The Popov Criterion For Consensus Between Delayed Agents ⋆

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Abstract: We consider consensus algorithms for multi-agent networks with high-order and delayed dynamics of agents. The topology is assumed to be fixed and undirected, however the couplings may be nonlinear and uncertain, we assume only the symmetry condition to be valid. We obtain conditions of stability for such algorithms that are similar in spirit to the celebrated Popov criterion for the stability of Lurie systems.

Keywords: Networks, synchronization, consensus, nonlinear systems, absolute stability.

1. INTRODUCTION.

Controlled *consensus* or synchronism achieved via local interactions between subsystems (or *agents*) of a complex system lies at the heart of numerous natural phenomena and engineering solutions. Examples include, but are not limited to, flocking, swarming, and other forms or regular motion of biological or technical systems [Olfati-Saber et al. 2007, Ren and Beard 2008, Ren and Cao 2011], synchronization in complex networks Wu [2007] etc. The core ideas of many consensus policies take their origin from distributed algorithms in computer science and decision making procedures in expert communities developed in applied statistics.

Nowadays consensus protocols with linear couplings are widely and deeply investigated (see e.g.,[Olfati-Saber et al. 2007, Ren and Beard 2008, Ren and Cao 2011, Wieland et al. 2011] and references therein). For the first and second order agents, this research typically relies on Lyapunov arguments (including LMI approach) and various results on convergence of infinite products of stochastic matrices; in some cases, linear networks with high-order agents may be reduced to first-order systems via decomposition of the Laplacian matrix Fax and Murray [2004] or using special dynamic controllers Wieland et al. [2011]. However, many applications involve synchronization via nonlinear couplings. For example, this holds for networks of various oscillators, e.g., Kuramoto networks Chopra and M.W.Spong [2009], where agents are typically coupled by means of periodic functions. Nonlinear couplings naturally arise in motion coordination under range-restricted communication in order to maintain the group connectivity Lin et al. [2007a], Tanner et al. [2007], Su et al. [2009]. In a practical setting, linear algorithms may acquire nonlinearities because of distortions caused by saturations, imprecise measurements, analog-to-digital transformations, quantization effects etc.

At the same time, recent intensive interest to nonlinear consensus theory was mainly focused on low-order agents such as first-order and second-order integrators [Moreau 2005, Lin et al. 2007b, Ajorlou et al. 2011, Lin et al. 2007a, Abdessameud and Tayebi 2010, Ren and Beard 2008, Ren and Cao 2011] or passive agents [Chopra and Spong 2006, Arcak 2007].

In recent paper [Proskurnikov 2013] a wide class of nonlinear consensus algorithms for linear agents of arbitrary order was considered. The couplings were assumed to satisfy a symmetry condition resembling the Newton's Third Law and conventional *sector inequalities* with known slopes Gelig et al. [2004], and the consensus criterion was given in terms of agents' transfer matrices, sector slopes and properties of topology but not the couplings themselves, which thus may be uncertain. This criterion in fact ensures that consensus is not only established by also robust in the class of uncertain couplings, bounded by a sector with known slopes. Intended for investigation of time-variant networks, the consensus criterion from [Proskurnikov 2013] may be rather conservative for networks with fixed topology and time-invariant couplings. For such a network, however, a serious improvement of this criterion may be obtained, which is the main result of the paper. While the criterion from [Proskurnikov 2013] is close in spirit to the circle criterion in the absolute stability theory, our criterion is analogous to the well-known *Popov criterion* Popov [1973]. For the sake of brevity and to simplify matters, we consider the case of discrete delays in the agent model, more general models with distributed delays may be treated in the same way.

The outline of the paper is as follows. Section 2 introduces some necessary concepts of the graph theory and auxiliary notations. The problem statement and main assumptions are given in Section 3. Section 4 presents the main results of the paper, which are illustrated by applying to agents of special type in Section 5.

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2. PRELIMINARIES AND NOTATIONS

Throughout the paper $\underline{N}$ stands for the set $\{1, 2, \ldots, N\}$. We put $\mathbb{R}_+ := [0; +\infty)$ and $\mathbb{C}_+ := \{z \in \mathbb{C} : Re\, z \geq 0\}$. For a matrix $A \in \mathbb{C}^{m\times n}$ we denote with $A^* \in \mathbb{C}^{n\times m}$ its complex-conjugate transpose, for $a \in \mathbb{C}$ we have $a^* = \bar{a}$.

The components of a vector $u \in \mathbb{R}^m$ are denoted with upper indices, thus $u = (u^1, \ldots, u^m)^T$. Given several column vectors $u_1, \ldots, u_N$, their column union is denoted with $col(u_1, \ldots, u_N)$. The symbol $diag(A_1, \ldots, A_N)$ denotes the block-diagonal matrix composed of square matrices A_j.

A *weighted graph* is a triple $\mathcal{G} = [V, E, \mathcal{W}(\cdot)]$ consisting of finite sets V and $E \subset V \times V$, which are referred respectively to as the set of *nodes* and *arcs*, and a "weighting" map $\mathcal{W}(\cdot) : E \to (0; +\infty)$. A node v is connected to a node w if $(v, w) \in E$, the sequence of nodes $v_1, \ldots, v_k$ with $(v_i, v_{i+1}) \in E\ \forall i$ is called the *path* between v_1 and v_k. The graph is *connected* if a path between any two nodes exists. Throughout the paper we deal only with *undirected* weighted graphs, which means that $(v, v') \in E \Rightarrow (v', v) \in E\,\&\,\mathcal{W}(v, v') = \mathcal{W}(v', v)$, that contain no self-loops: $(v, v) \notin E$. The symbol $\mathbb{G}_N$ stands for the class of all such graphs $\mathcal{G} = (\underline{N}, E, \mathcal{W})$ with the node set $\underline{N}$.

The graph $\mathcal{G} \in \mathbb{G}_N$ may be identified with its *adjacency* matrix $G = (g_{jk})_{j,k=1}^N$, where $g_{jk} := \mathcal{W}(j, k)$ if j and k are connected and $g_{jk} := 0$ otherwise. The properties of the graph $\mathcal{G}$ are closely related to the structure of its *Laplacian*

$$L[\mathcal{G}] := diag(D_1, \ldots, D_N) - G, \text{ where } D_j := \sum_{k=1}^N g_{jk}. \quad (1)$$

The second term in the ascending sequence of eigenvalues $\lambda_1(\mathcal{G}) = 0 \leq \lambda_2(\mathcal{G}) \leq \ldots \leq \lambda_N(\mathcal{G})$ of the symmetric matrix $L[\mathcal{G}] \geq 0$ is called *the algebraic connectivity* of the graph and may be alternatively defined [Fiedler 1973] by

$$\lambda_2(\mathcal{G}) = N \min_{z\in\Upsilon} \frac{\sum_{j,k=1}^N g_{jk}(z_k - z_j)^2}{\sum_{j,k=1}^N (z_k - z_j)^2} \quad (2)$$

where $\Upsilon := \left\{z \in \mathbb{R}^N : z_k \neq z_j \text{ for some } j, k\right\}$. The graph $\mathcal{G}$ is connected if and only if $\lambda_2(\mathcal{G}) > 0$ [Fiedler 1973, Olfati-Saber et al. 2007, Ren and Beard 2008].

3. PROBLEM STATEMENT

Consider a group of agents indexed 1 through N and governed by equations

$$\begin{aligned} \dot{x}_j(t) &= \sum_{l=0}^M \left[A_{lj} x_j(t - \tau_l) + B_{lj} u_j(t - \tau_l)\right], \\ y_j(t) &= \sum_{l=0}^M C_{lj} x_j(t - \tau_l),\ t \in \mathbb{R}_+ \quad j \in \underline{N}. \end{aligned} \quad (3)$$

Here $0 = \tau_0 < \tau_1 < \ldots < \tau_M$ are constant delays, the vectors $x_j(t) \in \mathbb{R}^{n_j}$, $u_j(t) \in \mathbb{R}^m$ and $y_j(t) \in \mathbb{R}^m$ stand respectively for the state, control and output of the j-th agent. We suppose the initial data to satisfy the condition

$$|x_j(\cdot)|, |u_j(\cdot)| \in L_\infty[-\tau_M; 0]\,\forall j \in \underline{N}. \quad (4)$$

The inputs u_j are affected by interactions among the agents, via communication, sensoring or otherwise. The interaction topology is assumed to be fixed, undirected and described by a weighted graph $\mathcal{G} \in \mathbb{G}_N$, the j-th agent has access to the output $y_k(t)$ if j and k are connected. We examine the following distributed protocols

$$u_j^q(t) = \sum_{k=1}^N g_{jk} \varphi_{jk}^q (y_k^q(t) - y_j^q(t)),\ j \in \underline{N}, q \in \underline{m}. \quad (5)$$

Therefore each scalar input u_j^q depends only on the scalar outputs y_k^q, however, we do not assume those outputs to be necessarily independent (it is possible e.g. that $y_k^1 \equiv y_k^2 \equiv \ldots \equiv y_k^m$), thus actual dimension of the output may be less than the number of control inputs. Here $G = (g_{jk}) = G^T$ stands for the adjacency matrix of the weighted graph $\mathcal{G}$ and the mappings $\varphi_{jk}^q : \mathbb{R} \to \mathbb{R}$ are called *couplings*.

Our goal is to disclose conditions under which the protocol (5) provides the consensus in one of the following senses.

Definition 1. The protocol (5) establishes *asymptotic consensus* if $y_k(t) - y_j(t) \to 0$ as $t \to +\infty$ for any j, k. It establishes L^2*-consensus* if $\int_0^\infty |y_k(t) - y_j(t)|^2\, dt < \infty\,\forall j, k$ and $\int_0^\infty |u_j(t)|^2\, dt < \infty$.

In general, L^2-consensus does not imply asymptotic consensus, and vice versa. However, asymptotic consensus follows from L^2-consensus if the agents have *common unstable dynamics* in the following sense.

Lemma 2. Suppose the functions $\chi_j(z) := \det\left(zI_{n_j} - \sum_{l=1}^M A_{lj} e^{-z\tau_l}\right)$ to have identical sets of zeros in $\mathbb{C}_+$ (taking the multiplicity into account). Then L^2-consensus in the network (3),(5) implies asymptotic consensus.

Idea of the proof. Evidently, it is sufficient to show that $\dot{y}_j - \dot{y}_k \in L_2$ for any j, k. If the agents are stable, the latter statement is true since $x_j \in L_2$ and therefore $\dot{y}_j \in L_2$ due to (3). Otherwise, for any j we have $\chi_j(z) = \chi_*(z)\tilde{\chi}_j(z)$, where $\chi_*(z)$ is unstable polynomial whose zeros are common unstable poles of the agents and the function $\tilde{\chi}_j(z)$ is analytic at any $z \in \mathbb{C}_+$. Taking arbitrary Hurwitz polynomial $\rho(z)$ with $\deg\rho = \deg\chi_* - 1$, we have $W_j(z) = \frac{\rho(z)}{\chi_*(z)}\tilde{W}_j(z)$, where $\tilde{W}_j \in H_\infty$ and hence

$$\chi_*\left(\frac{d}{dt}\right) y_j(t) = \rho\left(\frac{d}{dt}\right) \tilde{u}_j(t), \quad \tilde{u}_j(t) \in L_2[0; \infty]. \quad (6)$$

Since (6) is equivalent to a controllable and observable state-space model, one easily sees that $y_j - y_k \in L_2$ and $u_j - u_k \in L_2$ imply $\dot{y}_j - \dot{y}_k \in L_2$ which ends the proof.□

In this paper we are interested primarily in the case when the couplings φ_{jk}^q are nonlinear and the full information about them is not available. We only assume them to be odd and satisfy a stronger version of the *sector condition* [Gelig et al. 2004, Khalil 1996]. Precisely, we suppose that $\varphi_{jk}^q \in \mathfrak{S}[\alpha; \beta]$, where $0 \leq \alpha < \beta < \infty$ and the class $\mathfrak{S}[\alpha; \beta]$ consists of odd and continuous functions $\varphi(\cdot)$, such that

$$\alpha < \frac{\varphi(y)}{y} < \beta\ \forall y \neq 0, \quad \varliminf_{y\to 0} \frac{\varphi(y)}{y} > 0,\ \varliminf_{y\to\infty} \varphi(y) > 0. \quad (7)$$

(the second and third conditions automatically hold if $\alpha > 0$). The first inequality in (7) implies that the graph of the function $\varphi = \varphi(y)$ lies strictly in the sector between the lines $\varphi = \alpha y$ and $\varphi = \beta y$ everywhere except the origin, while the second and third one prohibit too fast decreasing of the coupling near zero and infinity points.

Our consensus conditions may be also considered as criteria for *robust* consensus, because they imply consensus *for any* couplings of the described type, that satisfy our last *symmetry* assumption:

Assumption 3. For any $j \neq k$ one has $\varphi_{jk}(y) = \varphi_{kj}(y)$.

Combined with the suppositions that $\mathcal{G} \in \mathbb{G}_N$ and the couplings φ^d_{jk} are odd functions, Assumption 3 implies that $g_{jk}\varphi_{jk}(y_k - y_j) = -g_{kj}\varphi_{kj}(y_j - y_k)$, i.e. the interactions between the agents satisfy "The Newton Third Law". Actually, in the case of physical coupling (e.g. the Kuramoto networks [Strogatz 2000]) the mentioned assumptions hold due to exactly this law.

4. MAIN RESULT: POPOV-LIKE CONSENSUS CRITERION.

To start with, we investigate the case when consensus may be achieved for "arbitrarily weak" couplings $\varphi^q_{jk} \in \mathfrak{S}[0;\beta]$ with some fixed $\beta \in (0;\infty)$ to which case the Subsection 4.1 is devoted. More complicated situation of exponentially unstable agents will be considered in Subsection 4.2.

4.1 Consensus with weak couplings

It looks natural that weak couplings can hardly provide to consensus in networks where the agents can not be stabilized with a weak feedback, so we adopt the following.

Assumption 4. For any $j \in \underline{N}$ there exists $\varepsilon^*_j > 0$ such that the feedback $u_j = -\varepsilon y_j$ stabilizes the j-th agent whenever $\varepsilon \in (0;\varepsilon^*_j)$. In other words, for $\varepsilon \in (0,\varepsilon^*_j)$ one has $\det\big(zI_{n_j} - \hat{A}_j(z) + \varepsilon\hat{B}_j(z)\hat{C}_j(z)\big) \neq 0 \forall z \in \mathbb{C}_+$.

Here and throughout the paper $\hat{A}_j(z) := \sum_l A_{lj}e^{-z\tau_l}$, $\hat{B}_j(z) := \sum_l B_{lj}e^{-z\tau_l}, \hat{C}_j(z) := \sum_l C_{lj}e^{-z\tau_l}$.

Assumption 4 becomes necessary for robust in $\mathfrak{S}[0;\beta]$ with $\beta > 0$ consensus if the agents are identical.

Remark 5. Let the agents be identical ($A_{lj} \equiv A_l$, $B_{lj} \equiv B_l$, $C_{lj} = C_l$) and $g_{jk} \neq 0$ for some j,k. Suppose that the protocol (5) with linear couplings $\varphi^q_{jk}(y) := \varkappa y$ establishes the asymptotic or L^2-consensus for any $\varkappa \in (0;\beta)$. Then Assumption 4 holds with $\varepsilon^*_j = \varepsilon^* := \beta\lambda_N(\mathcal{G})$.

Indeed, slightly adapting the arguments from [Fax and Murray 2004, Theorem 3] to our case, we obtain that a feedback $u = -\varkappa y$ with $\varkappa \in (0;\beta)$ must stabilize plants

$$\begin{aligned}\dot{x}(t) &= \sum_l A_l x(t-\tau_l) + \lambda B_l u(t-\tau_l),\\ y(t) &= \sum_l A_l x(t-\tau_l),\ \lambda \in \{\lambda_2(\mathcal{G}),\dots,\lambda_N(\mathcal{G})\}.\end{aligned} \quad (8)$$

Since $\lambda_N(\mathcal{G}) > 0$, Assumption 4 holds.

To proceed, we introduce the agents' transfer matrices

$$W_j(z) := \hat{C}_j(z)\big(zI_{n_j} - \hat{A}_j(z)\big)^{-1}\hat{B}_j(z),\ j \in \underline{N}.$$

The following gives a sufficient conditions for consensus.

Theorem 6. Suppose that $\varphi^q_{jk} \in \mathfrak{S}[0;\beta]$, Assumptions 3,4 hold and the graph $\mathcal{G} \in \mathbb{G}_N$ is connected. Let diagonal matrices $\mathfrak{P} = diag(\mathfrak{p}_1,\dots,\mathfrak{p}_m)$, $\mathfrak{Q} = diag(\mathfrak{q}_1,\dots,\mathfrak{q}_m)$ with $\mathfrak{p}_k \neq 0, \mathfrak{q}_k > 0$ exist such that

$$(\mathfrak{Q} + \imath\omega\mathfrak{P})W_j(\imath\omega) + W_j(\imath\omega)^*(\mathfrak{Q} - \imath\omega\mathfrak{P}) + \frac{1}{\beta D_j}\mathfrak{Q} \geq 0 \quad (9)$$

for any $j \in \underline{N}$ and $\omega \in \mathbb{R}$, such that $W_j(\imath\omega)$ is well defined. Here D_j are defined by (1). Then the protocol (5) establishes L_2-consensus. If the agents have common unstable dynamics in the sense of Lemma 2 it also establishes asymptotic consensus.

The proof of Theorem 6 may be obtained in the way analogous to that from [Proskurnikov 2011], it will appear in extended version of this paper [Proskurnikov 2014] and is available upon request.

Throughout the remainder of this section we make several remarks concerning the use of this result and its relation to the classical Popov criterion.

Remark 7. The frequency-domain inequality (9) may be reformulated as follows: $F_j(\imath\omega;\tilde{u}) \geq 0$ for any $\tilde{u} \in \mathbb{C}^m$, where a Hermitian form $F_j(\imath\omega;\cdot)$ is defined as follows

$$F_j(i\omega;\tilde{u}) = Re\,\tilde{u}^*\left[\mathfrak{Q} + \imath\omega\mathfrak{P}\right]W(\imath\omega)\tilde{u} + (2\beta D_j)^{-1}\tilde{u}^*\mathfrak{Q}\tilde{u}.$$

Remark 8. If $y_j, u_j \in \mathbb{R}$ ($m = 1$), then the inequality (9) reduces to the conventional Popov inequality (after dividing by $\mathfrak{q} > 0$ and redesignating $\mathfrak{p}/\mathfrak{q}$ with $\mathfrak{p}$):

$$Re\left[W_j(\imath\omega) + \mathfrak{p}\imath\omega W_j(\imath\omega)\right] + \mu_j^{-1} \geq 0,\ \mu_j := 2\beta D_j. \quad (10)$$

In the scalar case, discussed in the latter Remark 8, the relation between the classical Popov criterion and Theorem 6 may be further illustrated by considering a system of two ($N = 2$) identical linear SISO agents

$$\dot{x}_1 = Ax_1 + Bu_1, y_1 = Cx_1; \quad \dot{x}_2 = Ax_2 + Bu_2, y_2 = Cx_2,$$

interconnected by means of the "protocol" $u_1(t) = \varphi(y_2(t) - y_1(t))$, $u_2(t) = \varphi(y_1(t) - y_2(t))$. Here $\varphi(\cdot) \in S[0;\beta]$ and we assume for simplicity the agents to be controllable and observable linear systems. Introducing auxiliary functions $X(t) := x_2(t) - x_1(t)$, $Y(t) := y_2(t) - y_1(t)$, $\Phi(y) := -2\varphi(y)$, one comes to a Lurie system

$$\dot{X}(t) = Ax(t) - B\Phi(Y(t)), Y(t) = CX(t) \quad (11)$$

with a nonlinearity $\Phi \in S[0;\mu]$, $\mu := 2\beta$. Assumption 4 says that for $\Phi(y) = \varepsilon y$ with ε sufficiently small the system (11) is stable (since $Y(t) \to 0$ we have $X(t) \to 0$ due to controllability and observability). Applying the Popov criterion [Gelig et al. 2004, Khalil 1996], one easily shows that the inequality (10) (applied for $D_1 = D_2 = 1$) guarantees the stability of the Lurie system (11) (and the asymptotic consensus $Y(t) \to 0$) for any $\Phi \in \mathfrak{S}[0;\mu]$.

5. CONSENSUS AMONG UNSTABLE AGENTS

In this section we discuss the situation where Assumption 4 may be violated, for instance, the agents (3) are exponentially unstable. Synchronization in such a network requires sufficiently strong couplings hence one can not expect robust consensus in the class of nonlinear couplings $\mathfrak{S}[0;\beta]$, but only in $S[\alpha;\beta]$ with $\alpha > 0$ sufficiently large. Unlike Theorem 6, the result below which addresses this case (Theorem 10) is applicable only for identical agents, and its extension to heterogeneous agents remains a nontrivial open problem.

Throughout this section the agents are identical so that $A_j \equiv A, B_j \equiv B, C_j \equiv C$. The consensus for any couplings from $S[\alpha;\beta]$ certainly can not be proved if is not reached for linear couplings $\varphi_{jk}^q(y) := \varepsilon y$ where $\varepsilon \in (\alpha;\beta)$, or, equivalently, the feedback $u = -\varepsilon y$ stabilizes each of the systems (8) for any $\varepsilon \in (\alpha;\beta)$ (this condition is also sufficient, if the graph $\mathcal{G}$ is connected). In the following, we assume a formally weaker condition to be valid:

Assumption 9. There exists $\mu \in (\alpha;\beta)$ such that $\det(zI_n - \hat{A}(z) - \mu\lambda_j(\mathcal{G})\hat{B}(z)\hat{C}(z)) \neq 0\, \forall z \in \mathbb{C}_+, j = 2,\dots,N$.

We also introduce two constants

$$\delta := \frac{\alpha\beta}{\alpha+\beta},\ \gamma := \frac{1}{\alpha+\beta}. \tag{12}$$

The following theorem gives a sufficient condition for consensus among identical agents.

Theorem 10. Suppose that the agents are identical, $\varphi_{jk}^q \in \mathfrak{S}[\alpha;\beta]$ with $\alpha > 0$, Assumptions 3,9 hold, the graph $\mathcal{G} \in \mathbb{G}_N$ is connected and there exist diagonal matrices $\mathfrak{P} = diag(\mathfrak{p}_1,\dots,\mathfrak{p}_m)$, $\mathfrak{Q} = diag(\mathfrak{q}_1,\dots,\mathfrak{q}_m)$ with $\mathfrak{p}_j \neq 0, \mathfrak{q}_j > 0$ such that

$$\begin{gathered}(\mathfrak{Q} + \imath\omega\mathfrak{P})W(\imath\omega) + W(\imath\omega)^*(\mathfrak{Q} - \imath\omega\mathfrak{P}) + \\ +\delta\lambda_2(\mathcal{G})W(\imath\omega)^*\mathfrak{Q}W(\imath\omega) + \frac{\gamma}{2D_{max}}I_m \geq 0,\end{gathered} \tag{13}$$

where $D_{max} := \max D_k$ and D_k are defined by (1). Then the protocol (5) establishes asymptotic consensus.

The proof of Theorem 6 may be obtained in the way analogous to that from [Proskurnikov 2011], it will appear in extended version of this paper [Proskurnikov 2014] and is available upon request.

Remark 11. Unlike the inequalities (9) that are fully "decentralized" in the sense that the j-th inequality involves only properties of the j-th node in the network such as transfer matrix W_j and a "weighted degree" D_j, the inequality (13) involves some global information about the network, namely, the algebraic connectivity $\lambda_2(\mathcal{G})$ and the maximal degree D_{max}. However, those quantities are multiplied by non-negatively definite matrices, and the inequality remains sufficient for consensus, replacing $\lambda_2(\mathcal{G})$ with its lower and D_{max} with its upper bound (e.g. $D_{max} \leq (N-1)\max_{j,k} w_{jk}$). A lot of non-conservative estimates for λ_2 coming from the algebraic graph theory may be put in use [Fiedler 1973, Merris 1994] in the case when its precise computation is troublesome.

6. EXAMPLES

We now illustrate the potential of Theorems 6 and 10 by considering several types of SISO agents.

6.1 Consensus in heterogeneous network of first-order and second-order delayed agents.

Despite the consensus problems for first-order agents have been deeply investigated during recent decade, the effects caused by input delays seem to be explored only for special situations. The most significant progress in this area was achieved by using the Lyapunov-Krasovskii method, which comes to non-trivial and high-dimensional LMIs [Sun and Wang 2009, Lin and Jia 2011] and various frequency-domain techniques [Tian and Liu 2008, Bliman and Ferrari-Trecate 2008, Muënz et al. 2010, Lestas and Vinnicombe 2010]. However, all of those results address the case of linear networks, while the effects of self-actuation delays in nonlinear consensus protocols remain almost unexplored (unlike the case of so-called *communication delay* which affects only transmitted data but not the own state of the agent [U.Muënz et al. 2011]). The foregoing is also applicable to higher-order agents, where only protocols with linear delayed couplings seem to be investigated [Lin and Jia 2010, Qin et al. 2011].

In the present section we give sufficient criteria for consensus in heterogeneous network of first and second-order nonlinearly-coupled delayed agents, which extend previously obtained results for identical first-order integrators [Proskurnikov 2011]. Analogous networks for the discrete-time case were considered in [Liu and Liu 2011], as for the continuous-time case, only undelayed agents and protocols seem to be considered [Zheng and Wang 2012].

We consider a team of N_1 first-order agents

$$\dot{y}_j(t) = \varkappa_j u_j(t-\tau_j) \in \mathbb{R},\ j \in \underline{N_1} \tag{14}$$

and N_2 second-order counterparts as follows

$$\ddot{y}_j(t) + l_j\dot{y}_j(t) = \varkappa_j u_j(t-\tau_j) \in \mathbb{R}, j = N_1+1,\dots,N_2. \tag{15}$$

Here $\tau_j \geq 0,\ k_j, l_j > 0$. Our goal is disclose conditions under which the protocol (5) with $\varphi_{jk}^1 \in \mathfrak{S}[0;\beta]$ establishes asymptotic consensus. Note that an agent (15) which applies the control strategy (5) is equivalent to a double integrator agent $\ddot{y}_j = w_j$ which has direct access to its velocity but delayed measurements of the position, and its control input is taken in the form $w_j = -l_j\dot{y}_j + \varkappa_j \sum_{k=1}^N g_{jk}\varphi_{jk}(y_k(t-\tau_j) - y_j(t-\tau_j))$.

To apply Theorem 6, one has to verify the frequency-domain inequality (10) for each of the agents. The following lemmas gives conditions for its validity in the case of agents (14) and (15).

Lemma 12. For existense of a number $\mathfrak{p} \neq 0$ such that (10) holds for the agent (14) it is necessary and sufficient that

$$\tau_j \leq \frac{\pi}{4\varkappa_j\beta D_j}. \tag{16}$$

Proof. For $\tau_j = 0$ (10) holds for any small $\mathfrak{p} \neq 0$, so we focus on the case when $\tau_j > 0$. Since $Re\, W_j(\imath\omega) = -\varkappa_j \sin\omega\tau_j/\omega$ and $Re\, \imath\omega W_j(\imath\omega) = \varkappa_j\cos\omega\tau_j$, we have

$$Re\,(W_j(\imath\omega) + \mathfrak{p}\imath\omega W_j) = -\varkappa_j\tau_j\rho\big(\omega\tau_j, \mathfrak{p}/\tau_j\big), \text{ where}$$

$$\rho(\omega, A) := \frac{\sin\omega}{\omega} - A\cos\omega, \rho(0,A) := -A.$$

It was shown in [Proskurnikov 2011] that $\max_{\omega\in\mathbb{R}}\rho(\omega,A) \geq \frac{2}{\pi}$ and equality is achieved for $A = 4/\pi^2$. Thus (10) holds for some $\mathfrak{p} \neq 0$ if and only if $2\varkappa_j\tau_j/\pi \leq (2\beta D_j)^{-1}$.

Lemma 13. For existense of a number $\mathfrak{p} \neq 0$ such that (10) holds for the agent (15) it is sufficient that

$$\tau_j \leq \frac{l_j}{2\varkappa_j\beta D_j}. \tag{17}$$

Proof. A direct computation shows that ,

$$Re\, W_j = \varkappa_j\, Re\, \frac{e^{-\imath\omega\tau_j}}{(\imath\omega)(\imath\omega + l_j)} = -\varkappa_j \frac{\omega \cos\omega\tau_j + l_j \sin\omega\tau_j}{\omega(l_j^2 + \omega^2)},$$

$$Re\, \imath\omega W_j = \varkappa_j\, Re\, \frac{e^{-\imath\omega\tau_j}}{\imath\omega + l_j} = \varkappa_j \frac{l_j \cos\omega\tau_j - \omega_j \sin\omega\tau_j}{l_j^2 + \omega^2},$$

$$Re\, W_j(1 + \mathfrak{p}\imath\omega) = \varkappa_j \frac{\mathfrak{p}_j l_j - 1}{l_j^2 + \omega^2} \cos\omega\tau_j - \varkappa_j \frac{l_j + \mathfrak{p}_j \omega^2}{l_j^2 + \omega^2} \frac{\sin\omega\tau_j}{\omega}$$

By noticing that $\sin\omega/\omega \leq 1$ for any $\omega \in \mathbb{R}$, one obtains that (10) is fulfilled with $\mathfrak{p}_j := l_j^{-1}$ if $\varkappa_j\tau_j/l_j < (2\beta D_j)^{-1}$.

Since our agents satisfy Assumption 4, Theorem (6) entails the following corollary.

Corollary 14. Suppose that $\varphi_{jk}^1 \in \mathfrak{S}[0;\beta]$, Assumption 3 holds and the graph $\mathcal{G} \in \mathbb{G}_N$ is connected. If each of the agents (14),(15) satisfy the inequalities (16) and (17) respectively, the protocol (5) establishes asymptotic consensus.

6.2 *Consensus among unstable first-order agents*

In this subsection we demonstrate the use of Theorem 10 for unstable agents. We consider a team of identical agents

$$\dot{y}_j - a y_j = u_j(t - \tau) \in \mathbb{R}, \tag{18}$$

where $a > 0$ and $\tau \geq 0$. We are interested in finding a criteria for consensus in the networked system (18),5 where $\varphi_{jk}^1 \in \mathfrak{S}[\alpha;\beta]$, $\beta > \alpha > 0$.

To start with, we determine conditions under which Assumption 9 holds. We need the following lemma, which may be derived from [Hale 1977, Appendix, Theorem A.5] so its proof is omitted here.

Lemma 15. The feedback $u_j = -K y_j$ stabilizes the agent (18) if and only if $K > a$ and

$$\frac{\arccos \frac{a}{K}}{\sqrt{K^2 - a^2}} < \tau. \tag{19}$$

It may be shown that the left-hand side of (19) monotonely increases when $K > a$ so (19) is satisfied in and only if $K > K_*(a,\tau)$. Assumption 9 may be formulated as follows: $\lambda_2(\mathcal{G}) > K_*(a,\tau)/\alpha$.

The following result gives a sufficient condition for consensus among the agents (18).

Corollary 16. Suppose that $\varphi_{jk}^q \in \mathfrak{S}[\alpha;\beta]$ with $\alpha > 0$, Assumption 3 holds, the graph $\mathcal{G} \in \mathbb{G}_N$ is connected, moreover, $\lambda_2(\mathcal{G}) > K_*(a,\tau)/\alpha$ and $2(\alpha+\beta)D_{max} < a$, where $D_{max} := \max D_k$ and D_k are defined by (1). Then the protocol (5) establishes asymptotic consensus.

Remark 17. The conditions of Corollary 16 require the coupling weights g_{jk} to be sufficiently large (lower bound for $\lambda_2(\mathcal{G})$) to meet Assumption 9 and sufficiently small to satisfy the frequency-domain inequality (upper bound for D_{max}). It is evidently possible to satisfy both requirements for $\tau = 0$ and thus for all sufficiently small τ, however, to get analytic bound for maximal possible τ in terms of a, α and β seems to be a non-trivial problem.

Proof We notice that $W(z) = e^{-z\tau}/(z - a)$ and thus

$$Re\, W(\imath\omega) = Re\, \frac{e^{-\imath\omega\tau_j}}{\imath\omega - a} = -\frac{a\cos\omega\tau + \omega\sin\omega\tau}{a^2 + \omega^2}$$

$$Re\, \imath\omega W(\imath\omega) = \frac{\omega^2 \cos\omega\tau - a\omega\sin\omega\tau}{a^2 + \omega^2}$$

$$Re\, W(1 + \mathfrak{P}\imath\omega) = \frac{\mathfrak{P}\omega^2 - a}{a^2 + \omega^2}\cos\omega\tau - \frac{(1 + a\mathfrak{P})\omega}{a^2 + \omega^2}\sin\omega\tau$$

In particular, taking $\mathfrak{P} := -a^{-1}$, one obtains that if $2(\alpha+\beta)D_{max} < a$, then $Re\, W(1 + \mathfrak{p}\imath\omega) + \gamma/(2D_{max}) = (2(\alpha+\beta)D_{max})^{-1} - a\cos\omega\tau \geq 0$ which implies (13) with $\mathfrak{Q} = 1$.

7. CONCLUSION

We address the problem of synchronization (consensus) in multi-agent networks with fixed undirected topology and nonlinear couplings which may be uncertain and assumed only to satisfy conventional sector inequalities. The agents are assumed to obey linear stationary delay equations of arbitrary order and may be either heterogeneous without strictly unstable poles or identical. We offer easily verifiable synchronization criteria, based on the Popov method from absolute stability theory and close in spirit to circle and Popov stability criteria.

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