathbb{P}\{\omega \in \Omega : \mathbf{v}_i(\omega) \in A\}$ is well defined and independent of i for each A in the Borel σ-field over $\mathbb{R}^m$, denoted $\mathbf{B}(\mathbb{R}^m)$, the distribution function $\mu : \mathbf{B}(\mathbb{R}^m) \to [0,1]$ is defined as $\mu(A) := \mathbb{P}\{\omega \in \Omega : \mathbf{v}_i(\omega) \in A\}$. Uniqueness of solutions to (4) is not assumed. Indeed, because the flow and jump sets may have non-trivial overlap, the flow map may be set valued, and the jump map may be set valued, uniqueness of solutions in not typical in these models. We refer the reader to [Teel, 2013] for the definition of solutions. Stability results are reviewed in the Appendix.

3.1 The nominal algorithm

We propose a stochastic, hybrid algorithm for global almost sure synchronization. Let r, s be distinct integers, and define $\mathbb{I}_s : \{r,s\} \to \{0,1\}$ by $\mathbb{I}_s(q) := |q - r|/|s - r|$. The algorithm uses four parameters: $\kappa > 0$, $T > 0$ and

$$0 < \theta_s < \theta_r < 2. \tag{5}$$

The closed-loop stochastic hybrid system has state

$$x := (\xi_1, \xi_2, q_1, q_2, w_1, w_2, \tau_1, \tau_2) \in \mathbb{R}^{10},$$

which evolves in

$$X := \left(\mathbb{S}^1\right)^2 \times \{r,s\}^2 \times \{-1,1\}^2 \times [0,T]^2, \tag{6}$$

and generates the continuous-time control laws

$$u_i = (1 - \mathbb{I}_s(q_i))\kappa \xi_j^T J \xi_i + \mathbb{I}_s(q_i) w_i. \tag{7}$$

The closed-loop hybrid dynamics are generated as follows. Recall the definition of W in (1). Define the flow set C via

$$\widetilde{C}_s := \left\{\xi \in \mathbb{S}^1 \times \mathbb{S}^1 : W(\xi) \ge \theta_s\right\} \tag{8a}$$
$$\widetilde{C}_r := \left\{\xi \in \mathbb{S}^1 \times \mathbb{S}^1 : W(\xi) \le \theta_r\right\} \tag{8b}$$
$$\widehat{C}_1 := \cup_{q_1 \in \{r,s\}} \left(\widetilde{C}_{q_1} \times \{q_1\} \times \{r,s\}\right) \tag{8c}$$
$$\widehat{C}_2 := \cup_{q_2 \in \{r,s\}} \left(\widetilde{C}_{q_2} \times \{r,s\} \times \{q_2\}\right) \tag{8d}$$
$$C := \left(\widehat{C}_1 \cap \widehat{C}_2\right) \times \{-1,1\}^2 \times [0,T]^2 \tag{8e}$$

and the jump set D via

$$\widetilde{D}_s := \left\{\xi \in \mathbb{S}^1 \times \mathbb{S}^1 : W(\xi) \le \theta_s\right\} \tag{9a}$$
$$\widetilde{D}_r := \left\{\xi \in \mathbb{S}^1 \times \mathbb{S}^1 : W(\xi) \ge \theta_r\right\} \tag{9b}$$
$$\widehat{D}_1 := \bigcup_{q_1 \in \{r,s\}} \left(\widetilde{D}_{q_1} \times \{q_1\} \times \{r,s\}\right) \times \{-1,1\}^2 \times [0,T]^2 \tag{9c}$$
$$\widehat{D}_2 := \bigcup_{q_2 \in \{r,s\}} \left(\widetilde{D}_{q_2} \times \{r,s\} \times \{q_2\}\right) \times \{-1,1\}^2 \times [0,T]^2 \tag{9d}$$
$$D_{s,1} := \widetilde{C}_s \times \{s\} \times \{r,s\} \times \{-1,1\}^2 \times (\{0\} \times [0,T]) \tag{9e}$$
$$D_{s,2} := \widetilde{C}_s \times \{r,s\} \times \{s\} \times \{-1,1\}^2 \times ([0,T] \times \{0\}) \tag{9f}$$
$$D := D_{s,1} \cup D_{s,2} \cup \widehat{D}_1 \cup \widehat{D}_2. \tag{9g}$$

We get the dynamics (we suppress equations where derivatives are zero and updates do not change the state)

$$x \in C \quad \begin{cases} \dot{\xi}_1 = ((1 - \mathbb{I}_s(q_1))\kappa \xi_2^T J \xi_1 + \mathbb{I}_s(q_1) w_1) J \xi_1 \\ \dot{\xi}_2 = ((1 - \mathbb{I}_s(q_2))\kappa \xi_1^T J \xi_2 + \mathbb{I}_s(q_2) w_2) J \xi_2 \\ \dot{\tau}_1 = -\mathbb{I}_s(q_1) \\ \dot{\tau}_2 = -\mathbb{I}_s(q_2) \end{cases} \tag{10a}$$

$$x \in \widehat{D}_1 \quad \begin{cases} q_1^+ = \mathbb{I}_s(q_1) r + (1 - \mathbb{I}_s(q_1)) s \\ \tau_1^+ = 0 \end{cases} \tag{10b}$$

$$x \in \widehat{D}_2 \quad \begin{cases} q_2^+ = \mathbb{I}_s(q_2) r + (1 - \mathbb{I}_s(q_2)) s \\ \tau_2^+ = 0 \end{cases} \tag{10c}$$

$$x \in D_{s,1} \quad \begin{cases} w_1^+ = v_1 \\ \tau_1^+ = v_3 \end{cases} \tag{10d}$$

$$x \in D_{s,2} \quad \begin{cases} w_2^+ = v_2 \\ \tau_2^+ = v_4. \end{cases} \tag{10e}$$

We make the following assumptions on the random variables used to generate the inputs v_i, $i \in \{1, \ldots, 4\}$.

Assumption 1. The inputs v_1 and v_2 are generated by i.i.d. random variables uniformly distributed in $\{-1,1\}$ and the inputs v_3 and v_4 are generated by i.i.d. random variables distributed on $[0,T]$ with positive probability of being positive. □

We use $x \mapsto f(x)$ to denote the flow mapping, which is a function (single valued) on its domain, the latter taken to be the flow set C. We use $(x,v) \mapsto G(x,v)$ to denote the jump map, which is a set-valued mapping by virtue of the fact that the sets used to define the jump map overlap at some points. At such points, the jump map is the union of the values specified above.

3.2 Result for nominal algorithm

The main result about the above algorithm uses the condition (5) together with the following two conditions on the parameters $(\kappa, T, \theta_s, \theta_r)$:

$$\theta_s \le W(\xi) \le \theta_r \implies \kappa|\xi_2^T J \xi_1| > 1 \tag{11a}$$

$$\left.\begin{array}{r}\theta_s \le W(\xi)\\ \tau \in [-2T, 2T]\end{array}\right\} \implies -\xi_1^T \exp(J\tau)\xi_2 > 0. \tag{11b}$$

The condition (5) together with $\theta_s \le W(\xi) \le \theta_r$ implies that $|\xi_2^T J \xi_1|$ is bounded away from zero. Thus, given (5), there always exists κ sufficiently large so that (11a) holds. The condition (11b) requires that $\theta_s > 1$ and that T is sufficiently small. Indeed, $1 < \theta_s \le W(\xi)$ implies that $-\xi_1^T \xi_2 \ge \theta_s - 1$ and since $\exp(J \cdot 0) = I$, we have $-\xi_1^T \exp(J\tau)\xi_2 > 0$ for all $\tau \in [-2T, 2T]$ when T is sufficiently small. The main result about the behavior of the algorithm in the absence of perturbations is stated next. Its proof follows from the proof of a subsequent result in Section 3.4 pertaining to perturbed (inflated) dynamics.

Theorem 1. Define X as in (6). If Assumption 1 holds and the parameter θ_s, θ_r, $\kappa > 0$ and $T > 0$ satisfy (5) and (11) then the set $\mathcal{X} := \left\{x \in X : \xi_1^T \xi_2 > 1 - \theta_r, q_1 = q_2 = r\right\}$ is forward invariant and the set

$$\mathcal{A} := \left\{x \in X : \xi_1^T \xi_2 = 1, q_1 = q_2 = r\right\}$$

is locally asymptotically stable and globally asymptotically stable in probability for the stochastic hybrid system with data (C, f, D, G) from Section 3.1. ■

3.3 Partial simulation results

The parameters $T > 0$ and θ_s have the strongest effect on the average time in which the algorithm spends in its "stochastic" mode. Figures 1-3 provide histograms generated by running 100 solutions; the histograms indicate the percentage of solutions that have reached different threshold levels after a certain amount of time when each agent is running in its stochastic mode. The three different figures correspond to using $T \in \{0.1, 0.3, 0.5\}$. In each case, we have used a uniform distribution (on $[0, T]$) for the jumps of the timer variables. In each figure, there are curves corresponding to the threshold levels for $\xi_1^T \xi_2$ greater than or equal to -0.9, -0.75 and -0.5; in other words, for $W(x)$ less than or equal to $\theta \in \{1.9, 1.75, 1.5\}$.

3.4 Results for a perturbed system

Let the real numbers $\underline{\theta}_s$, $\overline{\theta}_s$, $\underline{\theta}_r$, $\overline{\theta}_r$ satisfy

$$0 < \underline{\theta}_s < \theta_s < \overline{\theta}_s < \underline{\theta}_r < \theta_r < \overline{\theta}_r < 2. \tag{12}$$

In order to consider inflated data, we analyze the nominal system with θ_s replaced by $\underline{\theta}_s$ in (8a), θ_r replaced by $\overline{\theta}_r$ in (8b), θ_s replaced by $\overline{\theta}_s$ in (9a) and θ_r replaced by $\underline{\theta}_r$ in (9b). We denote the corresponding flow set by $C_{\nearrow}$ and the corresponding jump set by $D_{\nearrow}$. We also allow perturbations to the controls as follows.

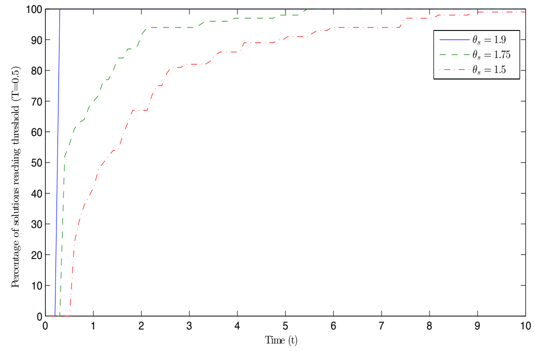

Fig. 1. Histograms (100 simulations) for $T = 0.5$.

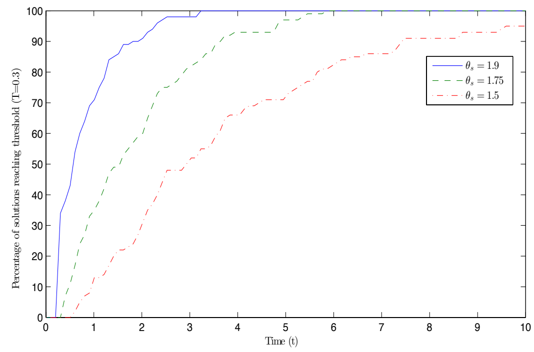

Fig. 2. Histograms (100 simulations) for $T = 0.3$.

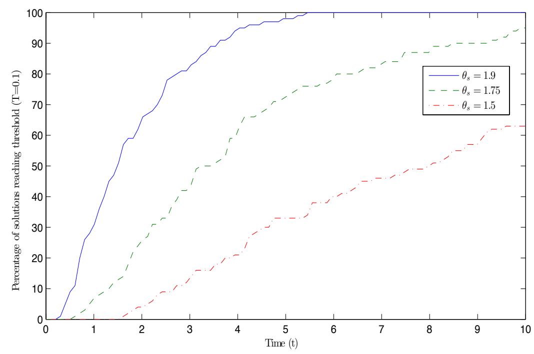

Fig. 3. Histograms (100 simulations) for $T = 0.1$.

Assumption 2. The values $\xi_j^T J \xi_i$ in (7) are replaced by the intervals $\left\{\xi_j^T J \xi_i\right\} + [-\delta, \delta]$ where $\delta > 0$ is sufficiently small; we denote the corresponding flow map as F_δ. □

The jump map itself remains unchanged because its values are either discrete or formed from random variables.

Our main result for the inflated system uses (12) together with the conditions

$$\underline{\theta}_s \le W(\xi) \le \overline{\theta}_r \implies \kappa|\xi_2^T J \xi_1| > 1 \tag{13a}$$

$$\left.\begin{array}{r}\underline{\theta}_s \le W(\xi)\\ \tau \in [-2T, 2T]\end{array}\right\} \implies -\xi_1^T \exp(J\tau)\xi_2 > 0. \tag{13b}$$

We note that if (11a)-(11b) hold then the conditions (13a)-(13b) hold for $\underline{\theta}_s < \theta_s$ sufficiently close to θ_s and $\overline{\theta}_r > \theta_r$ sufficiently close to θ_r.

The next result generalizes Theorem 1 and is proved in the next section.

Theorem 2. Let X be defined in (6). If Assumptions 1 and 2 hold and the parameter $\underline{\theta}_s$, $\overline{\theta}_s$, $\underline{\theta}_r$, $\overline{\theta}_r$, $\kappa > 0$ and $T > 0$ satisfy (12) and (13) then there exists $\delta^* > 0$ such that, for all $\delta \in (0, \delta^*]$, the set

$$\mathcal{X} := \left\{x \in X : \xi_1^T \xi_2 > 1 - \underline{\theta}_r, q_1 = q_2 = r\right\}$$

is forward invariant and, for each $\varepsilon > 0$, there exists $\delta \in (0, \delta^*]$ such that the compact set

$$\mathcal{A} := \left\{x \in X : \xi_1^T \xi_2 \geq 1 - \varepsilon, q_1 = q_2 = r\right\}$$

is locally asymptotically stable (in the sense of non-stochastic hybrid systems) and globally asymptotically stable in probability for the stochastic hybrid system with data $(C_{\nearrow}, F_\delta, D_{\nearrow}, G)$ defined above. ■

4. PROOF OF THEOREM 2

4.1 Lyapunov function candidate

The proof relies on Theorem 3 (from the Appendix) using the Lyapunov function candidate

$$\begin{aligned} V(x) := & \, \mathbb{I}_s(q_1)\mathbb{I}_s(q_2)\exp(\lambda(\tau_1+\tau_2)) \cdot \\ & \quad [k_0 + W(\exp(Jw_1\tau_1)\xi_1, \exp(Jw_2\tau_2)\xi_2)] \\ & + [1 - \mathbb{I}_s(q_1)\mathbb{I}_s(q_2)]\, W(\xi)\alpha(W(\xi)) \\ & + [1 - \mathbb{I}_s(q_1)]\,[1 - \mathbb{I}_s(q_2)]\,\gamma(W(\xi)) \\ & + \varepsilon_0 \exp(-\lambda(\tau_1+\tau_2)) \cdot \\ & \quad \Big([1 - \mathbb{I}_s(q_1)]\,\mathbb{I}_s(q_2) + [1 - \mathbb{I}_s(q_2)]\,\mathbb{I}_s(q_1)\Big) \end{aligned}$$

where

1. $\lambda > 0$ is sufficiently small,
2. $\gamma : [0, 2] \to \mathbb{R}_{\geq 0}$ is smooth, nondecreasing, $\gamma(s) = 0$ for $s \in [0, \overline{\theta}_s]$, and $\gamma(s) > 0$ for $s \in (\overline{\theta}_s, 2]$,
3. $\alpha : [0, 2] \to \mathbb{R}_{>0}$ is smooth, nondecreasing, $\alpha(s) \leq k_1$ for $s \in [0, \overline{\theta}_s]$ and $\alpha(t) \geq k_2$ for $t \in [\underline{\theta}_r, 2]$,
4. $\overline{\theta}_s k_1 + \varepsilon_0 < k_0$, $k_0 + 2 < \underline{\theta}_r k_2$, $0 < \varepsilon_0 < \gamma(\underline{\theta}_r)$.

This Lyapunov function is zero if and only if $W(\xi) = 0$ and $q_1 = q_2 = r$. We initially consider the case $\varepsilon = \delta = 0$ in the theorem statement. The case $\varepsilon > 0$ and $\delta > 0$ sufficiently small follows readily from the $\varepsilon = \delta = 0$ analysis and the form of the Lyapunov function $V(x)$.

4.2 Lyapunov analysis: flows

Case 1: $q_1 = q_2 = s$. In this case

$$V(x) = \exp(\lambda(\tau_1+\tau_2))\,[k_0 + W(\exp(Jw_1\tau_1)\xi_1, \exp(Jw_2\tau_2)\xi_2)]$$

and, for $i \in \{1, 2\}$,

$$\dot{\xi}_i = w_i J \xi_i\ , \quad \dot{w}_i = 0\ , \quad \dot{\tau}_i = -1.$$

We claim that $x \mapsto \exp(Jw_i\tau_i)\xi_i$ is constant along flows for each $i \in \{1, 2\}$. Indeed, by the chain rule,

$$\begin{aligned} \frac{d}{dt}(\exp(Jw_i\tau_i)\xi_i) = & \exp(Jw_i\tau_i)\, w_i J\xi - \exp(Jw_i\tau_i)\, w_i J\xi_i \\ = & \, 0. \end{aligned}$$

Since $\dot{\tau}_1 + \dot{\tau}_2 = -2$, it follows that, in this case, $\langle \nabla V(x), f(x)\rangle = -2\lambda V(x)$.

Case 2: $q_1 = q_2 = r$. In this case

$$V(x) = W(\xi)\alpha(W(\xi)) + \gamma(W(\xi))$$

and

$$\dot{\xi}_i = (\kappa \xi_j^T J \xi_i) J \xi_i \quad \forall i \in \{1, 2\}\ ,\ j \neq i.$$

Then

$$\begin{aligned} & \langle V(x), f(x)\rangle \\ & = (\alpha(W(\xi)) + W(\xi)\alpha'(W(\xi)) + \gamma'(W(\xi)))\left(-2\kappa(\xi_2^T J \xi_1)^2\right) \\ & \leq -2\alpha(0)\kappa \cdot (\xi_2^T J \xi_1)^2. \end{aligned}$$

In the set $W(\xi) \leq \overline{\theta}_r < 2$, which is a requirement for flowing when $q_1 = q_2 = r$, we have that $\xi_2^T J \xi_1 = 0$ if and only if $\xi_1 = \xi_2$. Thus, the derivative is negative definite.

Case 3: $q_1 \neq q_2$. In this case

$$V(x) = W(\xi)\alpha(W(\xi)) + \varepsilon_0 \exp(-\lambda(\tau_1+\tau_2)) \qquad (14)$$

and, by symmetry, we can assume $q_1 = r$, $q_2 = s$, and

$$\begin{aligned} \dot{\xi}_1 &= (\kappa \xi_2^T J \xi_1) J \xi_1 & \dot{\xi}_2 &= w_2 J \xi_2 \\ \dot{\tau}_1 &= 0 & \dot{\tau}_2 &= -1 \end{aligned}$$

where $w_2 \in \{-1, 1\}$. Then, letting $\widetilde{f}$ denote the component of the vector field satisfying $\dot{\xi} = \widetilde{f}(\xi)$, we get

$$\begin{aligned} \langle \nabla W(\xi), \widetilde{f}(\xi)\rangle &= -\kappa(\xi_2^T J \xi_1)^2 - w_2 \xi_1^T J \xi_2 \\ &\leq -|\xi_2^T J \xi_1|\left(\kappa|\xi_2^T J \xi| - 1\right). \end{aligned}$$

Note that we must have $0 < \underline{\theta}_s \leq W(\xi) \leq \overline{\theta}_r < 2$ when $q_1 = r$ and $q_2 = s$. It follows from (13a) and the properties of α that, with $\lambda > 0$ sufficiently small, the derivative of V is negative.

4.3 Lyapunov analysis: jumps

Case 1: $x \in \widehat{D}_1$

1. Suppose q_1 toggles from s to r. Thus $W(\xi) \leq \overline{\theta}_s$.
   - If $q_2 = r$ then
     $$\begin{aligned} V(x^+) &= W(\xi)\alpha(W(\xi)) + \gamma(W(\xi)) \\ &< W(\xi)\alpha(W(\xi)) + \varepsilon_0 \exp(-2\lambda T) \\ &\leq V(x) \end{aligned}$$
     where the middle inequality follows from the properties of γ and ε_0. Thus, the Lyapunov function decreases at the jump in this case.
   - If $q_2 = s$ then
     $$\begin{aligned} V(x^+) &\leq W(\xi)\alpha(W(\xi)) + \varepsilon_0 \leq \overline{\theta}_s k_1 + \varepsilon_0 < k_0 \\ &\leq \exp(\lambda(\tau_1+\tau_2)) \cdot \\ &\quad [k_0 + W(\exp(Jw_1\tau_1)\xi_1, \exp(Jw_2\tau_2)\xi_2)] \\ &= V(x). \end{aligned}$$
     Thus, the Lyapunov function decreases at the jump in this case.
2. Suppose q_1 toggles from r to s. Thus $W(\xi) \geq \underline{\theta}_r$.
   - If $q_2 = r$ then
     $$\begin{aligned} V(x^+) &\leq W(\xi)\alpha(W(\xi)) + \varepsilon_0 \\ &< W(\xi)\alpha(W(\xi)) + \gamma(\underline{\theta}_r) \\ &\leq W(\xi)\alpha(W(\xi)) + \gamma(W(\xi)) = V(x). \end{aligned}$$
     Thus, the Lyapunov function decreases at the jump in this case.

- If $q_2 = s$ then

$$\begin{aligned} V(x^+) &= \exp(\lambda\tau_2)\left[k_0 + W(\xi_1, \exp(Jw_2\tau_2)\xi_2)\right] \\ &\leq \exp(\lambda T)(k_0+2) < \underline{\theta}_r k_2 \\ &\leq W(\xi)\alpha(W(\xi)) + \exp(-2\lambda T)\varepsilon_0 \leq V(x) \end{aligned}$$

where we have used $\lambda > 0$ sufficiently small in the middle inequality. Thus, the Lyapunov function decreases at the jump in this case.

Case 2: $x \in \widehat{D}_2$ Same as Case 1, by symmetry.

Case 3: $x \in D_{s,1}$ The values of q_1 and q_2 do not change during these jumps.

(1) If $q_1 \neq q_2$ then

$$V(x) = W(\xi)\alpha(W(\xi)) + \varepsilon_0 \exp(-\lambda\tau_2) \tag{15}$$

and

$$V(x^+) = W(\xi)\alpha(W(\xi)) + \varepsilon_0 \exp(-\lambda(v_3 + \tau_2)).$$

The average value is

$$W(\xi)\alpha(W(\xi)) + \varepsilon_0 \exp(-\lambda\tau_2)\int_0^T \exp(-\lambda v_3)\mu(dv_3) < V(x)$$

where the last inequality follows from (15) and the last part of Assumption 1.

(2) The condition $q_1 = q_2 = r$ is not possible for $x \in D_{s,1}$.

(3) Let $q_1 = q_2 = s$. Necessarily $W(\xi) \geq \underline{\theta}_s$. According to (13b), there exists $\varepsilon_2 > 0$ such that $-\xi_2^T \exp(J\tau)\xi_1 \geq \varepsilon_2 > 0$ for all $\tau \in [-2T, 2T]$. In particular, $-(\exp(Jw_2\tau_2)\xi_2)^T \exp(Jv_1v_3)\xi_1 \geq \varepsilon_2$ for all $v_1, w_2 \in \{-1, 1\}$ and $v_3, \tau_2 \in [0, T]$. We have $\tau_1 = 0$ and $\tau_2 \in [0, T]$ before the jump. Let p be such that $\cos(p) = -(\exp(Jw_2\tau_2)\xi_2)^T\xi_1$. Then, before the jump,

$$V(x) = \exp(\lambda\tau_2)\left[k_0 + 1 + \cos(p)\right] \tag{16}$$

and after the jump

$$V(x^+) = \exp(\lambda(v_3 + \tau_2))\left[k_0 + 1 + \cos(p + v_1v_3)\right] \tag{17}$$

where, by assumption $\cos(p + v_1v_3) \geq \varepsilon_2$ for all $v_1 \in \{-1, 1\}$ and $v_3 \in [0, T]$. Define $\eta := \int_0^T \cos(v_3)\mu(dv_3)$, which is less than one due to the last part of Assumption 1 and the constraint on T give above. Then, using (16) and (17), the average of $V(x^+)$ is bounded by

$$\begin{aligned} &\exp(2\lambda T)\cdot \\ &\left[k_0 + 1 + 0.5\int_0^T [\cos(p + v_3) + \cos(p - v_3)]\mu(dv_3)\right] \\ &= \exp(2\lambda T)\left[k_0 + 1 + \cos(p)\int_0^T \cos(v_3)\mu(dv_3)\right] \\ &= \exp(2\lambda T)\left[k_0 + 1 + \cos(p)\eta\right] \\ &= V(x)\exp(2\lambda(T - \tau_2))\frac{k_0 + 1 + \cos(p)\eta}{k_0 + 1 + \cos(p)} \\ &\leq V(x)\exp(2\lambda T)\frac{k_0 + 1 + \varepsilon_2\eta}{k_0 + 1 + \varepsilon_2}. \end{aligned}$$

We pick $\lambda > 0$ sufficiently small to make less than one the coefficient multiplying $V(x)$, which is possible since $\eta < 1$ and $\varepsilon_2 > 0$.

Case 4: $x \in D_{s,2}$ Same as Case 3, by symmetry.

5. CONCLUSION

The tradeoffs between sure almost global synchronization and global almost sure synchronization on nonlinear spaces are interesting to explore. When it comes to robustness, global almost sure synchronization algorithms have an advantage. We have illustrated this advantage mathematically for the case of two agents evolving on the unit circle. The performance of stochastic hybrid algorithms in the face of causal adversaries motivates having a clear theory for the analysis of stochastic hybrid systems with disturbances, also known as stochastic hybrid inclusions. New results in this area [Teel, 2013] facilitated the development of the algorithm presented here and its analysis.

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Appendix A. STOCHASTIC HYBRID SYSTEMS

A.1 Stability concepts

Stability concepts are expressed in terms of probabilities on solution graphs. To save on notation, we suppress the ω dependence of a random solution $\mathbf{x}$ when working with probabilities. Moreover, by abuse of notation, we write "$\mathbf{x}(t,j) \in S$ for $(t,j) \in \text{dom } \mathbf{x}$" in place of "$\mathbf{x}_\omega(t,j) \in S$ for $(t,j) \in \text{dom } \mathbf{x}_\omega$" where $\mathbf{x}_\omega := \mathbf{x}(\omega)$. The compact set $\mathcal{A} \subset \mathbb{R}^n$ is *Lyapunov stable in probability for (4)* if for each $\varepsilon > 0$ and $\varrho > 0$ there exists $\delta > 0$ such that

$$\xi \in \mathcal{A} + \delta\mathbb{B},\ \mathbf{x} \in \mathcal{S}_{\mathbf{r}}(\xi) \quad \Longrightarrow$$
$$\mathbb{P}\Big(\text{graph}(\mathbf{x}) \subset \big(\mathbb{R}^2 \times (\mathcal{A} + \varepsilon\mathbb{B}^\circ)\big)\Big) \geq 1 - \varrho. \qquad \text{(A.1)}$$

The condition $\text{graph}(\mathbf{x}) \subset \big(\mathbb{R}^2 \times (\mathcal{A} + \varepsilon\mathbb{B}^\circ)\big)$ is equivalent to having $\mathbf{x}(t,j) \in \mathcal{A} + \varepsilon\mathbb{B}^\circ$ for all $(t,j) \in \text{dom } \mathbf{x}$. The inequality (A.1) asks that this condition on $\mathbf{x}$ holds with probability at least $1 - \varrho$. The value $\delta > 0$ is chosen sufficiently small to accommodate $\varepsilon > 0$ and $\varrho > 0$.

The compact set $\mathcal{A}$ is *Lagrange stable in probability for (4)* if for each $\delta > 0$ and $\varrho > 0$ there exists $\varepsilon > 0$ such that (A.1) holds. Here $\varepsilon > 0$ is chosen large to accommodate $\delta > 0$ and $\varrho > 0$. The set $\mathcal{A}$ is *globally stable in probability for (4)* if it is both Lyapunov stable and Lagrange stable in probability for (4). The set $\mathcal{A}$ is *uniformly globally attractive in probability for (4)* if for each $\varepsilon > 0$, $\varrho > 0$, and $R > 0$ there exists $\tau \geq 0$ so that

$$\xi \in \mathcal{A} + R\mathbb{B},\ \mathbf{x} \in \mathcal{S}_{\mathbf{r}}(\xi) \implies$$
$$\mathbb{P}\Big((\text{graph}(\mathbf{x}) \cap (\Gamma_{\geq\tau} \times \mathbb{R}^n)) \subset \big(\mathbb{R}^2 \times (\mathcal{A} + \varepsilon\mathbb{B}^\circ)\big)\Big) \geq 1 - \varrho. \qquad \text{(A.2)}$$

By convention, the empty set is a subset of any set. The condition $(\text{graph}(\mathbf{x}) \cap (\Gamma_{\geq\tau} \times \mathbb{R}^n)) \subset \big(\mathbb{R}^2 \times (\mathcal{A} + \varepsilon\mathbb{B}^\circ)\big)$ asks that $\mathbf{x}(t,j) \in \mathcal{A} + \varepsilon\mathbb{B}^\circ$ for all $(t,j) \in \text{dom } \mathbf{x}$ satisfying $t + j \geq \tau$. The inequality (A.2) asks that this condition on $\mathbf{x}$ holds with probability at least $1 - \varrho$. Here τ is chosen large to accommodate $\varepsilon > 0$, $\varrho > 0$, and $R > 0$. The set $\mathcal{A} \subset \mathbb{R}^n$ is *uniformly globally asymptotically stable (UGAS) in probability for (4)* if it is globally stable in probability for (4) and uniformly globally attractive in probability for (4).

A.2 Stochastic hybrid basic conditions

Regularity conditions are used to establish existence of random solutions and to guarantee that integrals appearing in the study of the system (4) are well defined. The conditions can also be used to establish that Lyapunov conditions for uniform global asymptotic stability in probability are robust; see [Teel, 2013]. The first set of regularity assumptions are inherited from non-stochastic hybrid systems, as provided by [Goebel and Teel, 2006].

Assumption 3. (Hybrid Basic Conditions)

(1) The sets $C \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^n$ are closed.
(2) The set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous (that is, it has a closed graph), locally bounded (that is, the image of each bounded set is a bounded set), and for each $x \in C$ the value $F(x)$ is nonempty and convex.
(3) The set-valued mapping $G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is locally bounded and, for each $v \in \mathbb{R}^m$, $x \mapsto G(x,v)$ is outer semicontinuous. □

$G(x,v)$ may be empty for some $x \in D$. With $\mathcal{V} := \cup_{\omega\in\Omega, i\in\mathbb{Z}_{\geq 0}} \mathbf{v}_{i+1}(\omega)$, solutions can take values in the set $C \cup D \cup G(D \times \mathcal{V})$. This set is equal to $C \cup D$ when $G(D \times \mathcal{V}) \subset C \cup D$.

We also impose a condition on how G depends on v:

Assumption 4. (Stochastic Hybrid Basic Condition) The mapping $v \mapsto \text{graph}(G(\cdot,v))$ is measurable, where $\text{graph}(G(\cdot,v)) := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : y \in G(x,v)\}$. □

Assumption 4 guarantees that $v \mapsto G(\psi(v),v)$ is a measurable mapping when ψ is a measurable function. See [Rockafellar and Wets, 1998, Thm. 14.13(b)]. Assumption 4 holds if the domain of $v \mapsto \text{graph}(G(\cdot,v))$ is countable or if $(x,v) \mapsto G(x,v)$ is outer semicontinuous. It also holds if G is single valued and such that $x \mapsto G(x,v)$ is continuous and $v \mapsto G(x,v)$ is measurable, as shown in the details of [Rockafellar and Wets, 1998, Ex. 14.15]. Assumption 4 implies $v \mapsto G(x,v)$ is measurable for each $x \in \mathbb{R}^n$.

A.3 Lyapunov conditions

A function $V : \text{dom } V \to \mathbb{R}$ is a *certification candidate for* (C,D,G,μ) if (recall the definition $\mathcal{V} := \cup_{\omega\in\Omega, i\in\mathbb{Z}_{\geq 0}} \mathbf{v}_{i+1}(\omega)$)

C1. $C \cup D \cup G(D \times \mathcal{V}) \subset \text{dom } V$,
C2. $0 \leq V(x)$ for all $x \in C \cup D \cup G(D \times \mathcal{V})$, and
C3. the quantity $\int_{\mathbb{R}^m} \sup_{g\in G(x,v)} V(g)\mu(dv)$ is well defined for each $x \in D$, using the convention that $\sup_{g\in G(x,v)} V(g) = 0$ when $G(x,v) = \varnothing$, justified by the preceding item.

Lemma 1. Under Assumptions 3-4, if $V : \text{dom } V \to \mathbb{R}$ is upper semicontinuous and satisfies C1-C2 then it satisfies C3. ■

Let $L_V(0) := \{x \in \text{dom } V : V(x) = 0\}$. The function V is a *partially Lipschitz certification candidate for* (C,D,G,μ) if V is locally Lipschitz on an open set containing $C \backslash L_V(0)$. Given a real-valued function V that is locally Lipschitz on an open set $\mathcal{U} \subset \mathbb{R}^n$ a point $x \in \mathcal{U}$, and a vector $f \in \mathbb{R}^n$ we use $V^\circ(x;f)$ to denote the Clarke generalized directional derivative of V at x in the direction f. When V is C^1 near x, this quantity reduces to $\langle \nabla V(x), f\rangle$. See [Clarke, 1990].

We define a Lyapunov function and assert that it certifies uniform global asymptotic stability in probability. Let $\mathcal{A} \subset \mathbb{R}^n$ be compact. A partially Lipschitz certification candidate for (C,D,G,μ) is a *Lyapunov function* for $\mathcal{A}$ for (4) if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a continuous, positive definite function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \quad \forall x \in C \cup D \cup G(D \times \mathcal{V})$$
$$V^\circ(x;f) \leq -\rho(|x|_{\mathcal{A}}) \quad \forall x \in C\backslash L_V(0), f \in F(x)$$
$$\int_{\mathbb{R}^m} \sup_{g\in G(x,v)} V(g)\mu(dv) \leq V(x) - \rho(|x|_{\mathcal{A}}) \quad \forall x \in D.$$

Theorem 3. Let $\mathcal{A} \subset \mathbb{R}^n$ be compact and let Assumptions 3-4 hold. Under these conditions, if there exists a Lyapunov function for $\mathcal{A}$ for (4) then $\mathcal{A}$ is uniformly globally asymptotically stable in probability for (4). ■