The Circle Criterion for Synchronization in Nonlinearly Coupled Networks.

Anton V. Proskurnikov

* *St.-Petersburg State University and Institute for Problems of Mechanical Engineering, St.-Petersburg, Russia E-mail: avp1982@gmail.com*

Abstract: The problem of synchronization (consensus) in nonlinearly coupled network is addressed. The agents of the network are assumed to identical and linear, however, they may have arbitrary order and be unstable. The interaction topology may switch and the couplings are uncertain, assumed only to satisfy conventional quadratic constraints. We offer easily verifiable synchronization criteria, based on the Kalman-Yakubovich-Popov lemma and extending a number of known result for agents with special dynamics. Those criteria are close in spirit to the celebrated circle criterion for the stability of Lurie systems.

Keywords: Synchronization, consensus, complex networks, absolute stability, nonlinear systems.

1. INTRODUCTION.

The problems of synchronization in complex networks Wu (2007) attract enormous attention of different research communities. A complex network may be considered as a group of subsystems (or *agents*), that interact (via communication, physical coupling or otherwise) in accordance with some pattern, typically depicted as a graph or *topology* of the network. The interest in synchronization problems is focused on the global synchronism (known also as the consensus or agreement) that is established as a result of local interactions between the agents. The effect of consensus lies in the heart of many natural phenomena and engineering approaches. Examples include numerous forms of collective behavior in complex biological and artificial multi-vehicle systems, such as herding, flocking, swarming, schooling etc. (see e.g. Olfati-Saber et al. (2007); Ren and Beard (2008); Bullo et al. (2009) and references therein) and synchronization of oscillator ensembles Strogatz (2000); Chopra and M.W.Spong (2009); Ren (2008). Distributed control policies (called also consensus or synchronization protocols) which render independent agents to act synchronously take their origin from averaging procedures in the theory of stochastic matrices and other distributed algorithms in computer science, see Olfati-Saber et al. (2007) for detailed review.

While synchronization protocols with linear couplings are widespread and deeply investigated, see Olfati-Saber et al. (2007); Ren and Beard (2008); Seo et al. (2009); Wieland et al. (2011); Scardovi and Sepulchre (2009) and references therein, a vast number of applications require nonlinear consensus algorithms which can not be tackled by linear techniques (such as convergence criteria for matrix products, Laplacian decomposition, frequency-domain methods etc.) Such algorithms come from, for instance, oscillator networks with periodic couplings Strogatz (2000); Chopra and M.W.Spong (2009) and coordination with range-restricted sensoring Tanner et al. (2007); Lin et al. (2007a); Su et al. (2009); Lin et al. (2007b). In the real-world applications linear protocols may de facto become nonlinear because of quantization effects and other data distortions. These challenges motivated the development of synchronization theory for nonlinearly coupled networks. One of the major achievements in this direction concerns nonlinear averaging procedures for the first-order agents Moreau (2005); Lin et al. (2007a,b); Ajorlou et al. (2011) that provide the convex hull of the agents to be nested and use the diameter of this convex hull as a Lyapunov function of the system. Another class of results for nonlinear consensus protocols deals with passive agents Chopra and Spong (2006); Arcak (2007) and exploits the fact of decreasing of the total energy of the system (sum of individual storage functions) along the trajectories. For non-passive nonlinearly coupled agents, e.g. double integrators, the consensus is usually proved for quite special types of nonlinearities Ren and Beard (2008); Su et al. (2009); Abdessameud and Tayebi (2010).

While most of recent results on synchronization in nonlinearly coupled networks focus on agents with special dynamics (passive, multiple integrators etc.), below the networks of general MIMO agents (possibly, exponentially unstable) are considered, and the topology may have zero dwell-time and is not necessarily piecewise-continuous. Our main result is a synchronization criterion for the networks with undirected interaction topology and nonlinear couplings, satisfying symmetry conditions (resembling the Newton's Third Law). It appears, however, that the same conditions guarantee the consensus for linearly coupled networks with directed topology, that satisfy a balance condition. In both cases, the full knowledge of couplings may be unavailable, they are supposed only to satisfy some conventional quadratic constraint Gelig et al. (2004).

[1] Supported by RFBR, grants 11-08-01218, 12-01-00808, 13-08-01014 and the Russian Federal Program "Cadres" (contract 8855)

Our criterion, based on the absolute stability techniques, offers in fact a condition for the robust consensus (in the mentioned class of uncertainties), which is closely related to the celebrated circle criterion Lurie systems Gelig et al. (2004); Khalil (1996). The proposed criterion generalizes a number of known results on consensus in nonlinearly coupled networks. It extends, for instance, the results from Chopra and Spong (2006); Arcak (2007) to the case of non-passive MIMO agents, allows to obtain synchronization criterion for harmonic oscillators from Ren (2008) and new consensus results for double integrator agents.

The paper extends results of previous papers Proskurnikov (2011, 2013a), where the circle consensus criterion for SISO agents and scalar sectorial nonlinearities was obtained, in several ways. First of all, we consider general MIMO agents and correspondent quadratic constraints. Moreover, in both SISO and MIMO cases we obtain sufficient conditions for consensus with exponential convergence rate.

The paper is organized as follows. Section 2 introduces some basic concepts from the graph theory. Section 3 is devoted to the formulation of the problem, the main assumptions are given in Section 4. Section 5 presents the main result of the paper (the sufficient condition for consensus), and Section 6 illustrates some applications to agents with special dynamics.

2. PRELIMINARIES FROM THE GRAPH THEORY

A pair $G = (V, E)$ of two finite set V (the set of nodes) and $E \subset V \times V$ (the set of arcs) is called a *(directed) graph*. The node v is *connected to* the node w in G if $(v, w) \in E$. Any sequence of nodes $v_1, v_2, \ldots, v_k$ with $(v_i, v_{i+1}) \in E$ for $i = 1, 2, \ldots, k-1$ is called *a path* between v_1 and v_k. The graph is *strongly connected* if a path between any two different nodes exists. Given a graph $G = (V, E)$, the *mirror graph* $\hat{G} = (V, \hat{E})$ is obtained by inserting all inverted arcs, i.e. $\hat{E} = E \cup \{(w, v) : (v, w) \in E\}$. A graph coinciding with its mirror is said to be *undirected*. Throughout the paper $\mathbb{G}_N$ stands for the class of all graphs $G = (V_N, E)$ with the node set $V_N = \{1, 2, \ldots, N\}$ and set of arcs E containing no loops ($(v, v) \notin E$ for any $v \in V_N$). Define the *adjacency* matrix $(a_{jk}(G))$ of $G \in \mathbb{G}_N$ as follows: $a_{jk}(G)$ is 1 if $(k, j) \in E$ and 0 otherwise. The *Laplacian matrix* of G is given by

$$L(G) = \begin{bmatrix} \sum_{j=1}^{N} a_{1j} & -a_{12} & \ldots & -a_{1N} \\ -a_{21} & \sum_{j=1}^{N} a_{2j} & \ldots & -a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N1} & -a_{N2} & \ldots & \sum_{j=1}^{N} a_{Nj} \end{bmatrix}. \quad (1)$$

If G is undirected then $L(G) = L(G)^T$ and $\lambda_1(G)$ (the least eigenvalue of $L(G)$) equals to 0, thus $L(G) \geq 0$ Olfati-Saber et al. (2007). The second eigenvalue $\lambda_2(G)$ is called the *algebraic connectivity* of G and may be defined by Fiedler (1973):

$$\lambda_2(G) = N \min_z \frac{\sum_{i,j=1}^{N} a_{ij}(G)(z_j - z_i)^2}{\sum_{i,j=1}^{N} (z_j - z_i)^2}. \quad (2)$$

The minimum in (2) is over the set of all $z \in \mathbb{R}^N$ with $z_k \neq z_j$ for some j, k. One has $\lambda_2(G) > 0$ if and only if the undirected graph G is connected. Moreover, denoting by $e(G) \geq 0$ the minimal number of arcs one has to delete to break the graph connectivity, the inequality holds Fiedler (1973)

$$\lambda_2(G) \geq 2e(G)\left(1 - \cos\frac{\pi}{N}\right). \quad (3)$$

Following Olfati-Saber et al. (2007), we define the algebraic connectivity of a *directed* graph G by (2) and denote it as $\lambda_2(G)$ despite it is no longer eigenvalue of $L(G)$. Since $a_{jk}(G) + a_{kj}(G) \geq a_{jk}(\hat{G}) = a_{kj}(\hat{G})$ and thus $\lambda_2(G) \geq \frac{1}{2}\lambda_2(\hat{G})$, for a strongly connected graph $G \in \mathbb{G}_N$ one has

$$\lambda_2(G) \geq e(\hat{G})\left(1 - \cos\frac{\pi}{N}\right) \geq 1 - \cos\frac{\pi}{N}. \quad (4)$$

The estimates (3),(4) and many others Merris (1994) may be useful whenever precise computation of λ_2 is complicated (e.g. the graph is unknown or its size is too large for straightforward computation of the algebraic connectivity).

3. PROBLEM FORMULATION.

Throughout the paper we deal with a team of identical agents, indexed 1 through $N \geq 2$ and governed by a common MIMO state-space model

$$\dot{x}_j(t) = Ax_j(t) + Bu_j(t),\ y_j(t) = Cx_j(t). \quad (5)$$

Here $t \geq 0$, $x_j \in \mathbb{R}^d$, $u_j \in \mathbb{R}^m$, $y_j \in \mathbb{R}^k$ are the state, control, and output of the j-th agent, respectively. The model (5) is assumed to be controllable and observable. The control inputs are results of interactions (via communication, mechanical links or otherwise) between the agents. The interaction topology is time-varying and at time $t \geq 0$ is described by a graph $G(t) = [V_N, E(t)] \in \mathbb{G}_N$: the output $y_k(t)$ of k-th agent exerts influence on j-th one the if and only if $(k, j) \in E(t)$.

Specifically, we examine distributed control protocols of the following type:

$$u_j(t) = \sum_{k:(k,j) \in E(t)} \varphi_{jk}(t, y_k(t) - y_j(t)). \quad (6)$$

The mappings $\varphi_{jk} : [0; +\infty) \times \mathbb{R}^k \to \mathbb{R}^m$ are referred as *couplings* and determine interaction "strength" between the agents. The aim of the paper is to give conditions under which such a protocol establishes consensus among the agents in the following sense.

Definition 1. The protocol (6) provides the output consensus if the following two claims hold for all initial states $(x_j(0))$ and some constant $M > 0$:

$$\lim_{t \to +\infty} |y_j(t) - y_k(t)| = 0 \quad \forall k, j; \quad (7)$$

$$\mathcal{W}(t) := \sum_{j \neq k} |x_j(t) - x_k(t)|^2 \leq M\mathcal{W}(0)\ \forall t \geq 0. \quad (8)$$

If additionally $\mathcal{W}(t) \to 0$ as $t \to +\infty$, we say that the protocol (6) provides *the state consensus*. We say that the *exponential state consensus* is established, if $\mathcal{W}(t) \leq Me^{-\alpha t}\mathcal{W}(0)$ for some constants $M, \alpha > 0$.

The state consensus implies the output consensus. The converse is true under non-restrictive in practice condition of uniform continuity at zero point.

Remark 2. The output consensus implies the output consensus if $\lim_{y \to 0} \sup_{t \geq 0} |\varphi_{jk}(t, y)| = 0$ for any j, k.

Proof. Due to (6) and (7), the output consensus and the assumption $\lim_{y\to 0}\sup_{t\geq 0}|\varphi_{jk}(t,y)| = 0$ imply that $|u_j(t)| \to 0$ as $t \to +\infty$. Taking by definition $X_{jk} := x_k - x_j$, $U_{jk} := u_k - u_j$, and $Y_{jk} := y_k - y_j$, one obtains $\dot{X}_{jk} = AX_{jk} + BU_{jk}$ and $Y_{jk} = CX_{jk}$, where the pairs (A, B) and (A, C) are controllable and observable, respectively, $U_{jk}(t) \to 0$ and $Y_{jk}(t) \to 0$ as $t \to +\infty$. Thus $X_{jk}(t) \to 0$ for any k, j q.e.d. □

4. MAIN ASSUMPTIONS

The main assumptions basically come to the connectivity of the interaction topology (Assumption 3), the symmetry or balance condition on the couplings (Assumption 4) and the sector conditions for the couplings. We start with the assumption about the underlying graph.

Assumption 3. The graph $G(t)$ is strongly connected for all $t \geq 0$. The function $G(\cdot)$ is Lebesgue measurable, i.e. $G^{-1}(\Gamma) \subset \mathbb{R}$ is a measurable set for any $\Gamma \in \mathbb{G}_N$.

Maintaining connectivity (in some sense) is clearly necessary to prevent the agents from dissemination into separate clusters that do not interact and thus cannot be synchronized.

Our next assumptions concern the couplings φ_{jk}.

Assumption 4. The closed-loop system (5), (6) has a solution defined for all $t \geq 0$ for any initial states $x_j(0)$, and at least one of the following two statements holds whenever $j \neq k$ and $t \geq 0$:
a) The graph $G(t)$ is undirected and $\varphi_{jk}(t, y) = -\varphi_{kj}(t, -y)$;
b) The couplings are linear $\varphi_{jk}(t, y) = w_{jk}(t)y$ and the gains $w_{jk}(t) \in \mathbb{R}^{k\times m}$ satisfy the balance condition

$$\sum_{k:(k,j)\in E(t)} w_{jk}(t) = \sum_{k:(j,k)\in E(t)} w_{kj}(t)\ \forall j. \tag{9}$$

The symmetry condition from a) resembles the Newtons Third Law for couplings (since $\varphi_{jk}(t, y_k - y_j) = -\varphi_{kj}(t, y_j - y_k)$), whereas the balance condition is similar in spirit to the mass or energy preservation law (the summary inflow at the node equals the cumulative outflow). Moreover, in some applications, e.g. in oscillator networks Strogatz (2000); Chopra and M.W.Spong (2009); Ren (2008) Assumption 4 holds due to exactly these laws.

Although in general the output consensus condition (7) says nothing about the asymptotics of individual outputs y_j, under Assumption 4 the condition (7) implies the *average output consensus* Olfati-Saber et al. (2007): $y_j(t) - Ce^{tA}\tilde{x}_0 \to 0$ as $t \to \infty$ for all j, where $\tilde{x}_0 = \frac{1}{N}\sum_{j=1}^{N} x_j(0)$. To establish this, notice that $\sum_{j=1}^{N} u_j = 0$ due to Assumption 4. By summing up the equations from (5), we see that $\sum_{j=1}^{N} y_j(t) = Ce^{tA}\tilde{x}_0$, thus our claim is evident from (7).

In the present paper we focus on the case when the couplings may be unknown but satisfy some *quadratic constraint* Gelig et al. (2004). In other words, $\varphi_{jk} \in \mathfrak{S}(\mathcal{F})$ for all j, k where $\mathcal{F} : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$ is a Hermitian form and $\mathfrak{S}(\mathcal{F})$ stands for the set of functions $\varphi : [0; +\infty) \times \mathbb{R}^k \to \mathbb{R}^m$ such that the claims hold:

i) $\varphi(t, 0) \equiv 0$ and $\varphi(t, y)$ is measurable in t for all y and continuous in y for almost all t;

ii) For any compact subset $K \subset \mathbb{R}\setminus\{0\}$,
$$\inf_{y\in K, t\geq 0} \mathcal{F}(\varphi(t, y), y) > 0. \tag{10}$$

In particular, the graph of $\varphi(t, \cdot)$ lies strictly in the cone $\mathcal{F}(\varphi, y) > 0$ except for the origin.

Thus the consensus criterion should be given in terms of $\mathcal{F}$ and the coefficients A, B, C, but not the couplings themselves. Such a criterion automatically ensures consensus for all couplings from the class $\mathfrak{S}(\mathcal{F})$ that satisfy Assumption 4. We note that one of the simplest and widespread class of quadratically constrained function is constituted by scalar-valued functions satisfying the conventional sector condition with given slopes Gelig et al. (2004); Khalil (1996), see paragraph 5.2 for details.

5. MAIN RESULTS.

In the paragraph 5.1 below the main result of the paper is presented which establishes sufficient conditions for consensus in general MIMO case. Those conditions are especially transparent and convenient to use in the scalar case ($k = m = 1$) when the quadratic constraint represents the sector inequality. This particular case is subject of the paragraph 5.2

5.1 Consensus criterion for the MIMO agents case.

We start with introducing some auxiliary notations. Let $W_x(\lambda) = (\lambda I - A)^{-1}B$ and $W_y(\lambda) = CW_x(\lambda)$ be the transfer matrices of the plant (5) from u to y, x respectively and θ stand for the minimal algebraic connectivity of the interaction topology:

$$\theta = \min_{t\geq 0} \lambda_2\big(G(t)\big) \tag{11}$$

Given a Hermitian form $\mathcal{F}(u, y)$ as follows
$$\mathcal{F}(\varphi, y) = Re(\varphi^* q y) - y^* Q y - \varphi^* R \varphi, \tag{12}$$
(where $y \in \mathbb{C}^k$, $\varphi \in \mathbb{C}^m$ and $q \in \mathbb{R}^{m\times k}$, $Q = Q^* \in \mathbb{R}^{k\times k}$, $R = R^* \in \mathbb{R}^{m\times m}$) and a matrix $\Lambda \in \mathbb{C}^{k\times m}$, let

$$\Pi_{\mathcal{F}}(\Lambda) = q\Lambda + \Lambda^* q^* + \theta\Lambda^* Q\Lambda + \frac{1}{2(N-1)}R. \tag{13}$$

The following theorem is the main result of the paper.

Theorem 5. Suppose that Assumptions 3 and 4 hold and $\varphi_{jk} \in \mathfrak{S}(\mathcal{F})\ \forall j, k$ where $\mathcal{F}$ is a quadratic form (12) with $Q \geq 0, \Gamma \geq 0$. Assume also that

(i) There exists matrix $K \in \mathbb{R}^{m\times k}$ such that $A - NBKC$ is Hurwitz and $\mathcal{F}(Ky, y) > 0\ \forall y \neq 0$;
(ii) for any frequency $\omega \in \mathbb{R}$ such that $\det(i\omega I - A) \neq 0$, the following inequality is true
$$\Pi_{\mathcal{F}}(\Lambda) \geq 0 \text{ for } \Lambda := W_y(i\omega). \tag{14}$$

Then the protocol (6) provides the output consensus. Furthermore, if $\varepsilon > 0$ exists such that $\Pi(i\omega) \geq \varepsilon|W_x(i\omega)|^2$ for any ω, then the exponential state consensus is established.

The proof of this theorem may be given using techniques analogous to that from Proskurnikov (2013a), it is omitted due to space limitations, available upon request and is going to appear in Proskurnikov (2013b).

By Theorem 9, conditions (a) and (b) in fact imply the robust consensus in the sense that (7) holds for all couplings $\varphi_{jk} \in \mathfrak{S}(\mathcal{F})$ and time-varying interaction graphs $G(t)$ satisfying Assumptions 3 and 4. In practice (a) is almost unavoidable for such a robust consensus. The existence of a matrix K such that the map $y \mapsto Ky$ belongs to $\mathfrak{S}(\mathcal{F})$ is typically a non-restrictive assumption which is fulfilled, for instance, for scalar case and sector inequalities (see paragraph 5.2). The following remark shows that all of those matrices, if exist, should satisfy (a).

Remark 6. Under the robust consensus, (a) holds with any $K \in \mathbb{R}^{m\times k}$ such that $\mathcal{F}(Ky, y) > 0\, \forall y \neq 0$.

Indeed, the output consensus for $G(t) \equiv G_0$ with complete graph G_0 and $\varphi_{jk}(t, y) = Ky$ implies the state consensus by Remark 2. So for a nonzero eigenvalue λ of the Laplacian $L(G_0)$, the matrix $A - \lambda BKC$ is Hurwitz Fax and Murray (2004). It remains to note that $\lambda = N$ for the complete graph.

Remark 7. In the conditions for consensus given by Theorem 5, the underlying topology of the network is concerned only by the multiplier θ in (13). Since by assumption $Q \geq 0$, formal replacement of θ by its lower estimate retains sufficiency for consensus. This observation is useful whenever the exact computation of θ is complicated. Then many available constructive estimates of the algebraic connectivity, like (3) or (4), may be used; we refer the reader to Merris (1994) for their survey. Moreover, the value of θ is unimportant if $Q = 0$.

5.2 Consensus for scalar sectorial couplings.

In this paragraph we interpret the result of Theorem 5 for SISO agents and scalar couplings which satisfy *the sector inequalities* with known slopes Gelig et al. (2004); Khalil (1996). In other words, $k = m = 1$ and $\varphi_{jk} \in S[\alpha;\beta]$ where $S[\alpha;\beta]$ is a set of functions $\varphi : [0;+\infty) \times \mathbb{R} \to \mathbb{R}$ such that $\varphi(t, 0) \equiv 0$, $\varphi(t,\sigma)$ is measurable in t for all σ, continuous in σ for almost all $t \geq 0$ and

$$\alpha < \inf_{\sigma\in K, t\geq 0} \frac{\varphi(t,\sigma)}{\sigma} \leq \sup_{\sigma\in K, t\geq 0} \frac{\varphi(t,\sigma)}{\sigma} < \beta, \tag{15}$$

for any compact set $K \subset \mathbb{R} \setminus \{0\}$. It follows from (15) that the graph of $\varphi(t,\cdot)$ lies strictly between the lines $\xi = \alpha\sigma$ and $\xi = \beta\sigma$ everywhere except for the origin.

We note that if $\alpha < 0$ and $\beta > 0$, the protocol (6) obviously does not provide consensus unless A is a Hurwitz matrix, although all assumptions are satisfied. Since we are mostly interested in agents with unstable open-loop dynamics, we exclude the case $\alpha < 0, \beta > 0$ from consideration, thus focusing on the cases where either $0 \leq \alpha < \beta$ or $\alpha < \beta \leq 0$. Moreover, since the second of them is reduced to the first one by the substitution $(\alpha, \beta, B) \mapsto (-\beta, -\alpha, -B)$, we shall consider only the first case.

We first transform the inequalities (15) into quadratic constraint. We introduce the constants

$$\gamma = \frac{1}{\beta + \alpha} \geq 0, \quad \delta = \frac{\alpha}{1 + \alpha\beta^{-1}} \geq 0, \tag{16}$$

and the quadratic forms as follows:

$$\mathcal{F}_{\alpha;\beta}(\varphi, y) := \varphi y - \delta y^2 - \gamma\varphi^2,\ \varphi, y \in \mathbb{R} \tag{17}$$

$$\Pi_{\alpha;\beta}(\lambda) = Re\,\lambda + \theta\delta|\lambda|^2 + \frac{\gamma}{2(N-1)},\ \lambda \in \mathbb{C}. \tag{18}$$

Proposition 8. $S[\alpha;\beta] = \mathfrak{S}(\mathcal{F}_{\alpha;\beta})$ and $\Pi_{\mathcal{F}_{\alpha;\beta}} \equiv \Pi_{\alpha;\beta}$.

Indeed, the sector inequalities (15) may be rewritten as $(\varphi - \alpha y)(y - \beta^{-1}\varphi) = (1 + \alpha\beta^{-1})\mathcal{F}_{\alpha;\beta}(\varphi, y) > 0$, where $\varphi := \varphi(t, y)$ and the latter inequality is uniform in $t \geq 0$ and $y \in K$ (for any compact $K \subset \mathbb{R} \setminus \{0\}$). The second claim is easily seen from (13),(18). □

The proposition 8 allows one to obtain the following consensus criterion for SISO agents case.

Theorem 9. Suppose that Assumptions 3 and 4 are satisfied, $\varphi_{jk} \in S[\alpha;\beta]\ \forall j, k$ where $0 \leq \alpha < \beta \leq \infty$, and the following two claims hold:

(a) There exists $\mu \in (\alpha;\beta)$ such that the matrix $A - \mu NBC$ is Hurwitz;
(b) for any frequency $\omega \in \mathbb{R}$ such that $\det(i\omega I - A) \neq 0$, the following inequality is true

$$\Pi_{\alpha;\beta}(\lambda) \geq 0 \text{ for } \lambda := W_y(i\omega). \tag{19}$$

Then the protocol (6) provides the output consensus. Moreover, if $\varepsilon > 0$ exists such that $\Pi_{\alpha;\beta}(W_y(i\omega)) \geq \varepsilon|W_x(i\omega)|^2$, the exponential state consensus is established.

This result immediately follows from Theorem 5 (applied for $\mathcal{F} := \mathcal{F}_0$, $R := \gamma$, $Q := \delta$) and Proposition 8. Indeed, (a) coincides with the assumption (i) from Theorem 5 and (b) is nothing more than (ii) from the same theorem.

Inequality (19) is non-trivial since the Hermitian form $\Pi_{\alpha;\beta}$ is not non-negative definite: its discriminant equals to $\frac{1}{4}\left[\frac{2\delta\gamma\theta}{N-1} - 1\right] \leq 0$ since $\theta \leq N$ by (2) and $\delta\gamma \leq 1/4$.

Remark 10. Condition (b) means that the Nyquist curve $\{W_y(i\omega)\}$ lies outside the set $\mathcal{D}$ which is the open half-plane $\{z \in \mathbb{C} : Re\,z < -\frac{\gamma}{2(N-1)}\}$ for $\alpha = 0$ and the disk $\mathcal{D} = \{z \in \mathbb{C} : |z - z_0| < \rho_0\}$ for $\alpha > 0$,

$$z_0 = -\frac{1}{2\delta\theta}, \quad \rho_0 = \frac{1}{2\delta\theta}\sqrt{1 - \frac{2\theta\delta\gamma}{N-1}}.$$

Here the first claim is trivial since $\alpha = 0 \Rightarrow \delta = 0$ by (16). The second claim is valid since $Re\,z + \delta\theta|z|^2 + \frac{\gamma}{2(N-1)} \geq 0 \Leftrightarrow |z|^2 - 2z_0\,Re\,z + |z_0|^2 - \rho_0^2 \geq 0 \Leftrightarrow z \notin \mathcal{D}$.

The geometrical interpretation given by Remark 10 highlights the similarity between the conditions presented by Theorem 9 and the celebrated circle criterion for stability of Lurie systems Gelig et al. (2004); Khalil (1996). Theorem 9 can be considered as an extension of the circle criterion in the following sense. Consider two agents ($N = 2$) applying the protocol (6) with two-directional communication (E(t)={(1,2),(2,1)}) and the coupling functions $\varphi_{12} = -\varphi_{21} \in S[\alpha;\beta]$. Taking $X(t) = x_2(t) - x_1(t)$, $Y(t) = y_2(t) - y_1(t)$, $U(t) = u_2(t) - u_1(t)$ and $\Phi(t, Y) = -2\varphi_{12}(t, Y)$, we have $\Phi \in S[-2\beta; -2\alpha]$ and

$$\dot{X}(t) = AX(t) + BU(t), U(t) = \Phi(t, Y(t)). \tag{20}$$

By the circle criterion Gelig et al. (2004) the equilibrium $X = 0$ of the system (20) is exponentially stable if $A + kBC$ is Hurwitz for some $k \in [-2\beta; -2\alpha]$ (which follows from (a) since $\theta = 2$) and the Nyquist curve $\{W_y(i\omega) : \omega \in \mathbb{R}\} \subset \mathbb{C}$ lies strictly outside the disk based on the diameter $[(-2\alpha)^{-1} + 0i; (-2\beta)^{-1} + 0i]$ (or the half-plane $\{z : Re\,z < (-2\beta)^{-1}\}$ in the case of $\alpha = 0$). This disk (or half-plane) coincides with the set $\mathcal{D}$ introduced above.

6. ILLUSTRATIVE EXAMPLES

Now we illustrate that Theorems 5,9, applied to agents with special dynamics, provide improvements of recent results in the area.

6.1 Consensus among passive agents

Let the transfer matrix W_y be square (thus $k = m$) and satisfy the positive realness condition:

$$W_y(i\omega) + W_y(i\omega)^* \geq 0 \qquad \forall\omega \in \mathbb{R}. \tag{21}$$

As is well known, this implies their *passivity* Chopra and Spong (2006); Khalil (1996) provided that (a) from Theorem 9 holds. Consensus and synchronization of passive systems have earned a considerable interest; see e.g. Chopra and Spong (2006); Arcak (2007) among the others. Most of the related results deal with nonlinear agents but the topology is either time-invariant or with positive dwell time, and the couplings are time-invariant. Though this paper is concerned with only linear agents, on the positive side, it considers the most general case of measurable time-varying interaction topology and admits non-stationary couplings.

Taking the quadratic form $\mathcal{F}(\varphi, y) := \varphi^T y$ and observing that (21) implies (14) for any θ and $\mathcal{F}(Ky, y) = \frac{1}{2}y^T(K + K^T)y/2\ \forall K \in \mathbb{R}^{m\times m}$, we obtain the following corollary of Theorem 5.

Theorem 11. Let Assumptions 3 and 4 be valid, (21) hold, $\varphi_{jk} \in \mathfrak{S}(\mathcal{F})$ with $F(\varphi, y) = \varphi^T y$, and there exists matrix $K \in \mathbb{R}^{m\times m}$ such that $K + K^T > 0$ and $A - BKC$ is Hurwitz. Then the protocol (6) establishes the output consensus.

In the scalar case the result of Theorem 11 becomes especially simple.

Theorem 12. Let $k = m = 1$,Assumptions 3 and 4 be valid, (21) hold, $\varphi_{jk} \in S[0; +\infty]$, and $A-\mu BC$ be Hurwitz for some $\mu > 0$. Then the protocol (6) establishes the output consensus.

Theorem 12 generalizes the results of Chopra and Spong (2006); Lin et al. (2007b); Ren and Beard (2008), etc., which concern networks of the first-order integrators $\dot{y}_j = u_j \in \mathbb{R}$, as well as those of Chopra and M.W.Spong (2009); Olfati-Saber et al. (2007), which deal with networks of identical Kuramoto oscillators with initial phases from $(-\pi/2; \pi/2)$. Another example is the result of Ren (2008). It concerns a network of identical harmonic oscillators $\dot{q}_j = y_j \in \mathbb{R}, \dot{y}_j = -\omega_0^2 q_j + u_j$ and asserts that the y-output consensus is established by the linear balanced protocol (6) (Assumption 4b holds) with constant gains $w_{jk} > 0$. Theorem 12 generalizes this result on time-varying gains and nonlinear couplings. Indeed, $W_y(z) = z/(z^2 - \omega_0^2)$ and thus $W_y(i\omega) + W_y(i\omega)^* = 2\,Re\,W_y(i\omega) = 0$, and it is obvious that the feedback $u_j = -\mu y_j$ stabilizes the j-th agent for any $\mu > 0$.

6.2 Consensus among strictly passifiable agents

A natural generalization of passivity is *passifiability*: a linear agent is called *passifiable* if it may be made passive by an appropriate linear feedback. This is implied by the *strict passifiability* Fradkov (2003). The problem of consensus among passifiable agents seems to have been unexplored up to now in the literature. To simplify matters, we bound ourselves with SISO agents only: $k = m = 1$. Under such an assumption, strict passifiability of the agent means that $CB > 0$ and the polynomial $\psi(\lambda) = \det(\lambda I - A)W_y(\lambda)$ is Hurwitz (i.e., the agent is minimum-phase).

The following theorem shows that such consensus holds whenever the couplings are strong enough.

Theorem 13. Suppose that the agents are strictly passifiable, Assumptions 3 and 4 hold, and $\varphi_{jk} \in S[\alpha; +\infty]$, where for $\theta = \min_{t\geq 0} \lambda_2\big(G(t)\big)$ one has

$$\alpha > \alpha_* := \frac{1}{\theta} \sup_{\omega\in\mathbb{R}} f(\omega),\ f(\omega) := -\frac{Re\,W_y(i\omega)}{|W_y(i\omega)|^2}. \tag{22}$$

Then the exponential state consensus is provided.

The proof of this theorem is analogous to that of Theorem 9 in Proskurnikov (2013a) (notice, however, that the latter result states only output consensus but not exponential convergence rate) and follows from the fact that $\Pi_{\alpha;\infty}(W_y(i\omega)) \geq (\alpha - \alpha_*)\theta|W_y(i\omega)|^2$. Since $CB > 0$ and $\det(i\omega I - A) \neq 0$, a constant $\xi > 0$ exists such that $|W_y(i\omega)| \geq \xi|W_x(i\omega)|$ for all $\omega \in \mathbb{R}$. This proves the claim since $\Pi_{\alpha;\infty}(W_y(i\omega)) \geq \varepsilon|W_x(i\omega)|^2$ for $\varepsilon := \xi^2\theta(\alpha - \alpha_*)$.

6.3 Consensus among double-integrator agents.

Consensus problem for networks of double integrators have recently attracted considerable interest because of various applications to multi-vehicle formation control; see e.g., Ren and Beard (2008); Tanner et al. (2007); Abdessameud and Tayebi (2010) among the others. In this case, the j-th agent is described by the following equations

$$\ddot{z}_j = u_j,\ y_j = q_0 z_j + q_1 \dot{z}_j, \tag{23}$$

where y_j is the output, and $q_0, q_1 \in \mathbb{R}$ are constants. So $W_y(i\omega) = q_0(i\omega)^{-2} + q_1(i\omega)^{-1}$.

It is easy to see that (19) shapes into $Kq_0^2\omega^{-4} + (Kq_1^2 - q_0)\omega^{-2} \geq 0$, where $K := \alpha\theta$. This permits us to apply Theorem 9, which gives rise to the following.

Corollary 14. Suppose that the agents are described by equations (23) with $q_0, q_1 > 0$, Assumptions 3 and 4 hold, and $\varphi_{jk} \in S\left[\theta^{-1}q_1^{-2}q_0; +\infty\right]$, where $\theta := \inf_{t\geq 0} \lambda_2\big(G(t)\big)$. Then the protocol (6) establishes the output consensus.

It should be noticed that usually relative position and velocities $z_j - z_k$ and $\dot{z}_j - \dot{z}_k$ are measured rather than their linear combinations $y_j - y_k$. By choosing appropriate $q_0, q_1 > 0$, one may ensure establishing of the consensus for the coupling class $S[\alpha; +\infty]$ with arbitrary small $\alpha > 0$. However, to find q_0, q_1 an estimate of θ from below is required, moreover, Corollary 14 does not work for saturation-like bounded nonlinearities (that does not belong to $S[\alpha; \beta]$ unless $\alpha = 0$). These shortages may be overcome if absolute velocities $\dot{z}_j$ can be measured. In this case, the modified protocols may be applied Cheng et al. (2011):

$$u_j = -p\dot{z}_j + \sum_{(k,j)\in E(t)} \varphi_{jk}(t, y_k(t) - y_j(t)). \tag{24}$$

The closed-loop system (23),(24) is the same as if the standard protocol (6) were applied to modified agents

$$\ddot{z}_j + p\dot{z}_j = u_j,\ y_j = q_0 z_j + q_1 \dot{z}_j. \tag{25}$$

Notice that if $p > 0$, $q_0 > 0, q_1 \geq 0$ then feedback $u_j = -\mu y_j$ stabilizes the plant (25) for any $\mu > 0$. Since $W_y(i\omega) = (q_0 + q_1(i\omega))[(i\omega)^2 + p(i\omega)]^{-1}$ and $Re\, W_y(i\omega) = (q_1 p - q_0)(p^2 + \omega^2)^{-2}$ for the agent (25), it is easily seen that (19) holds for $\alpha = \theta = 0$ if

$$\inf_{\omega \in \mathbb{R}} Re\, W_y(i\omega) = \min\left(\frac{q_1 p - q_0}{p^2}, 0\right) \geq -\frac{\beta^{-1}}{2(N-1)}.$$

Theorem 9 implies the following result:

Corollary 15. Suppose that the agents are described by equations (23) with $q_1 \geq 0$, $q_0 > 0$, Assumptions 3 and 4 hold, $p > 0$ and $\varphi_{jk} \in S\,[0;\beta]$ with $\beta \leq +\infty$. Suppose also that $p^2\beta^{-1} + 2(N-1)(q_1 p - q_0) \geq 0$. Then the protocol (24) establishes the output consensus.

Corollary 15, in particular, guarantees consensus for $q_1 p \geq q_0$ and $\beta = +\infty$ (which agrees with Theorem 12 as agents (25) are passive) and for $q_1 = 0$, $\beta \leq p^2/(2q_0(N-1))$ (the relative velocities are not used at all). Also it is easily noticed from (25) that in assumptions of Remark 2 the protocol (24) provides condition $\dot{z}_j(t) \to 0$ besides the state consensus.

7. CONCLUSION

Output consensus among identical high-order linear agents was examined in the case where the interaction topology is uncertain and switching but assumed to preserve its connectivity. The couplings among the agents are uncertain as well and satisfy the conventional quadratic constraint. A new criterion for robust output consensus is established which is to be extended on the leader-following formation control, reference-tracking consensus, etc.

REFERENCES

Abdessameud, A. and Tayebi, A. (2010). On consensus algorithms for double-integrator dynamics without velocity measurements and with input constraints. *Syst. Contr. Letters*, 59, 812–821.

Ajorlou, A., Momeni, A., and Aghdam, A. (2011). Sufficient conditions for the convergence of a class of nonlinear consensus algorithms. *Automatica*, 47, 625–629.

Arcak, M. (2007). Passivity as a design tool for group coordination. *IEEE Trans. Autom. Control*, 52(8), 1380–1390.

Bullo, F., Cortes, J., and Martinez, S. (2009). *Distributed control of robotics networks*. Princeton University Press.

Cheng, L., Hou, Z.G., Tan, M., and Wang, X. (2011). Necessary and sufficient conditions for consensus of double-integrator multi-agent systems with measurement noises. *IEEE Trans. Autom. Control*, 53(8), 1958–1963.

Chopra, N. and M.W.Spong (2009). On exponential synchronization of Kuramoto oscillators. *IEEE Trans. Autom. Control*, 54(2), 353–357.

Chopra, N. and Spong, M. (2006). Passivity-based control of multi-agent systems. In S. Kawamura and M. Svinin (eds.), *Advances in Robot Control:*. Springer-Verlag, Berlin.

Fax, J. and Murray, R. (2004). Information flow and cooperative control of vehicle formations. *IEEE Trans. Automatic Control*, 49(9), 1465–1476.

Fiedler, M. (1973). Algebraic connectivity of graphs. *Czech. Math. Journal*, 23, 298–305.

Fradkov, A. (2003). Passification of non-square linear systems. *European J. of Control*, (6), 573–581.

Gelig, A., Leonov, G., and Yakubovich, V. (2004). *Stability of stationary sets in control systems with discontinuous nonlinearities*. World Scientific Publishing Co.

Khalil, H. (1996). *Nonlinear systems*. Prentice-Hall, Englewood Cliffs, NJ.

Lin, J., Morse, A., and Anderson, B. (2007a). The multi-agent rendezvous problem (2 parts). *SIAM Journal Control and Optimization*, 46(6), 2096–2147.

Lin, Z., Francis, B., and Maggiore, M. (2007b). State agreement for continuous-time coupled nonlinear systems. *SIAM Journ. of Control and Optimization*, 46(1), 288–307.

Merris, R. (1994). Laplacian matrices of graphs: A survey. *Linear Algebra and its Applications*, 197, 143–176.

Moreau, L. (2005). Stability of multiagent systems with time-dependent communication links. *IEEE Trans. Autom. Control*, 50(2), 169–182.

Olfati-Saber, R., Fax, J., and Murray, R. (2007). Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1), 215–233.

Proskurnikov, A. (2013a). Consensus in switching networks with sectorial nonlinear couplings: Absolute stability approach. *Automatica*, 49(2), 488–495.

Proskurnikov, A. (2013b). Nonlinear consensus algorithms with uncertain couplings. *Asian J. of Control (to appear)*.

Proskurnikov, A. (2011). Consensus in symmetric multi-agent networks with sector nonlinear couplings. In *Proceedings of 18-th IFAC Congress*, 8945–8950. Milano, Italy.

Ren, W. (2008). Synchronization of coupled harmonic oscillators with local interaction. *Automatica*, 44, 3195–3200.

Ren, W. and Beard, R. (2008). *Distributed consensus in multi-vehicle cooperative control: theory and applications*. Springer.

Scardovi, L. and Sepulchre, R. (2009). Synchronization in networks of identical linear systems. *Automatica*, 45, 2557–2562.

Seo, J., Shima, H., and Back, J. (2009). Consensus of high-order linear systems using dynamic output feedback compensator: Low gain approach. *Automatica*, 45, 3195–3200.

Strogatz, S. (2000). From Kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled oscillators. *Physica*, D(143), 648–651.

Su, H., Wang, X., and Lin, Z. (2009). Synchronization of coupled harmonic oscillators in a dynamic proximity network. *Automatica*, 45, 2286–2291.

Tanner, H., Jadbabaie, A., and Pappas, G. (2007). Flocking in fixed and switching networks. *IEEE Trans. Autom. Control*, 52(5), 863–868.

Wieland, P., Sepulchre, R., and Allgöwer, F. (2011). An internal model principle is necessary and sufficient for linear output synchronization. *Automatica*, 47, 1068–1074.

Wu, C. (2007). *Synchronization in complex networks of nonlinear dynamical systems*. World Scientific, Singapore.

