CONTROL OF SYSTEMS WITH TIME-VARYING DELAYS

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Abstract: The paper presents a model-based control design method linear systems with time-varying delays. The design is based on pole placement of input-output models. Furthermore, the design of an output feedback controller is divided into two independent design tasks: a predictor design and a feedback design for non-delayed output.

Keywords: time-varying system, delay, skew polynomials, observer, controller

1. INTRODUCTION

A typical networked control system is described in Fig.1. The main problem in their design is the time-varying and often unknown delays between the controlled system (process) and the controller (N. Vatanski and Jämsä-Jounela, 2007). Even though the delays can be measured the time-variance makes the control design difficult because of the need of time-varying design methods . In this paper a design methodology based on time-varying polynomial systems theory is presented and applied to the observer and controller design.

The consideration of delayed systems in continuous time with delay-differential models is complicated and mathematically difficult. In particular, this holds for time-varying systems. Instead, in discrete time the models can be presented, at least as a good approximation, with time-varying difference equations. This results in a simpler methodology, even though the lack of continuity of signals and parameters causes some difficulties. In spite of these, the discrete-time models are used in what follows. First some basic concepts and definitions of time-varying difference systems and their interconnections as presented in (Ylinen, 1975; Ylinen, 1980) are given. The mathematical descriptions are based on linear equations over *skew polynomials* in an operator. Then the pole placement designs of observers and feedback controllers are considered (Blomberg and Ylinen, 1978; Ylinen, 1980; Blomberg and Ylinen, 1983).

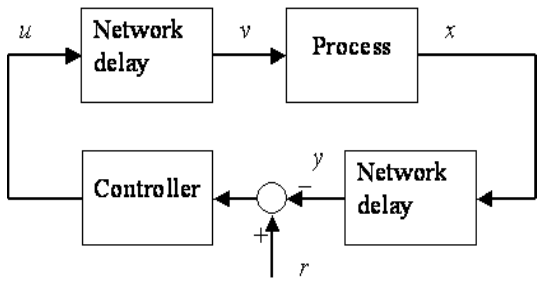

Fig. 1. Control system with network delays

The methodology is applied to the networked control system in Fig. 1. First, the overall model of the controlled system with input and output delays is constructed. Then the observers for prediction of the process output are considered and a Smith predictor type predictor is taken as an example. Finally, the feedbacks of the predicted output are parameterized.

Two numerical examples are presented. A simple process is controlled over network with time-varying delays. The operation of the time-varying Smith predictor is simulated both in open loop and in closed loop with a time-invariant PI-controller.

2. TIME-VARYING LINEAR DIFFERENCE SYSTEMS

2.1 Description of systems

Time-varying linear discrete time input-output systems are usually described by difference equations of the form

$$\sum_{i=0}^{n} a_i(k)y(k-i) = \sum_{i=0}^{m} b_i(k)u(k-i) \qquad (1)$$

where $k \in \mathbf{Z} \triangleq$ *time set* , $u,y \in \mathcal{X} \triangleq$ *signal space* $\subset \mathbf{C}^{\mathbf{Z}}$ and $a_i,b_i \in K \triangleq$ *coefficient space* $\subset \mathbf{C}^{\mathbf{Z}}$. $\mathbf{C}$, $\mathbf{Z}$ above denote the complex numbers and the integers, respectively, and $\mathbf{C}^{\mathbf{Z}}$ is the set of infinite bisequences over $\mathbf{C}$.

Provided that the signal space is closed with respect to the *(unit) delay operator* r

$$(rx)(k) = x(k-1) \qquad (2)$$

and to the pointwise multiplication by coefficients the equation (15) can be written as an operator equation

$$\underbrace{\sum a_i r^i}_{a(r)} \; y = \underbrace{\sum b_i r^i}_{b(r)} \; u \qquad (3)$$

Alternatively, the *(unit) prediction operator*

$$(qx)(k) = x(k+1) \qquad (4)$$

can be used leading to the model

$$\tilde{a}(q)y = \tilde{b}(q)u \qquad (5)$$

Note that in the case $\mathcal{X} = \mathbf{C}^{\mathbf{Z}}$ the operators r and q are invertible and $q^{-1} = r$.

Under some additional assumptions the operators $\sum c_i r^i$ constitute the (non commutative) ring $K[r; r_K, 0_K]$ of *skew polynomials* (or *skew polynomial forms*) with respect to addition

$$\sum a_i r^i + \sum b_i r^i = \sum (a_i + b_i) r^i \qquad (6)$$

and multiplication

$$(\sum a_i r^i)(\sum b_i r^i) = \sum c_i r^i \qquad (7)$$

which can be constructed by

$$rb = r_K(b) + 0(b) \qquad (8)$$

where $r_K \triangleq$ the unit delay operator on K and $0_K \triangleq$ the zero operator on K.

Similarly, the use of the (unit) prediction operator gives the skew polynomial ring $K[q; q_K, 0_K]$.

Most of the concepts and properties of ordinary polynomials can be applied to skew polynomials. Let $\mathcal{X}$ be 'sufficiently rich' to make the powers $r^0, r^1, r^2, \ldots$ *linearly independent* over K. Then the representation of a skew polynomial $a(r)$ is unique and its *degree* $\deg a(r)$ is well-defined. The choice $\mathcal{X} = \mathbf{C}^{\mathbf{Z}}$ obviously guarantees this. The same choice for coefficients K, unfortunately, leads to such kind of weak algebraic structures which do not offer any methodology and tools for consideration of system models above.

For instance, the *division algorithms*, e.g. the *(right) division algorithm (RDA)*

$$a(r) = b(r)c(r) + d(r), \; \deg d(r) < \deg b(r) \qquad (9)$$

are satisfied uniquely for *all* $a(r), b(r) \neq 0$ if and only if the coefficient ring K is a field. This is important because the division algorithms are needed for manipulation of skew polynomial matrices used in descriptions of multivariable systems. Usual coefficient rings are not fields, but often they can be *extended* to their *fields of fractions* and the signals to corresponding *rational* signals.

Note that a skew polynomial can be invertible as a skew polynomial only if it is of degree zero but there can exist skew polynomials of higher degree which are invertible as mappings (e.g r and q).

A matrix with skew polynomial entries, i.e. a skew polynomial matrix is *unimodular* if it is invertible as a skew polynomial matrix. Note that for skew polynomial matrices there is no determinant which could be used for testing the unimodularity. Furthermore, a skew polynomial matrix can be invertible as a mapping even though it is not unimodular.

Two skew polynomial matrices $A(r), B(r)$ are *row (column) equivalent* if there is a unimodular matrix $P(r)$ such that $A(r) = P(r)B(r)$ $(A(r) = B(r)P(r))$. Skew polynomial matrices can be brought to *row or column equivalent* forms e.g. to an upper triangular form using the *elementary operations*. These are: (i)add a row (column) multiplied from the left (right) by a skew polynomial to another row (column), (ii)interchange of two rows (columns), (iii)multiply a row (column) from the left (right) by an invertible skew polynomial.

2.2 Systems and compositions

A set of linear, time-varying difference equations can be written as matrix equations

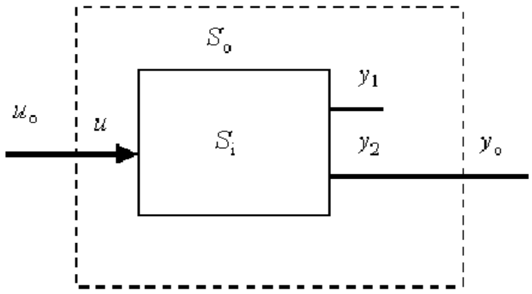

Fig. 2. General composition

$$A(r)y = B(r)u \Leftrightarrow [A(r) \vdots - B(r)] \begin{bmatrix} y \\ u \end{bmatrix} = 0 \quad (10)$$

where $u \in \mathcal{X}^r, y \in \mathcal{X}^s$ and $A(r)$, $B(r)$ are skew polynomial matrices. Then the multivariable *input-output (IO-) relation* generated by (10) is defined as the set

$$S = \{(u, y) | A(r)y = B(r)u\} \quad (11)$$

The matrix $[A(r) \vdots - B(r)]$ is called a *generator* for S. Generators for the same input-output relation are *input-output (IO-) equivalent.* Obviously, two row equivalent generators are IO-equivalent, but also multiplication by a matrix invertible as a mapping gives an IO-equivalent generator.

A *composition* of input-output relations consists of a set of input-output relations ('subsystems') or their generators, and a description of the interconnections between the (signals of the) subsystems. Every composition can be brought to the general form of Fig. 2, where S_i is the *internal IO-relation* and S_o the *overall IO-relation* generated by the composition. Conversely, the composition is then said to be a *decomposition* of S_o. Decompositions of the same IO-relation are *input-output (IO-) equivalent.*

For the internal IO-relation S_i it is always possible to construct a generator from the generators of the subsystems and the interconnections

$$\begin{bmatrix} A_1(r) & A_2(r) & \vdots & -B_1(r) \\ A_3(r) & A_4(r) & \vdots & -B_2(r) \end{bmatrix} \quad (12)$$

Instead, for the overall IO-relation

$$S_o = \{(u_o, y_o) | \exists y_1 [(u_o, (y_1, y_o)) \in S_i]\} \quad (13)$$

the construction of a generator is a more complicated task.

3. ANALYSIS OF SYSTEMS AND COMPOSITIONS

3.1 Systems

Realizability of a system model requires that the output of the model can be solved uniquely whenever the input and a sufficient, finite number of initial values of output are given. Then the model is said to *regular.*

If only past and and present values of the input are needed for solving the output, then the model is *nonanticipative* or *causal.* If the model is presented using the unit prediction operator q, then the model is causal if and only if it is *proper.* For a single-input-single-output (SISO) system (5) this means that the degree of $\tilde{a}(q)$ is not lower than the degree of $\tilde{b}(q)$.

An IO-relation S generated by $[A(r) \vdots - B(r)]$ is said to be *stable* if every solution y to $A(r)y = 0$ approaches 0 when the time t approaches the infinity.

It should be noted that in general the stability cannot be tested from the *'pointwise'* roots of $\det A(k)(r)$, where $A(k)(r)$ denotes the ordinary polynomial matrix obtained from $A(r)$ by replacing the coefficients by their values at time k.

Let S be generated by

$$[A(r) \vdots - B(r)] = L(r)[A_1(r) \vdots - B_1(r)] \quad (14)$$

Now, if $L(r)$ is not invertible, S contains modes related to $L(r)$ which cannot be affected by the input u. This means that S is not *controllable.*

3.2 Compositions

Consider the composition of Fig. 2 and suppose that the composition is regular, i.e. the internal IO-relation is regular. The generator (12) can be brought to upper triangular form

$$\begin{bmatrix} \widetilde{A}_1(r) & \widetilde{A}_2(r) & \vdots & -\widetilde{B}_1(r) \\ 0 & \widetilde{A}_4(r) & \vdots & -\widetilde{B}_2(r) \end{bmatrix} \quad (15)$$

Now if for each (u_o, y_o) satisfying the equation

$$\tilde{A}_4(r) y_o = \tilde{B}_2(r) u_o \quad (16)$$

there exists a y_1 such that $(u_o, (y_1, y_o))$ satisfies the equation

$$\tilde{A}_1(r) y_1 = -\tilde{A}_2(r) y_o + \tilde{B}_1(r) u_o \quad (17)$$

then the overall IO-relation S_o is generated by the equation (16) or by the generator $[\tilde{A}_4(r) \vdots - \tilde{B}_2(r)]$.

If $\tilde{A}_1(r)$ is invertible, then the y_1 satisfying (17) must be unique. In this case the composition is *observable*. If the system is causal, then it is always possible to take $\tilde{A}_1(r) = I$.

Consider again the composition of Fig.2. If the generator of S_i can be brought to the form

$$\begin{bmatrix} \widehat{A}_1(r) & 0 & \vdots & -\widehat{B}_1(r) \\ \widehat{A}_3(r) & I & \vdots & -\widehat{B}_2(r) \end{bmatrix} \tag{18}$$

the composition is called a *generalized state space decomposition* of S_o, and y_1 is the corresponding *generalized state.*

4. SYSTEM WITH TIME-VARYING DELAYS

4.1 Time-varying delays

Return to the system in Fig.1. Provided that the time-varying input delay is a multiple of the sampling interval, it can be presented as

$$v(k) = u(k - \theta(k)) = (r^{\theta(.)}u)(k) \tag{19}$$

This is not a skew polynomial representation but it can be written as a skew polynomial equation

$$v = (d_0 + d_1 r + ... + d_n r^n)u = d(r)u \tag{20}$$

where the coefficients are zero otherwise but

$$d_{\theta(k)}(k) = 1 \tag{21}$$

Another way to describe the time-varying delay is to use prediction

$$v(k + \vartheta(k)) = (q^{\vartheta(.)}v)(k) = u(k) \tag{22}$$

which can be written as a skew polynomial equation

$$(\tilde{c}_0 + \tilde{c}_1 q + ... + \tilde{c}_n q^n)v = \tilde{c}(q)v = u \tag{23}$$

If the delay θ and the prediction interval ϑ are related to each other by

$$\vartheta(k) = \theta(k + \vartheta(k)) \tag{24}$$

then

$$q^{\vartheta(.)}r^{\theta(.)}u = u \tag{25}$$

for all $u \in \mathcal{X}$. On the other hand, if

$$\theta(k) = \vartheta(k - \theta(k)) \tag{26}$$

then

$$r^{\theta(.)}q^{\vartheta(.)}v = v \tag{27}$$

but only if v belongs to the range of $r^{\theta(.)}$. Thus $r^{\theta(.)}$ and $q^{\vartheta(.)}$ as well as the corresponding skew polynomials are invertible mappings only if this range is the whole $\mathcal{X}$ which means that each value $u(k - \theta(k))$ appears only once in the values $v(k)$.

Similarly to the input delay above, the output delay can be written in two ways

$$y = (f_0 + f_1 r + ... + f_n r^n)x = f(r)x \tag{28}$$

$$(\tilde{e}_0 + \tilde{e}_1 q + ... + \tilde{e}_n q^n)y = \tilde{e}(q)y = x \tag{29}$$

4.2 Overall system

Now, if the process is described by (3), the controlled system is a series composition with internal model generated by

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & -d(r) \\ -b(r) & a(r) & 0 & \vdots & 0 \\ 0 & -f(r) & 1 & \vdots & 0 \end{bmatrix} \tag{30}$$

with variables (v, x, y, u). Using elementary row operations this can be brought to the form

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & -d(r) \\ 0 & a(r) & 0 & \vdots & -b(r)d(r) \\ 0 & -f(r) & 1 & \vdots & 0 \end{bmatrix} \tag{31}$$

so that v can be uniquely eliminated. Then the remaining model is written using the operator q

$$\begin{bmatrix} \tilde{a}(q) & 0 & \vdots & -\tilde{b}_d(q) \\ -1 & \tilde{e}(q) & \vdots & 0 \end{bmatrix} \tag{32}$$

Using again the elementary row operations gives

$$\begin{bmatrix} 1 & -\tilde{e}(q) & \vdots & 0 \\ 0 & \tilde{a}(q)\tilde{e}(q) & \vdots & -\tilde{b}_d(q) \end{bmatrix} \tag{33}$$

Thus x can be uniquely eliminated and the overall system is generated by

$$\begin{bmatrix} \tilde{a}(q)\tilde{e}(q) & \vdots & -\tilde{b}_d(q) \end{bmatrix} \tag{34}$$

5. OBSERVER DESIGN

5.1 General observer

Consider the composition of Fig. 2 and suppose that only the overall input $u_o = u$ and output $y_o = y_2$ are measured. The problem is to design a dynamic system, a so-called *observer* for continuous estimation of the internal output y_1, so that the estimation error $\tilde{y}_1 = y_1 - \widehat{y}_1$ behaves in a satisfactory way.

Let the internal IO-relation S_i be generated by the generator (15) of the upper triangular form and the observer $\widehat{S}$ to be designed by the generator $[C(r) \vdots -D_1(r) \quad -D_2(r)]$. In what follows, (r) (or (q)) is in some places omitted in order to shorten the notations.

If the observer is chosen to satisfy

$$\begin{bmatrix} C & -D_1 & -D_2 \\ 0 & \widetilde{A}_4 & -\widetilde{B}_2 \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} \widetilde{A}_1 & \widetilde{A}_2 & -\widetilde{B}_1 \\ 0 & \widetilde{A}_4 & -\widetilde{B}_2 \end{bmatrix} \quad (35)$$

for some T_1, T_2 then the error is generated by

$$C\tilde{y} = T_1\tilde{A}_1\tilde{y}_1 = 0 \quad (36)$$

(Ylinen, 1980; Blomberg and Ylinen, 1983). Thus the design problem has been changed to the construction of the matrices T_1, T_2. The matrix T_1 affects the stability of the estimation error and after T_1 of order high enough has been chosen the matrix T_2 is used to achieve a causal (proper, if q is used) observer. Both matrices can be constructed sequentially using the elementary row operations.

5.2 Smith predictor

The well-known Smith predictor is often used for compensating delays in control loops (N. Vatanski and Jämsä-Jounela, 2007). In the time-invariant case the design is simple but for time-varying systems more complicated. Consider the delayed system (30) and construct an observer for x using the input u and output y.

The design is started from (33). Multiplication by

$$\begin{bmatrix} \tilde{a}(q) & 1 \\ 0 & 1 \end{bmatrix} \quad (37)$$

results in the *open observer*

$$\begin{bmatrix} \tilde{a}(q) & 0 & \vdots & -\tilde{b}_d(q) \\ 0 & \tilde{a}(q)\tilde{e}(q) & \vdots & -\tilde{b}_d(q) \end{bmatrix} \quad (38)$$

Furthermore, multiplication of (33) by

$$\begin{bmatrix} \tilde{e}(q)\tilde{a}(q) & \tilde{e}(q) - 1 \\ 0 & 1 \end{bmatrix} \quad (39)$$

gives the Smith predictor

$$\begin{bmatrix} \tilde{e}(q)\tilde{a}(q) & -\tilde{a}(q)\tilde{e}(q) & \vdots & -(\tilde{e}(q) - 1)\tilde{b}_d(q) \\ 0 & \tilde{a}(q)\tilde{e}(q) & \vdots & -\tilde{b}_d(q) \end{bmatrix} \quad (40)$$

Example. Consider the system described by

$$(1 - 0.99r)x = 0.01rv \quad (41)$$

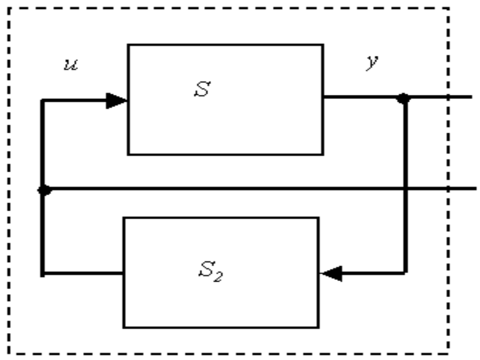

Fig. 3. General feedback control composition

The input and output delays are θ_i and θ, respectively, and corresponding predictions ϑ_i and ϑ. The resulting Smith predictor for x is

$$\begin{aligned} &\hat{x}(k) = 0.99\hat{x}(k-1) + y(k - \theta(k-1) \\ &+\vartheta(k - \theta(k-1))) - 0.99y(k-1) \\ &+0.01u(k - 1 - \theta_i(k-1)) - 0.01u(k - 1 \\ &-\theta(k-1) - \theta_i(k - 1 - \theta(k-1))) \end{aligned} \quad (42)$$

6. FEEDBACK COMPENSATOR DESIGN

6.1 General feedback

Consider the feedback composition in Fig.3 consisting of an IO-relation S to be compensated and a feedback compensator S_2 to be designed so that the resulting composition is stable, robust, realizable etc. Let S be controllable and generated by $[A \vdots -B]$ and the feedback IO-relation S_2 be generated by $[C \vdots -D]$. Then the feedback composition is generated by

$$\begin{bmatrix} A & -B \\ -D & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ T_3 & T_4 \end{bmatrix} \underbrace{\begin{bmatrix} A & -B \\ Q_3 & Q_4 \end{bmatrix}}_{Q} \quad (43)$$

where Q is invertible and can be constructed by elementary column operations and T_3, T_4 are appropriate matrices (Blomberg and Ylinen, 1983; Ylinen, 1980). The dynamic behaviour of the system depends on T_4 . Thus the feedback compensator can be designed starting from a suitable T_4 and constructing then T_3 so that the resulting feedback compensator is causal and the whole composition is robust against the parameter variations. The construction can be carried out step by step using elementary row operations.

The generalized state representations (18) can be controlled by state feedback. Instead of the state y_1 the corresponding estimate $\widehat{y}_1$ determined by the observer (35) can be used for feedback control.

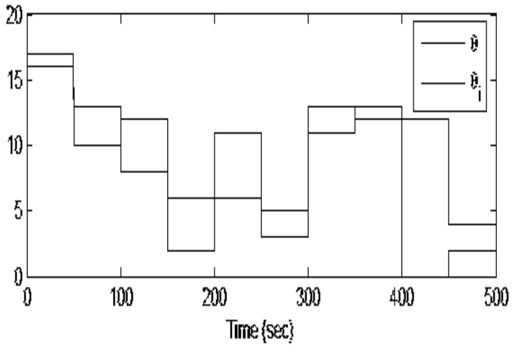

Fig. 4. Input delay θ_i and output prediction $\vartheta = \theta$

6.2 *Control of system with time-varying delays*

Consider again the system of Fig.1 and suppose that the non-delayed output x is available for feedback. Hence the design can be started from (31) or from the generator $[a(r) \vdots - b(r)d(r)]$. Suppose further that the system is controllable and the time-varying delay such that $d(r)$ is an invertible mapping.

Let the feedback controller be generated by $[l(r) \vdots - m(r)]$. Then the feedback composition is generated by

$$\begin{bmatrix} a & -bd \\ -m & l \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t_3 & t_4 \end{bmatrix} \begin{bmatrix} a & -bd \\ q_3 & q_4 d \end{bmatrix} \qquad (44)$$

Thus all possible feedback controllers can be generated by

$$[l \vdots - m] = [-t_3 bd + t_4 q_4 d \vdots t_3 a + t_4 q_3] \qquad (45)$$

varying the parameters $t_3(r)$ and $t_4(r)$.

Example. Consider a second order system described by (3) with

$$a(r) = 1 - 0.83r + 0.0055r^2 \qquad (46)$$
$$b(r) = 0.15r + 0.033r^2$$

and with time-varying input delay θ_i and output prediction $\vartheta = \theta$ presented in Fig. 4. The system is controlled by a PI controller

$$(1 - r)u = (0.055 - 0.0098r)(y_{ref} - y) \qquad (47)$$

A Smith predictor is used to compensate the output delay and it is generated by

$$\begin{aligned} &\hat{x}(k) = 0.83\hat{x}(k-1) - 0.0055\hat{x}(k-2) \\ &+y(k - \delta_1(k)) - 0.83y(k - 1 - \delta_2(k)) \\ &+0.0055y(k-2) + 0.15u(k - 1 - \delta_3(k)) \\ &+0.033u(k - 2 - \delta_4(k)) - 0.15u(k - 1 - \delta_5(k)) \\ &-0.033u(k - 2 - \delta_6(k)) \end{aligned} \qquad (48)$$

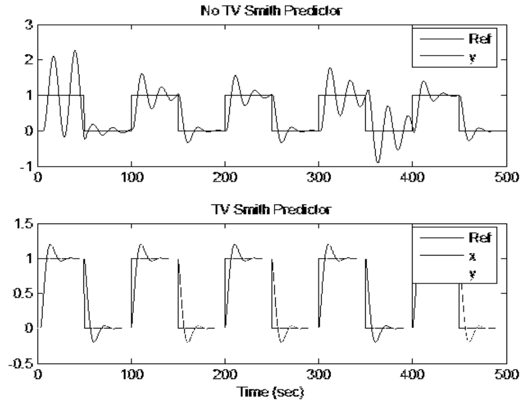

Fig. 5. Responses of PI controlled system

where $\delta_1, \delta_2, ..., \delta_6$ are functions of delays. A simulated responses of the closed loop system without and with the Smith predictor are presented in Fig.5.

7. CONCLUDING REMARKS

The time-varying polynomial systems theory gives tools for the analysis and design of linear estimators and feedback controllers. In this paper the methodology has been applied to design of predictors and controllers for delayed systems with time-varying but measurable delays. The main problems in the design are related to complicated symbolic calculation of skew polynomials. For multivariable systems the calculation must be done using special software for symbolic mathematics. Anyway, the theory gives tools to analyze the situation with respect to time-variance.

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