A cluster control of nonlinear network systems with external inputs

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Abstract: In this paper, we consider the synchronization problem in networks of identical nonlinear systems with delayed couplings and external inputs. We show that the existence of external inputs can generate partial synchronizations in networks, and the synchronization patterns based on the notion of equitable partitions introduced in graph theory under sufficiently large coupling strength and sufficiently small time-delays. Two illustrated examples with numerical simulations are given to show the validity of the obtained results. The results indicate that synchronization patterns can be controlled by applying external inputs.

Keywords: Clustering; Synchronization; Nonlinear systems; Network systems; Time delays

1. INTRODUCTION

Recently, large scale network systems attract a great deal of attention in many research fields including applied physics, mathematical biology, social sciences, control theory and interdisciplinary fields. The behaviors of large scale network systems are determined by the interaction of a number of subsystems, it is not so easy to analyze the behavior of each subsystem and control all subsystems in networks. Therefore it is highly important to establish clustering techniques and model reduction methods for large scale networks. Clustering techniques and model reduction methods provide us smaller order dynamics than original large scale network systems.

On the other hand, synchronization in networks of coupled systems is an interesting phenomenon, and we can observe various synchronization patterns such as partial synchronization and full synchronization in networks. By identifying the synchronized systems with the identical system, it may be possible to reduce the number of subsystems in networks. Concerning chaos network systems with delay couplings, Mimura et al. [2011] and Steur et al. [2012] have proposed an estimation method of the synchronization conditions by using a scaling law, independently.

In this paper, we propose a clustering method based on synchronization of nonlinear network systems with delay couplings. The proposed method is a combination of a full synchronization condition for delay network systems and a notion of equitable partitions of graph theory. The notion of equitable partitions has been already introduced for a clustering of integrator network systems with a single external input by Martini et al. [2008]. After that, Yazicioglu [2012] extended the result for multiple external inputs systems. In this paper, we extend these techniques for nonlinear network systems with delay couplings and multiple external inputs by combining the notion of synchronization in networks.

The following sections are organized as follows. In Section 2, we introduce network systems to be considered in this paper and show the boundedness of the solutions under the assumption that each system is semipassive. Section 3 reviews an estimation method of the synchronization condition for nonlinear network systems without external inputs. In Section 4, we propose a clustering method of nonlinear network systems with external inputs. The proposed result shows that the pattern of partial synchronization depends on the maximal equitable partition. Section 5 shows illustrative examples with numerical simulations to verify the validity of the derived result.

2. NONLINEAR NETWORK SYSTEMS WITH EXTERNAL INPUTS

2.1 System description

We consider the following N identical nonlinear systems:

$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) + Bu_i(t) \\ y_i(t) = Cx_i(t), \end{cases} \tag{1}$$

for $i = 1, \ldots, N$, where $x_i(t) \in \mathbb{R}^n$, $u_i(t)$, $y_i(t) \in \mathbb{R}$ are the state, the input and the output of the system i, respectively, $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field, and B, C are constant matrices of appropriate dimensions. Each system is interconnected through

$$u_i(t) = k \sum_{\substack{j=1 \\ i \neq j}}^{N} a_{ij}(y_j(t-\tau) - y_i(t-\tau)) + \sum_{l=1}^{m} \gamma_{il} v_l(t) \tag{2}$$

where $\tau \geq 0$ is a constant delay, $k > 0$ denotes a coupling strength and a_{ij} is the (i, j)-entry of the adjacency matrix which represents a network structure, that is,

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$$a_{ij} = a_{ji} = \begin{cases} 1 \text{ if there exists a coupling between } i \text{ and } j \\ 0 \text{ otherwise.} \end{cases}$$

$v_l(t)$ for $l = 1, \cdots, m$ denote m different external inputs, that is, $v_i \neq v_j$ for $i \neq j$ and the coefficients are given by $\gamma_{il} = 1$ if $v_l(t)$ is added to system i and $\gamma_{il} = 0$ otherwise. From these assumptions, networks without external inputs to be considered in this paper are connected and undirected. The total system (1) coupled with (2) can be written in the following form:

$$\dot{x}(t) = F(x) - k(L(\mathcal{G}) \otimes BC)x(t-\tau) + (\Gamma \otimes B)v(t) \quad (3)$$

where $x = [x_1^\top, \cdots, x_N^\top]^\top$, $F(x) = [f(x_1)^\top, \cdots, f(x_N)^\top]^\top$, $v = [v_1, \cdots, v_m]^\top$, operator $\otimes$ denotes the Kronecker product, matrix $L(\mathcal{G}) \in \mathbb{R}^{N\times N}$ is the graph Laplacian encoding the interconnected relationships in the graph $\mathcal{G}$ and $\Gamma \in \mathbb{R}^{N\times m}$ is the matrix such that $[\Gamma]_{ij} = \gamma_{ij}$. Since this paper deals with networks which are connected and undirected, the eigenvalues λ_i of graph Laplacian $L(\mathcal{G})$ satisfy the following relation:

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N.$$

For systems (3), we consider synchronization caused by applying external inputs to a part of systems and then we show a clustering method based on synchronization.

2.2 Semipassive systems

Throughout this paper, we assume that each system (1) has the following semipassive property.

Definition 1. (Pogromsky et al. [1998]). Consider a system

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t)), \end{cases} \quad (4)$$

where state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^p$, output $y(t) \in \mathbb{R}^p$, sufficiently smooth functions $f(\cdot) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ and $h(\cdot) : \mathbb{R}^n \to \mathbb{R}^p$. Consider a nonnegative definite storage function $V(\cdot) \in \mathcal{C}^r : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $V(0) = 0$, $r \geq 1$, such that the following dissipation inequality

$$\dot{V}(x(t)) \leq y^\top(t)u(t) - H(x(t)) \quad (5)$$

holds where $H(\cdot) : \mathbb{R}^n \to \mathbb{R}$. The system (4) is called

1) $\mathcal{C}^r$-semipassive if there exists a storage function $V(\cdot) \in \mathcal{C}^1$ and a function $H(\cdot)$ such that (5) holds with a function $H(\cdot) \geq 0$ outside a ball $\mathcal{B} \subset \mathbb{R}^n$ with radius R centered around 0, i.e.

$$\exists R > 0, |x| \geq R \Rightarrow H(x) \geq \varrho(|x|),$$

with some nonnegative continuous function $\varrho(\cdot)$ defined for all $|x| \geq R$ where $|\cdot|$ refers to the Euclidean vector norm;

2) strictly $\mathcal{C}^r$-semipassive if there exists a storage function $V(\cdot) \in \mathcal{C}^1$ and a function $H(\cdot)$ such that (5) holds with $H(\cdot) > 0$ outside a ball $\mathcal{B} \subset \mathbb{R}^n$ centered around 0.

Assuming the semipassive property for systems, the boundedness of the solutions of coupled systems without external inputs is guaranteed.

Lemma 1. (Steur et al. [2011]). Consider systems (1) coupled with

$$u_i(t) = k \sum_{\substack{j=1 \\ i\neq j}}^{N} a_{ij}(y_j(t-\tau) - y_i(t-\tau)). \quad (6)$$

Then the solutions of coupled systems (1) with (6) are ultimately bounded if each system (1) is strictly $\mathcal{C}^1$-semipassive with a radially unbounded storage function $V(\cdot) \in \mathcal{C}^1$ and a function $H(\cdot)$ such that $H(x_i(t)) - 2kd_i|y_i(t)|^2 > 0$ where $d_i = \sum_{j=1, i\neq j}^{N} a_{ij}$.

If external inputs of equation (2) are bounded, the solutions of coupled systems (1) with (2) are still bounded.

Lemma 2. Consider systems (1) coupled by (2) with any bounded external inputs. Then the solutions of the coupled systems (1) with (2) are ultimately bounded if each system (1) is strictly $\mathcal{C}^1$-semipassive with a radially unbounded storage function $V(\cdot) \in \mathcal{C}^1$ and a function $H(\cdot)$ such that $H(x_i(t)) - 2kd_i|y_i(t)|^2 > 0$.

This lemma can be proved in a similar way to Lemma 1 by replacing $H(\cdot)$ in (5) with a function of x and the upper bound of external inputs. Since the boundary of x depends on the upper bound of v_l, the bound shrinks to $\mathcal{B}$ of Lemma 1 as $|v_l| \to 0$.

3. SYNCHRONIZATION CONDITION FOR AUTONOMOUS NETWORK SYSTEMS

Before considering the clustering problem of network systems, we review synchronization in networks of delay-coupled systems. The formal definition of synchronization of delay-coupled systems is given as follows.

Definition 2. (Oguchi et al. [2011]). If there exists a positive real number r such that trajectories $x_i(t)$, $x_j(t)$ of the systems (1) with initial conditions φ_i, φ_j such that $||\varphi_i - \varphi_j||_C \leq r$ satisfy $|x_i(t) - x_j(t)| \to 0$ as $t \to \infty$, then the systems i and j are *asymptotically synchronized.* If $r = \infty$, then the systems i and j are *globally asymptotically synchronized.* Here $||\varphi||_c \equiv \max_{-\tau\leq\theta\leq 0}||\varphi(\theta)||$ stands for the norm of a vector function φ.

Moreover, if all systems in a network are synchronized, then the network system is said to be *fully synchronized*, and if a part of systems causes synchrony, then network system is *partially synchronized.*

Here we briefly explain an estimation method of the synchronization conditions based on a scaling law (Mimura et al. [2011] and Steur et al. [2012]). Assume that two coupled systems with (6) are synchronized for any pair (k, τ) in set $\mathcal{S}$. Then N coupled systems with (6) are fully synchronized for

$$(k, \tau) \in \bar{\mathcal{S}} \equiv \bigcap_{i=1}^{N-1} \mathcal{S}_i$$

where

$$\mathcal{S}_i = \{(k, \tau) | (\frac{\lambda_{i+1}k}{2}, \tau) \in \mathcal{S}\}$$

and λ_{i+1}, $i = 1, \cdots, N-1$ are nonzero eigenvalues of graph Laplacian. This means that synchronization condition for any networks can be estimated by scaling the region $\mathcal{S}$

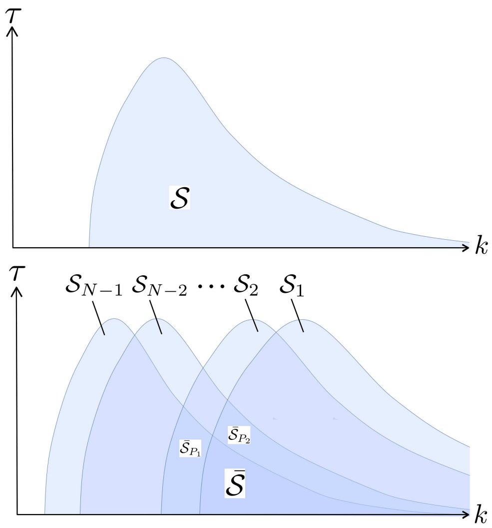

Fig. 1. Synchronization condition $\mathcal{S}$ of two coupled systems and all the scaled regions of network systems. $\bar{\mathcal{S}}$ is the full synchronization condition and, $\bar{\mathcal{S}}_{P_1}$ and $\bar{\mathcal{S}}_{P_2}$ are conditions such that partial synchronization may appear.

depending on the nonzero eigenvalues of graph Laplacian as shown in Fig. 1 and overlapping all the scaled regions.

Furthermore, partial synchronization may appear in intersections of a part of $\mathcal{S}_i$ such as the regions $\bar{\mathcal{S}}_{P_1}$ and $\bar{\mathcal{S}}_{P_2}$ in Fig. 1. Now, let $(k,\tau) \in \bar{\mathcal{S}}_P \equiv \cap_{i\in\mathcal{I}_P}\mathcal{S}_i$ where $\bar{\mathcal{S}}_P \neq \emptyset$ and $\mathcal{I}_P \subset \{1,\cdots,N-1\}$ with $|\mathcal{I}_P| < N-1$ is an index set. The existence of partial synchronization patterns is given by the solution of simultaneous equation

$$(\mu_{i+1}^\top \otimes I_n)x(t) = 0, \ \forall i \in \mathcal{I}_P \tag{7}$$

where μ_{i+1} denotes the right-hand eigenvector corresponding to eigenvalue λ_{i+1}, that is, systems l and k are asymptotically synchronized if $x_l = x_k$ is a solution of (7). Additionally, there is no synchronization pattern except for the solution in (7) for network systems without external inputs. In Section 4, we show that other synchronization patterns may occur by applying external inputs to systems in network.

4. CLUSTERING AND CLUSTER CONTROL

In this section, we propose a clustering method based on synchronization. We introduce the notions of equitable partition and the maximal equitable partition (Martini et al. [2008], Yazicioglu [2012]).

Definition 3. (Equitable partition). A partition π of nodes set V for graph $\mathcal{G}$ with cells $C_1,\cdots,C_r$ is said to be *equitable* if each node in C_i has the same number of neighbors in C_j, $\forall i,\ j \in \{1,\cdots,r\}$, $i \neq j$ with $r = |\pi|$ which denotes the cardinality of the partition.

Definition 4. (The maximal equitable partition). A equitable partition $\pi_M = \pi_F \cup \pi_E$ is said to be *maximal* if each node for external inputs $v_1,\cdots,v_m$ belongs to singleton cell $C_1^E,\cdots,C_m^E$ of the partition π_E and $\pi_F = \{C_1^F,\cdots,C_s^F\}$ is equitable partition of nodes for subsystems such that the cardinality of π_F is minimal.

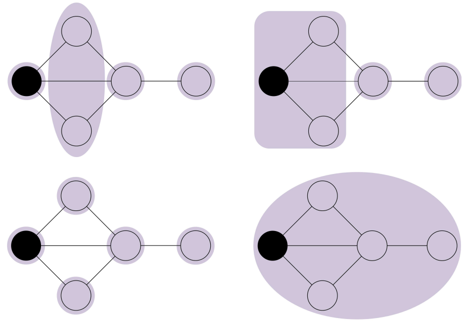

Fig. 2. Examples of equitable partitions

Fig. 2 shows examples of equitable partitions for an identical graph. The black nodes represent external inputs, and nodes are divided into shaded cells based on the equitable partition. As shown in Fig. 2, this network has only four patterns of equitable partitions. Among these partitions, the input node in a singleton cell is in the two left-most figures and the top-left equitable partition has the fewest number of cells. Thus, the top-left equitable partition is maximal.

By using these notions, we consider a clustering method for the identical systems (1) that can be transformed into the form

$$\dot{z}_i(t) = q(z_i(t), y_i(t)) \tag{8}$$

$$\dot{y}_i(t) = a(y_i(t), z_i(t)) + bu_i(t) \tag{9}$$

with $z_i(t) \in \mathbb{R}^{n-1}$, $y_i(t), u_i(t) \in \mathbb{R}$ and $b = CB$. Then we obtain the following result.

Theorem 1. Consider the coupled system (8), (9) with (2) and suppose the following conditions (**A1**), (**A2**):

(A1) Each system (8), (9) is strictly $\mathcal{C}^1$-semipassive with a radially unbounded positive definite storage function $V(\cdot) \in \mathcal{C}^1$ and a function $H(x_i(t)) > 2kd_i|y_i(t)|^2$.

(A2) There exists a positive definite function $V_0(\cdot) \in \mathcal{C}^2 : \mathbb{R}^{n-1} \to \mathbb{R}_{\geq 0}$, $V_0(0) = 0$, such that for all $z_i, z_j \in \mathbb{R}^{n-1}$ and all $y^* \in \mathbb{R}$

$$\nabla V_0^\top(\tilde{z}_{ij})(q(z_i,y^*) - q(z_j,y^*)) \leq -\alpha_0|\tilde{z}_{ij}|^2$$

with constant $\alpha_0 > 0$ and $\tilde{z}_{ij} = z_i - z_j$.

Then there exist constants $\bar{k}$ and $\bar{\gamma}$ such that for (k,τ) satisfying $k > \bar{k}$ and $k\tau < \bar{\gamma}$, systems i and j are globally asymptotically synchronized if nodes i, j belong to the same cell of π_M, and systems i and j are not synchronized if nodes i, j are in different cells of π_M.

Proof. First we consider synchronization of systems in the same cell. Consider that cells $C_1^F, C_2^F,\cdots,C_w^F$ have elements more than one and cells $C_{w+1}^F, C_{w+2}^F,\cdots,\ C_s^F$ have only one element in the maximal equitable partition π_M. Additionally, we denote by $r_{\hat{i}}$ the cardinality of each set $C_{\hat{i}}^F$ and consider that the first r_1 nodes belong to C_1^F, the second r_2 nodes belong to C_2^F, and so on. Then the graph Laplacian $L(\mathcal{G})$ is represented as

$$L(\mathcal{G}) = \left[\begin{array}{ccc|c} L_{1,1} & \cdots & L_{1,w} & L_{1,S} \\ \vdots & \ddots & & \vdots \\ L_{w,1} & & L_{w,w} & L_{w,S} \\ \hline L_{S,1} & \cdots & L_{S,w} & L_{S,S} \end{array}\right],$$

where $L_{\hat{i},S} = [L_{\hat{i},w+1}\ L_{\hat{i},w+2}\ \cdots\ L_{\hat{i},s}]$. Note that the block sub-matrices satisfy $L_{\hat{i},\hat{j}} = L^\top_{\hat{j},\hat{i}}$ from the symmetry of the graph Laplacian $L(\mathcal{G})$. Moreover, from Definition 3, $l_{i,j}$ which is entry of $L_{\hat{i},\hat{j}} \in \mathbb{R}^{r_{\hat{i}}\times r_{\hat{j}}}$ satisfies the following condition:

$$\sum_{k=1}^{r_{\hat{j}}} l_{i,k} = \sum_{k=1}^{r_{\hat{j}}} l_{j,k},\ \forall i,\ j \in C^F_{\hat{i}},$$

which means that all row sums in $L_{\hat{i},\hat{j}} \in \mathbb{R}^{r_{\hat{i}}\times r_{\hat{j}}}$ are equal. We represent the N coupled systems (8), (9) with input (2) by

$$\begin{aligned} \dot{z}(t) &= q(z(t), y(t)) \\ \dot{y}(t) &= a(y(t), z(t)) - bkL(\mathcal{G})y(t-\tau) + \Gamma v(t), \end{aligned}$$

where $z(t) \in \mathbb{R}^{N(n-1)}$, $y(t) \in \mathbb{R}^N$ are state vectors which have states of N systems in each entry and $q(z(t), y(t)) \in \mathbb{R}^{N(n-1)}, a(y(t), z(t)) \in \mathbb{R}^N$ are nonlinear functions such that each entry is nonlinear function $q(z_i(t), y_i(t))$ and $a(y_i(t), z_i(t))$ respectively. Now we get the following matrix:

$$M = \left[\begin{array}{c} M_0 \\ \hline M_1 \end{array}\right]$$

where $M_0 \in \mathbb{R}^{s\times N}$ is a matrix whose entries are given by

$$[M_0]_{\hat{i}j} = \begin{cases} 1 \text{ if } j = k_{\hat{i}} \\ 0 \text{ otherwise} \end{cases}$$

for such $k_{\hat{i}} = \min\left\{ i \mid i \in C^F_{\hat{i}} \right\}$, and $M_1 \in \mathbb{R}^{(N-s)\times N}$ is given by

$$M_1 = \left[\begin{array}{ccc|c} m_1 & & \mathbf{0} & \\ & \ddots & & \mathbf{0} \\ \mathbf{0} & & m_s & \end{array}\right],\ m_{\hat{i}} = \begin{bmatrix} 1 & -1 & & \mathbf{0} \\ \vdots & & \ddots & \\ 1 & \mathbf{0} & & -1 \end{bmatrix}$$

where $m_{\hat{i}} \in \mathbb{R}^{(r_{\hat{i}}-1)\times r_{\hat{i}}}$. Define a new coordinate as $\tilde{z} = (M \otimes I_{n-1})z,\ \tilde{y} = My$ where

$$\tilde{z} = \left[\begin{array}{c} z_{k_1} \\ \vdots \\ z_{k_s} \\ \hline \bar{z}_1 \\ \vdots \\ \bar{z}_w \end{array}\right],\ \tilde{y} = \left[\begin{array}{c} y_{k_1} \\ \vdots \\ y_{k_s} \\ \hline \bar{y}_1 \\ \vdots \\ \bar{y}_w \end{array}\right],$$

$$\bar{z}_{\hat{i}} = \begin{bmatrix} z_{k_{\hat{i}}} - z_{k_{\hat{i}}+1} \\ z_{k_{\hat{i}}} - z_{k_{\hat{i}}+2} \\ \vdots \\ z_{k_{\hat{i}}} - z_{k_{\hat{i}}+r_{\hat{i}}-1} \end{bmatrix},\quad \bar{y}_i = \begin{bmatrix} y_{k_{\hat{i}}} - y_{k_{\hat{i}}+1} \\ y_{k_{\hat{i}}} - y_{k_{\hat{i}}+2} \\ \vdots \\ y_{k_{\hat{i}}} - y_{k_{\hat{i}}+r_{\hat{i}}-1} \end{bmatrix}.$$

In this coordinate, if $\bar{z}(t)$ and $\bar{y}(t)$ converge to zero as $t \to \infty$, then the systems in the identical cell are asymptotically synchronized. By applying the coordinate transformation, matrices $L(\mathcal{G})$ and Γ are transformed as follows:

$$ML(\mathcal{G})M^{-1} = \left[\begin{array}{c|c} H & * \\ \hline \mathbf{0} & M_1 L(\mathcal{G})M_1^+ \end{array}\right],\ M\Gamma = \left[\begin{array}{c} M_0\Gamma \\ \hline \mathbf{0} \end{array}\right],$$

where M_1^+ is a pseudo inverse matrix of M_1 and

$$H = \left[\begin{array}{ccc|c} h_{1,1} & \cdots & h_{1,w} & h_{1,S} \\ \vdots & \ddots & & \vdots \\ h_{w,1} & \cdots & h_{w,w} & h_{w,S} \\ \hline h_{S,1} & \cdots & h_{S,w} & L_{S,S} \end{array}\right]$$

where $h_{\hat{i}\hat{j}}$ is the row sum of the first row of block matrix $L_{\hat{i},\hat{j}}$. Since matrix M is nonsingular, H and $\bar{L} = M_1 L(\mathcal{G})M_1^+$ preserve the eigenvalues of $L(\mathcal{G})$. Thus, H has one zero eigenvalue since $L(\mathcal{G})$ has one zero eigenvalue by the assumption of the connected graph. This means that all eigenvalues of $\bar{L}$ are positive. From these facts and analysis on global uniform asymptotical stability of $\bar{z} = 0$ and $\bar{y} = 0$ (proof of Corollary 6, Steur et al. [2011]), there exist threshold values $\bar{k}$ and $\bar{\gamma}$ such that if $k > \bar{k}$ and $k\tau < \bar{\gamma}$, systems i and j are globally asymptotically synchronized for nodes i, j in same cell of the maximal equitable partition.

Next we show that systems belonging to different cells don't synchronize, i.e. if $\hat{i} \neq \hat{j}$, systems $i \in C^F_{\hat{i}}$ and $j \in C^F_{\hat{j}}$ do not synchronize. To prove this by contradiction, we assume that even though systems i and j belong to difference cells, they show synchrony over the cells. In general, synchronized solutions $\zeta_{\hat{i}}$ of systems in cells $C^F_{\hat{i}}$ satisfy

$$\dot{\zeta}_{\hat{i}} = f(\zeta_{\hat{i}}) + k\sum_{\substack{\hat{k}=1 \\ \hat{i}\neq\hat{k}}}^{s} h_{\hat{i}\hat{k}}(\zeta_{\hat{k}}(t-\tau) - \zeta_{\hat{i}}(t-\tau)) + \sum_{l=1}^{m} \gamma_{il}v_l.$$

From the assumption, the dynamics of the synchronization error defined by $\zeta_{\hat{i}} - \zeta_{\hat{j}}$ must have the trivial solution for any external inputs, but this requires that

$$h_{\hat{i}\hat{k}} = h_{\hat{j}\hat{k}},\ \gamma_{il} = \gamma_{jl} \text{ for } \hat{k} \neq \hat{i}, \hat{j} \text{ and } l = 1, \cdots, m \quad (10)$$

However, conditions (10) mean that systems i and j are in the same cells, which is contradict to the assumption that systems i and j are in different cells. Therefore systems $i \in C^F_{\hat{i}}$ and $j \in C^F_{\hat{j}}$ for $\hat{i} \neq \hat{j}$ are not synchronized. □

Theorem 1 states that systems (8), (9) coupled with (2) are synchronized based on the maximal equitable partition under sufficiently large k and sufficiently small τ depicted in the region of Fig. 3. Theorem 1 also means that if we choose constant $\alpha \to 0$ for bounded external inputs satisfying $|v_l| \leq \alpha$, network system is fully synchronized because the maximal equitable partition has only one cell which includes all nodes of systems due to vanishment of external inputs. Therefore if external inputs with small bound α are applied to fully synchronized autonomous network systems, the network systems are partially synchronized. Moreover, the pattern of partial synchronization depends on the maximal equitable partition. Therefore the pattern of partially synchronization can be controlled by choosing systems adding external inputs.

Clustering methods for integrator network systems with external inputs have been already investigated by applying the maximal equitable partition (Martini et al. [2008], Egerstedt [2010], Egerstedt et al. [2012], Yazicioglu [2012]). Theorem 1 indicates that the identification of synchronization pattern by the maximal equitable partition is applicable to nonlinear network systems which have boundedness of the solutions regardless of stability of systems in network.

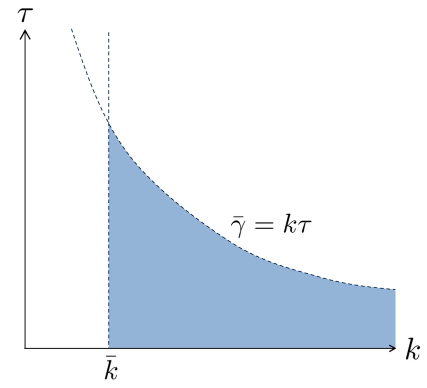

Fig. 3. Estimated synchronization region. For k and τ satisfying $k > \bar{k}$ and $k\tau < \bar{\gamma}$, network systems are partially synchronized depending on the maximal equitable partition.

Next, we consider a model reduction based on a partial synchronization pattern. Before that, we introduce the following characteristic matrix used in Mesbahi et al. [2010].

Definition 5. (Characteristic matrix). Characteristic vector $p_i \in \mathbb{R}^N$ of cell C_i for non-trivial partition is defined such that

$$[p_i]_j = \begin{cases} 1 & \text{if } j \in C_i \\ 0 & \text{otherwise,} \end{cases}$$

and characteristic matrix $P \in \mathbb{R}^{N \times r}$ of a partition π of nodes set V is a matrix with the characteristic vectors of the cell as its columns.

We can reduce the order of system (3) which denotes a whole system in network to the number of the cells in the maximal equitable partition. Let us consider the transformation $\xi = (P^+ \otimes I_n)x$ to the system (3) where $P^+ = (P^\top P)^{-1}P^\top$ is the left pseudo inverse matrix of P. This transformation gives the state corresponding to the average of the states of systems in each cell for the maximal equitable partition, that is, transformed system is reduced to cell-to-cell network system of the maximal equitable partition.

5. NUMERICAL SIMULATIONS

Consider the Hindmarsh-Rose neuron system given by

$$\begin{cases} \dot{y}_i(t) = -y_i^3(t) + 3y_i^2(t) + z_{i,1}(t) - z_{i,2}(t) + a + u_i(t) \\ \dot{z}_{i,1}(t) = 1 - 5y_i^2(t) - z_{i,1}(t) \\ \dot{z}_{i,2}(t) = b(4(y_i(t) + c) - z_{i,2}(t)) \end{cases}$$

which behaves chaotically for $a = 3.25$, $b = 0.005$ and $c = 1.618$, and the network topology with $N = 5$ nodes shown in Fig. 4. As numerical examples, we show two cases that (i) the external input $v_1(t) = \sin t$ is applied to subsystem 1 in Fig. 5(a) and (ii) $v_1(t) = \sin t$ is applied to subsystem 1 and $v_2 = 5$ is to subsystems 4 and 5 in Fig. 5(b). In Fig. 5(a) and 5(b), the corresponding maximal equitable partitions of the network are indicated by covering with shades. Throughout these examples, we choose a pair of $(k, \tau) = (1, 0.8)$ as simulations.

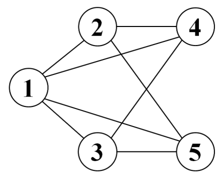

Fig. 4. The graph of network with $N = 5$ nodes

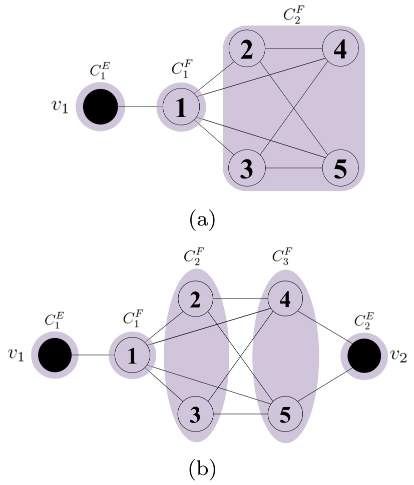

Fig. 5. (a) A case that an external input is added to subsystem 1; (b) A case that different external inputs are added to subsystems 1, 4 and 5

First, we consider the network system shown in Fig. 5(a). Fig. 6 shows that the synchronization errors among subsystems 2, 3, 4 and 5 in the same cell converge to 0 and therefore all subsystems in the same cell are asymptotically synchronized. On the other hand, Fig. 7 shows the synchronization error between subsystems 1 and 2. From this figure, we see that the synchronization error between subsystems in different cells does not converge to zero and synchronization between them does not occur. From these results, we can conclude that synchronization occurs just in the maximal equitable partition. Therefore, the network system can be divided into a unit of clusters by the maximal equitable partition.

Next, we consider the network system shown in Fig. 5(b). Fig. 8 shows the synchronization errors between subsystems 2, 3 and between 4 and 5 in the same cells and they are asymptotically converge to 0. While, Fig. 9 shows that the synchronization errors among subsystems 1, 2 or 4, and 2, 4 don't converge to zero. In the same way as the foregoing example, we can observe that synchronization occurs inside each cell divided by the maximal equitable partition. From the above-mentioned two results, we can deduce that pattern of clustering depends on how to apply external inputs. In addition, the clustering method proposed here can be applied to nonlinear network systems with strictly semipassivity including chaotic systems as well.

Finally, for the above two examples, these networks can be reduced to quotient graphs shown in Figs. 10, 11, which represent connections between cells. These graphs correspond to systems transformed by applying a coordinates transformation $\xi = (P^+ \otimes I_n)x$.

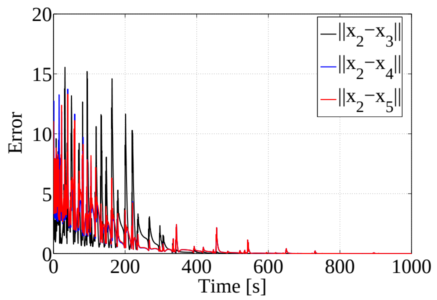

Fig. 6. Synchronization errors between subsystem 2 and subsystems 3, 4 or 5 for network in Fig. 5(a)

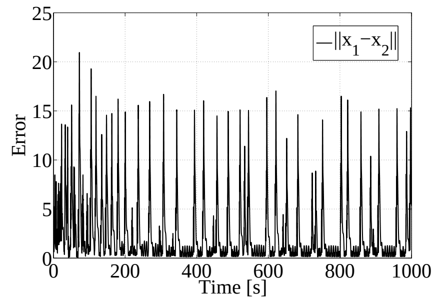

Fig. 7. Synchronization error between subsystem 1 and 2 for network in Fig. 5(a)

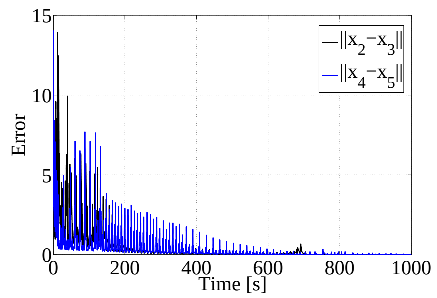

Fig. 8. Synchronization errors between subsystems 2 and 3 and between subsystems 4 and 5 for network in Fig. 5(b)

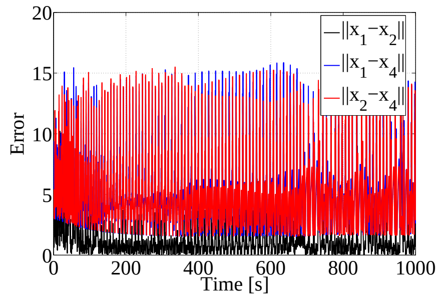

Fig. 9. Synchronization errors among subsystems 1, 2 or 4 and 2, 4 for network in Fig. 5(b)

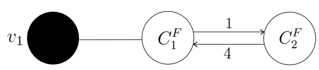

Fig. 10. Reduced model for network in Fig. 5(a)

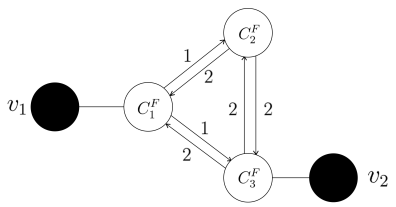

Fig. 11. Reduced model for network in Fig. 5(b)

6. CONCLUSIONS

This paper proposed a clustering method based on synchronization in networks of nonlinear systems with external inputs. In addition, we showed a synchronization pattern can be changed based on the maximal equitable partition by applying external inputs to systems in networks. The numerical examples support the validity of the proposed method. Throughout this paper, we considered systems with delayed couplings, but it is worth to note that the obtained result covers a case of $\tau = 0$.

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