MAXIMUM GAIN RULE FOR SELECTING CONTROLLED VARIABLES

Eduardo Shigueo Hori, Sigurd Skogestad

Department of Chemical Engineering
Norwegian University of Science and Technology, NTNU
N-7491, Trondheim, Norway
Email: skoge@chemeng.ntnu.no
Phone: +47-7359-4154

Abstract: The appropriate selection of controlled variables is one of the most important tasks in plantwide control. In this paper, we consider the choice of secondary controlled variables for indirect control. The objective is to keep the primary variables close to their desired setpoint at steady-state without controlling them directly. We use the maximum scaled gain rule (maximize minimum singular value) and compare it to the exact local method. The issue of input scaling has usually been neglected, but it is shown that it may be crucial for ill-conditioned plants. The application is the selection of control structures for a binary distillation column.

Keywords: self-optimizing control, distillation columns, plantwide, control system design, disturbance rejection

1. INTRODUCTION

The selection of controlled variables is one of the most important tasks in control structure design (Yi and Luyben, 1995) because this choice can limit the performance of the whole control system. In this paper, we refer to the regulatory control layer, where we are interested in selecting secondary controlled variables (y_2) that are able to reject disturbances and minimize the effects of implementation errors in the primary variables (y_1). Note that we only consider the common case with control of individual measurements y_2. Candidate measurements in this paper include temperatures, flows and flow ratios. For the optimal use of measurement combinations for indirect control, see Hori *et al.* (2005). Some desirable properties (requirements) for the controlled variables y_2 (Skogestad, 2000):

- We want small optimal variation in the selected variables
- We want variables with a large sensitivity (Tolliver and McCune, 1980)
- We want to be able to control the selected controlled variables tightly (small "implementation" error)

Moore (1992) proposed to select controlled variables y_2 using an SVD-analysis of the steady-state gain matrix G_{all} from the inputs u to all the candidate measurements y_2. After decomposing the gain matrix $G_{\text{all}} = U\Sigma V^T$, he proposed to use the ortonormal matrix U (matrix of left singular vectors) to locate the most sensitive measurements (with largest absolute values), which should be used as controled variables.

Halvorsen *et al.* (2003) derived rigorously the closely related method of selecting controlled variables that maximize the minimum singular value,

$\underline{\sigma}(G')$, of the appropriately scaled gain matrix from inputs u to the selected outputs y_2. This is the "maximum gain rule" (Halvorsen *et al.*, 2003): *Select controlled variables c such that we maximize the minimum singular value of the scaled gain matrix* G', $\underline{\sigma}(G')$, where

$$G' = S_1 G S_2 \quad (1)$$

Here G is the steady-state gain matrix from u (manipulated variables) to y_2 (controlled variables). S_1 and S_2 are the output and input scaling, respectively.

The output scalings S_1 are obviously important as the objective is to select the outputs (controlled variables), and the output scalings indirectly include the control objective through the optimal variation, see Eq. (5). However, the proper input scaling has in practice been neglected by assuming that S_2 is a diagonal or unitary matrix. The assumption seems to be of minor importance because the inputs are anyway given. The main goal of this paper is to reexamine this assumption. To do this, we compare the maximum gain rule with the exact local method (Halvorsen *et al.*, 2003) on a binary distillation column where $u = [\,L \quad V\,]$, $y_1 = [\,x^H_{\rm top} \quad x^L_{\rm btm}\,]$ and $c = y_2$ is a combination of two temperatures and/or flows. More precisely, the objective function J to be minimized is the relative steady-state deviation from the desired setpoint,

$$J = \Delta X^2 = \left(\frac{x^H_{top} - x^H_{top,s}}{x^H_{top,s}}\right)^2 + \left(\frac{x^L_{btm} - x^L_{btm,s}}{x^L_{btm,s}}\right)^2 \quad (2)$$

where x^H_{top} is the composition of the heavy key-component (H) in the top of the column and x^L_{btm} is the composition of the light key-component (L) in the bottom.

2. MAXIMUM GAIN RULE

The *maximum gain rule* is to select controlled variables that maximize the minimum singular value of $G' = S_1 G S_2$. Although this rule is not exact, especially for plants with an ill-conditioned gain matrix like distillation columns, it is very simple and it works well for most processes (Halvorsen *et al.*, 2003). Also, as the minimum singular value has the monotonic property, we can use a branch and bound algorithm to search for the configuration with largest minimum singular value, avoiding the evaluation of all possible configurations (Cao *et al.*, 1998).

To evaluate the maximum gain rule, we define the loss L as the difference between the actual value of the cost function ($J(y_2, d)$), obtained with a specific control strategy where y_2 is constant, and the truly optimal value of the cost function ($J_{\rm opt}(d)$), that is,

$$L = J(y_2, d) - J_{\rm opt}(d) \quad (3)$$

In our case, see Eq. (2), $J_{\rm opt}(d) = 0$.

2.1 Maximum gain rule: Output scaling (S_1)

An important part of the maximum gain rule is to scale the output variables appropriately. The outputs are scaled with respect to their "span", which is the sum of 1) the optimal variation due to disturbances d and 2) the effect of the implementation error (n^y). We have

$$S_1 = diag\{1/\text{span}\,(y_{2i})\} \quad (4)$$

where

$$\text{span}\,(y_{2i}) = |\text{opt.var.}| + |\text{implem.error}| = \\ = |y_2^{opt}|_i + |n^{y_2}|_i \quad (5)$$

The optimal variation may be obtained as follows. The linear steady-state model is:

$$y_1 = G_1 u + G_{d1} d \quad (6)$$

$$y_2 = G u + G_d d \quad (7)$$

where y_1 are the primary variables, y_2 are the measurements (candidated controlled variables), u are manipulated variables and d are disturbances.

In the presence of disturbances (d), perfect control of the primary variables ($y_1 = 0$) is obtained with

$$u^{\rm opt} = -G_1^{-1} G_{d1} d \quad (8)$$

The resulting optimal variation of the measurements (y_2) is

$$y_2^{\rm opt} = (-G G_1^{-1} G_{d1} + G_d) d \quad (9)$$

Remark: We here use perfect control ($y_1 = 0$) as the reference for computing the optimal variation. This is recommended even for cases where some manipulated variable is kept constant (e.g. reflux L is constant) so that $y_1 = 0$ is not possible, because perfect control is in any case the objective.

2.2 Maximum gain rule: Input scaling (S_2)

The best (correct) input "scaling" for the maximum gain rule is to select $S_2 = J_{\rm uu}^{-1/2}$ (Halvorsen *et al.*, 2003) where $J_{\rm uu}$ is the Hessian matrix of the cost function J (matrix of second derivatives of J with respect to u) (the term "scaling" is a bit misleading because $J_{\rm uu}^{-1/2}$ is generally not a diagonal matrix). However, it has been proposed a simplified scaling with S_2 assumed unitary.

2.2.1. $S_2 = J_{\mathrm{uu}}^{-1/2}$ (correct) For this case, Halvorsen *et al.* (2003) derived the worst-case loss as:

$$L_{\max} = \max_{\|e'_c\|_2 \le 1} L = \frac{1}{2\left(\underline{\sigma}\left(S_1 G J_{\mathrm{uu}}^{-1/2}\right)\right)^2} \quad (10)$$

where $e'_c = [\, d' \quad n^{y'} \,]^T$, d' is the expected disturbance and $n^{y'}$ is the implementation error of each controlled variable y_{2i}.

2.2.2. Simplified scaling (assuming J_{uu} unitary) We have $\underline{\sigma}(G') = \underline{\sigma}(S_1 G J_{\mathrm{uu}}^{-1/2}) \le \underline{\sigma}(J_{\mathrm{uu}}^{-1/2}) \underline{\sigma}(S_1 G)$. Using $\underline{\sigma}\left(J_{\mathrm{uu}}^{-1/2}\right) = 1/\bar{\sigma}\left(J_{\mathrm{uu}}\right)^{1/2}$, Eq. (10) becomes

$$L_{\max} = \max_{\|e'_c\|_2 \le 1} L \le \frac{\bar{\sigma}\left(J_{\mathrm{uu}}\right)}{2\left(\underline{\sigma}\left(S_1 G\right)\right)^2} \quad (11)$$

In Eq. (11) we have equality when we assume J_{uu} is a unitary matrix (simplified scaling). Since J_{uu} is independent of the outputs, we then minimize the loss $L_{\max}$ in Eq. (11) by maximizing $\underline{\sigma}\left(S_1 G\right)$, so the input scaling does not matter. As we will see, this may not be a good assumption for ill-conditioned plants, like distillation columns.

2.3 Exact local method

The exact local method was derived by Halvorsen *et al.* (2003). This method utilizes a Taylor series expansion of the loss function, so the exact value of the worst-case local loss becomes

$$L_{\max} = \max_{\|e'_c\|_2 \le 1} L = \left(\bar{\sigma}\left([\, M_d \quad M_n \,]\right)\right)^2 / 2 \quad (12)$$

where

$$M_d = J_{\mathrm{uu}}^{1/2}(J_{\mathrm{uu}}^{-1} J_{ud} - G^{-1} G_d) W_d \quad (13)$$

$$M_n = J_{\mathrm{uu}}^{1/2} G^{-1} W_n \quad (14)$$

The magnitude of the disturbances and implementation error enter into the diagonal matrices W_d and W_n. The steady-state gains G and G_d and the second order derivatives J_{uu} and J_{ud} may be obtained numerically by applying small perturbations in the inputs u. Actually, in our case, see cost function for the distillation columns in Eq. (2), J_{uu} and J_{ud} can be obtained more directly as shown next. Consider the special case where we have a quadratic cost function which can be represented by

$$J = y_1^T Q y_1 + u^T R u \quad (15)$$

where Q and R are weighting matrices (both are symmetric positive-definite).

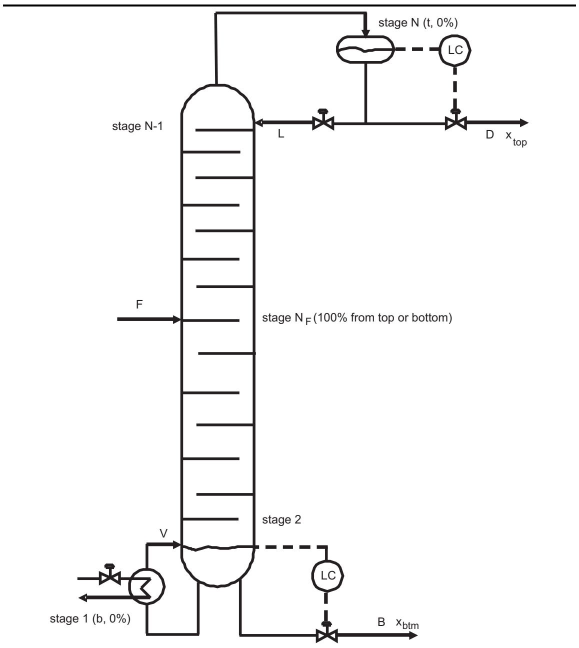

Fig. 1. Distillation column.

From Eq. (6) we then have

$$J_{\mathrm{uu}} = 2\left(G_1^T Q G_1 + R\right) \quad (16)$$

$$J_{ud} = 2 G_1^T Q G_{d1} \quad (17)$$

In our case, see Eq. (2), the weighting matrices are $R = 0$ and $Q = \left[\, 1/\left(x^H_{top,s}\right)^2 \quad 1/\left(x^L_{btm,s}\right)^2 \,\right]$.

Compared to the exact method, the branch and bound algorithm for minimizing $\underline{\sigma}(G')$ usually requires the evaluation of fewer combinations, and each evaluation is also less time consuming.

3. CASE STUDY: DISTILLATION COLUMN

The variable selection methods (maximum gain rule and exact local method) were applied to a binary distillation column. The components are assumed "ideal" and denoted A (light) and B (heavy). The relative volatility is equal to 1.5 ($\alpha_{AB} = 1.5$). The key components are denoted L for light and H for heavy. The main disturbances are the feed flow rate (F), feed enthalpy (q_F), and feed composition (z_F). The example is "column A" with a feed of 50% light component. The objective of the column is to keep 1% of heavy component in the top (and 99% lights) and 1% of light component in the bottom. The column has 41 stages (including the reboiler and the total condenser) and these stages are numbered as a percentage from top and bottom (both are 0%) to feed stage (100%) (see Figure 1).

A conventional distillation column with a given feed and pressure controlled using cooling has 4 degrees of freedom left: reflux flow rate (L), vapour boilup (V), destillate rate (D), and bottoms flowrate (B), i.e., $u_0 = [L \quad V \quad D \quad B]^T$. As we need to control two liquid levels to stabilize the column (which consumes two degrees of freedom because levels do not have steady-state effect), we are left with two steady-state degrees of freedom (Shinskey, 1984) available for composition control, which are here selected as $u = [L \quad V]$. Note that the steady-state gain matrix G_1 from $u = [L \quad V]$ to y_1 is generally ill-conditioned. In our case, the gain matrix is $G_1 = \begin{bmatrix} 1.085 & -1.098 \\ -0.875 & 0.862 \end{bmatrix}$ and $J_{\mathrm{uu}}^{-1/2} = \begin{bmatrix} 0.263 & 0.259 \\ 0.259 & 0.262 \end{bmatrix}$. The condition numbers are 145.6 for both matrices.

Our main objective, as shown by Eq. (2), is to use the two available degrees of freedom to keep the top and bottom compositions (primary variables y_1) close to their optimal values. As compositions are difficult to measure (due to long time delays, high cost, etc.), we want to use indirect control where temperatures are used as secondary controlled variables (y_2). For simplicity, we assume that the temperature T_i(°C) on each stage i is calculated as a linear function of the liquid composition in each stage (Skogestad, 1997)

$$T_i = 0x_{A,i} + 10x_{B,i} \qquad (18)$$

This may seem unrealistic, but results using detailed models show that this is actually of minor importance (Hori *et al.*, 2006). To compare the maximum gain rule with the exact method we use the maximum composition deviation, and note that

$$\Delta X_{\max} = \sqrt{J_{\max}} = \sqrt{L_{\max}} \qquad (19)$$

where $L_{\max}$ is calculated from Eq. (12) (exact method) or estimated from Eqs (10) and (11).

3.1 Output scaling (S_1)

The output scaling is $S_1 = 1/T_{opt}$ where

$$\Delta T_{opt} = |\tfrac{dT_{opt}}{dz_F}|\Delta z_F^E + |\tfrac{dT_{opt}}{dF}|\Delta F^E + |\tfrac{dT_{opt}}{dq_F}|\Delta q_F^E \qquad (20)$$

Here Δz_F^E, ΔF^E, and Δq_F^E are the expected (typical) disturbances and ΔT^n is the expected implementation/measurement error for controlling the temperature. The implementation error is assumed to be the same for all stages ($\Delta T^n = 0.5$°C). The expected magnitude of the disturbances are 20% for feed rate F, 10% for feed composition z_F and 10% for feed enthalpy q_F.

The resulting optimal variations were obtained using Eq. (9). The result is shown in Figure 2. It is possible to see that stages close to the feed stage are more sensitive to disturbances, while the stages close to the ends are affected more by the implementation error. Also, the figure shows that the main disturbance is in the feed composition. The optimal variation in temperature with feed flowrate is zero, because we have assumed constant efficiency for the column. This result confirms what is found in the literature (Luyben, 2005).

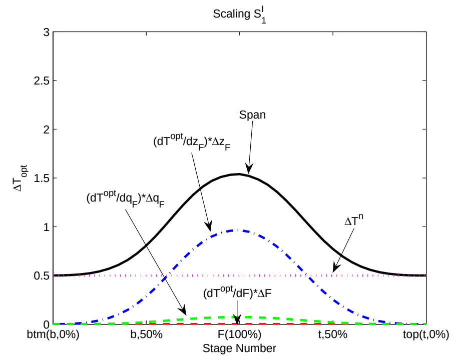

Fig. 2. Optimal variations and total span (output scaling S_1).

Remark on output scaling: Note that the output scaling was obtained using $u = [L \quad V]^T$ and Eq. (8) gives $y_1 = 0$. This means that "perfect control" is the basis.

3.2 Input scaling (S_2)

In this section we want to compare the maximum gain rule (with and without $S_2 = J_{\mathrm{uu}}^{-1/2}$) with the exact local method (section 2.3). The three methods were compared using the composition deviations estimated by Eqs (10) (maximum gain rule, $S_1 G J_{\mathrm{uu}}^{-1/2}$), (11) (simplified maximum gain rule, $S_1 G$), and (12) (exact local method) in Figure 3 for two fixed symmetrically located temperatures. The best locations have a composition deviation ΔX about 1 or less. This figure shows that the three methods give the same best temperatures, but note that the estimated ΔX is a factor 100 times higher when we use $\underline{\sigma}(S_1 G)$.

In Table 1 we consider the more general case where the candidates are all temperatures and flows (including flow ratios L/D, L/F, etc) and want to select two of these variables to be controlled. Table 1 shows that the simplified maximum gain rule using $\underline{\sigma}(S_1 G)$ gives completely wrong (too high) estimates for ΔX in most cases. The simplified method gives that the best control configuration is to keep L/F and V/B constant but, at least compared to the configurations with temperatures, this is a poor choice with an exact loss of 18.60.

Table 1. Steady-state composition deviation ΔX for distillation column for various configurations.

Configuration	Exact method	Maximum gain rule			
		Assume unitary J_{uu} (simplified)		$S_2 = J_{uu}^{-1/2}$ (correct)	
Fixed variables (y_2)	$\Delta X(Eq.12)$	$\underline{\sigma}(S_1G)$	Estimated $\Delta X(Eq.11)$	$\underline{\sigma}(S_1GJ_{uu}^{-1/2})$	Estimated $\Delta X(Eq.10)$
$T_{b,55\%}$-$T_{t,55\%}$	**0.530**	1.508	131	0.783	0.903
$T_{b,55\%}$-$T_{t,60\%}$	0.541	1.442	137	0.752	0.941
$T_{b,65\%}$-$T_{t,65\%}$	0.595	1.241	159	0.645	1.100
$T_{b,40\%}$-$T_{t,45\%}$	0.675	1.548	127	**0.792**	**0.893**
$T_{b,70\%}$-$T_{t,75\%}$	0.706	0.956	206	0.499	1.417
$T_{b,70\%}$-L/F	0.916	1.531	129	0.607	1.164
$T_{b,75\%}$-V/F	1.148	1.125	175	0.498	1.419
$T_{b,90\%}$-L	1.223	0.815	242	0.400	1.767
$T_{b,70\%}$-L/D	1.321	0.727	272	0.342	2.067
$T_{t,95\%}$-V	1.470	0.639	309	0.305	2.320
$T_{t,85\%}$-V/B	1.711	0.571	345	0.261	2.712
$T_{b,0\%}$-$T_{t,0\%}$	5.000	0.271	728	0.141	5.000
L/D-V/B	15.80	0.878	225	0.040	17.80
L/F-V/B	18.60	**1.603**	**123**	0.028	25.60
L-B	21.10	0.805	245	0.020	35.20
D-V	21.20	0.634	311	0.020	35.20
L/F-V/F	90.00	1.600	124	0.007	109.0

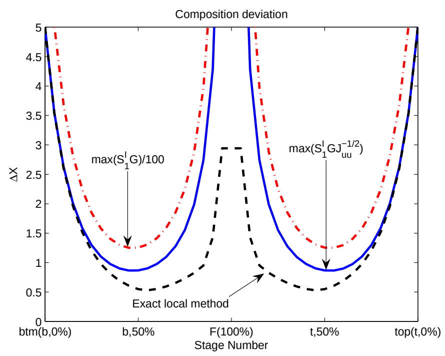

Fig. 3. Comparison of composition deviation estimates ($S_1GJ_{uu}^{-1/2}$, S_1G, and exact local method).

On the other hand, with the correctly scaled gain matrix $G' = S_1GJ_{uu}^{-1/2}$, the results are very close to the exact method, giving $T_{b,40\%}$-$T_{t,45\%}$ as the best set of controlled variables (exact loss of 0.675). This is close to the minimum steady-state composition deviation (ΔX) of 0.530 which is obtained when we control temperatures on stages 12 ($T_{b,55\%}$) and 30 ($T_{t,55\%}$), that is, with the temperatures symmetrically located on each side of the feed stage. Thus, although the maximum gain rule using $G' = S_1GJ_{uu}^{-1/2}$ is not exact, it gives results that are very similar to the exact method.

4. OPTIMAL COMBINATION OF MEASUREMENTS

In the section above we discussed the selection of single measurements for control. Another option would be the selection of a combination of measurements. The control of combination of variables was studied by Hori *et al.* (2005) and Alstad and Skogestad (2007). They state that, if the number of measurements (n_y) is larger or equal to the number of inputs (n_u) plus disturbances (n_d), it is possible to obtain optimal indirect control of the primary variables y_1 (rejet disturbances perfectly, at least at steady-state). This can be done combining the measurements into new secondary controlled variables $c = Hy_2$ (linear combinations of the available measurements y_2).

As the number of possible measurements is usually very large, we suggest a two-step approach to obtain the combination of variables:

(1) Choose the measurements applying the "Maximum Gain Rule" presented by Alstad and Skogestad (2007), i.e., maximize the minimum singular value of the matrix $\widetilde{G} = [\,G \quad G_d\,]$.
(2) Calculate matrix H, which represents the combination of variables, according to Hori *et al.* (2005)

$$H = \widetilde{G}_1\widetilde{G}^{-1} \tag{21}$$

where $\widetilde{G}_1 = [\,G_1 \quad G_{d1}\,]$.

For the distillation column case, we will not consider the feed flowrate as an important disturbance because its optimal variation in temperature is zero (see Section 3.1). Then, as we have 2 inputs ($n_u = 2$) and 2 disturbances ($n_d = 2$), we want to select 4 temperatures ($n_y = 4$) to obtain perfect disturbance rejection. So, applying the two-step approach above, we obtain that we should combine temperatures $T_{b,35\%}$, $T_{b,75\%}$, $T_{t,85\%}$, and $T_{t,40\%}$. The resulted combination of measurements is:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = H \begin{bmatrix} T_{b,35\%} \\ T_{b,75\%} \\ T_{t,85\%} \\ T_{t,40\%} \end{bmatrix} \tag{22}$$

where

$$H = \begin{bmatrix} 0.0018 & -\mathbf{0.0171} & 0.0004 & -0.0013 \\ -0.0016 & 0.0011 & -0.0008 & \mathbf{0.0148} \end{bmatrix} \tag{23}$$

The estimated maximum loss which in this case is caused by measurement error only, is equal to 0.20 (exact method). This is more than a factor two smaller than the loss (0.53) for the best choice of two single measurements (see Table 1).

Remark: From matrix H we can see that temperatures $T_{b,75\%}$ and $T_{t,40\%}$ are the most important measurements. This result is quite similar to the best temperatures obtained in Table 1 ($T_{b,55\%}$ and $T_{t,55\%}$).

5. CONCLUSIONS

In this paper we presented a systematic way to select secondary controlled variables using the maximum scaled gain rule. This method was compared with the exact local method and the results have shown that both methods give similar results (good control structures).

The exact local method (Eq. 12) gives, as the name already says, the best control configuration but it needs to be evaluated for all possible combinations.

The maximum gain rule involving $S_1 G J_{\mathrm{uu}}^{-1/2}$, see Eq. (10), that is, with the input scaling $S_2 = J_{\mathrm{uu}}^{-1/2}$, is preferred if we can obtain easily the Hessian J_{uu} (as in this example). The results are very close to the optimum (as can be seen in section 3.2) and it does not require the evaluation of all possible candidates, as for the exact local method. So, for large systems, the maximum gain rule is a preferred choice.

The simplified maximum gain rule involving $S_1 G$, see Eq. (11), that is, without input scaling, is the easiest to apply because it does not require an evaluation of the Hessian J_{uu}, which is sometimes hard to obtain, but unfortunately it can give a completely wrong result for ill-conditioned systems like distillation column (see Table 1). The problem could be avoided by choosing a different set of base variables u such that J_{uu} is close to unitary.

The output scaling (S_1) is an important factor, especially when we have different kinds of candidate controlled variables like temperatures and flows.

The two-step approach for selecting a *combination* of measurements is able to obtain a control structure with a small implementation loss (step 1) and zero loss (perfect indirect control) for disturbances (step 2).

REFERENCES

Alstad, V. and S. Skogestad (2007). Null space method for selecting optimal measurement combinations as controlled variables. *Ind. Eng. Chem. Res.* **46**(3), 846–853.

Cao, Y., D. Rossiter and D.H. Owens (1998). Globally optimal control structure selection using branch and bound method. In: *Preprints of DYCOPS'5*. Corfu, Greece. pp. 183–188.

Halvorsen, I.J., S. Skogestad, J. Morud and V. Alstad (2003). Optimal selection of controlled variables. *Ind. Eng. Chem. Res.* **42**(14), 3273–3284.

Hori, E.S., S. Skogestad and M.A. Al Arfaj (2006). Self-optimizing control configurations for two-product distillation columns. In: *Distillation and Absorption 2006*. IChemE. London, UK.

Hori, E.S., S. Skogestad and V. Alstad (2005). Perfect steady-state indirect control. *Ind. Eng. Chem. Res.* **44**(4), 863–867.

Luyben, W.L. (2005). Effect of feed composition on the selection of control structures for high-purity binary distillation. *Ind. Eng. Chem. Res.* **44**(20), 7800–7813.

Moore, C.F. (1992). Selection of controlled and manipulated variables. In: *Practical Distillation Control, W.L. Luyben (ed.)*. Van Nostrand Reinhold. pp. 140–177.

Shinskey, F.G. (1984). *Distillation control 2nd Edition*. McGraw-Hill.

Skogestad, S. (1997). Dynamics and control of distillation columns: a tutorial introduction. *Chem. Eng. Res. Des.* **75**(A6), 539–562.

Skogestad, S. (2000). Plantwide control: the search for the self-optimizing control structure. *J. Proc. Cont.* **10**(5), 487–507.

Tolliver, T.L. and L.C. McCune (1980). Finding the optimum temperature control trays for distillation-columns. *Instrum. Tech.* **27**(9), 75–80.

Yi, C.K. and W.L. Luyben (1995). Evaluation of plant-wide control structures by steady-state disturbance sensitivity analysis. *Ind. Eng. Chem. Res.* **34**(7), 2393–2405.