Efficient Computational Methods for Model Predictive Control

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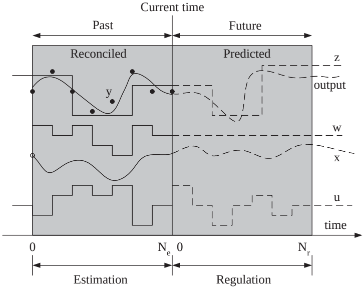
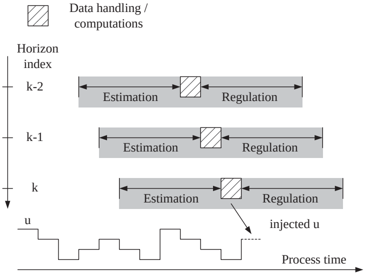
Model Predictive Control

Model Predictive Control Principle = Prediction + (Online) Optimization





Issues

- Model identification from data
- Output-based feedback (estimator)
- Regulator
- Optimization algorithm

Objectives

- Address ALL these issues of MPC by formulation of each subproblem as a Convex Optimization problem.
- Asses the achieved closed-loop performance in face of model uncertainty.

Outline

Constrained l_2-regression

Constrained l_2-regression

$$\min_x \quad \phi = \frac{1}{2}\|e\|_2^2$$
$$s.t. \quad e = Ax - b$$
$$Cx \geq d$$

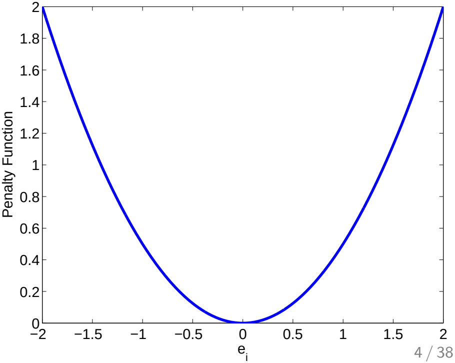
$$\phi = \frac{1}{2}\|e\|_2^2 = \sum_i \rho(e_i)$$

$$\rho = \rho(e_i) = \frac{1}{2}e_i^2$$

is a convex quadratic program

$$\min_x \quad \phi = \frac{1}{2}x'Hx + g'x + \gamma$$
$$Cx \geq d$$

$$H = A'A$$
$$g = -A'b$$
$$\gamma = \frac{1}{2}b'b$$

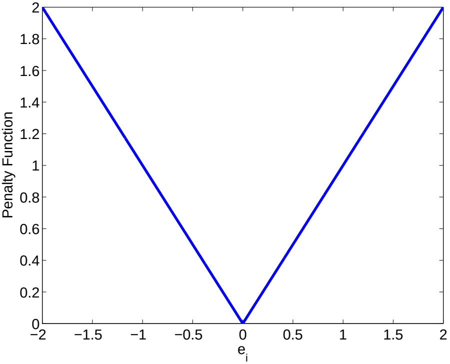


Constrained l_1-regression

Constrained l_1-regression

$$\min_x \quad \phi = \|e\|_1$$
$$s.t. \quad e = Ax - b$$
$$Cx \geq d$$

$$\phi = \|e\|_1 = \sum_i \rho(e_i)$$
$$\rho = \rho(e_i) = |e_i|$$



is a linear program

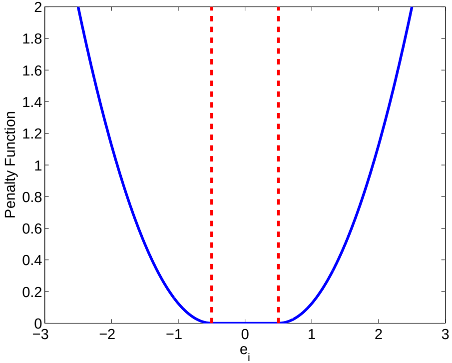
$$\min_{x,y} \quad \phi = \begin{bmatrix} 0 \\ e \end{bmatrix}' \begin{bmatrix} x \\ y \end{bmatrix}$$
$$s.t. \quad \begin{bmatrix} A & I \\ -A & I \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \geq \begin{bmatrix} b \\ -b \\ d \end{bmatrix}$$

l_2-regression with a dead zone

Constrained l_2-regression

$$\min_x \quad \phi = \sum_i \rho(e_i)$$
$$s.t. \quad e = Ax - b$$
$$Cx \geq d$$

$$\rho(e_i) = \begin{cases} 0 & |e_i| \leq \gamma \\ \frac{1}{2}(|e_i| - \gamma)^2 & |e_i| > \gamma \end{cases}$$



is a convex quadratic program

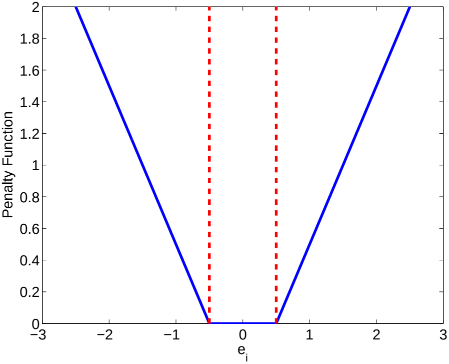
$$\min_{x,y} \quad \phi = \frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}' \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\begin{bmatrix} A & I \\ -A & I \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \geq \begin{bmatrix} b - \gamma \mathbf{1} \\ -b - \gamma \mathbf{1} \\ d \end{bmatrix}$$

Constrained l_1-regression with a dead zone

l_1-regression with deadzone

$$\min_x \quad \phi = \sum_i \rho(e_i)$$
$$s.t. \quad e = Ax - b$$
$$Cx \geq d$$

$$\rho = \rho(e_i) = \begin{cases} 0 & |e_i| \leq \gamma \\ |e_i| - \gamma & |e_i| > \gamma \end{cases}$$



is a linear program

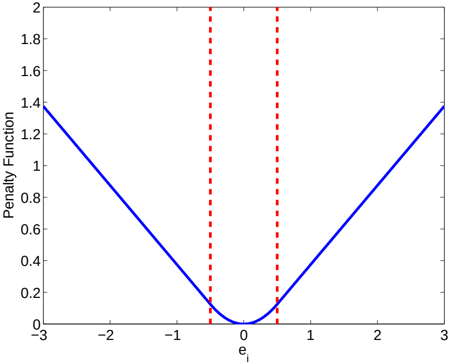
$$\min_{x,y} \quad \phi = \begin{bmatrix} 0 \\ \mathbf{1} \end{bmatrix}' \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\begin{bmatrix} A & I \\ -A & I \\ 0 & I \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \geq \begin{bmatrix} b - \gamma \mathbf{1} \\ -b - \gamma \mathbf{1} \\ 0 \\ d \end{bmatrix}$$

Huber Regression

Constrained l_2-regression

$$\min_x \quad \phi = \sum_i \rho(e_i)$$
$$s.t. \quad e = Ax - b$$
$$Cx \geq d$$

$$\rho = \rho(e_i) = \begin{cases} \frac{1}{2}e_i^2 & |e_i| \leq \gamma \\ \gamma|e_i| - \frac{1}{2}\gamma^2 & |e_i| > \gamma \end{cases}$$



is a convex quadratic program

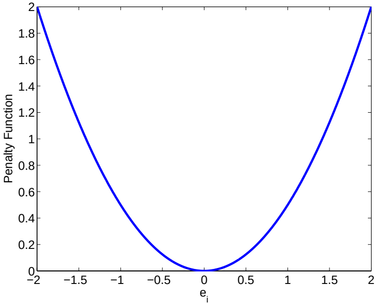
$$\min_{x,y,z} \phi = \frac{1}{2}\begin{bmatrix} x \\ y \\ z \end{bmatrix}' \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \gamma \mathbf{1} \end{bmatrix}' \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$s.t. \quad \begin{bmatrix} A & I & I \\ -A & I & I \\ 0 & 0 & I \\ C & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \geq \begin{bmatrix} b \\ -b \\ 0 \\ d \end{bmatrix}$$

Regularization

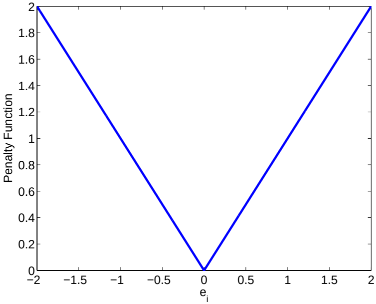
Regularization with the l_2-norm

$$\min_x \quad \phi = \sum_i \rho(e_i) + \frac{1}{2}\|Lx\|_2^2$$
$$s.t. \quad e = Ax - b$$
$$Cx \geq d$$



Regularization with the l_1-norm (Total Variation Regularization)

$$\min_x \quad \phi = \sum_i \rho(e_i) + \|Lx\|_1$$
$$s.t. \quad e = Ax - b$$
$$Cx \geq d$$



Semi Definite Programming (SDP)

$$\begin{aligned} \min_{X \in \mathcal{S}} \quad & \phi = \|AX - B\| \\ s.t. \quad & X \succeq 0 \end{aligned}$$

- This problem has application in data-driven tuning of Kalman Filters
- Same primal-dual interior point algorithm as for other convex optimization problems
- Used off-line in contrast to other MPC applications

Åkesson, Jørgensen, Poulsen, Jørgensen:
A generalized autocovariance least-squares method for Kalman filter tuning
Journal of Process Control, **18**, 2008: 769-779

Modern Convex Optimization

$$\min_{x} \quad \phi = \sum_{k=1}^{N} \rho(e_k)$$

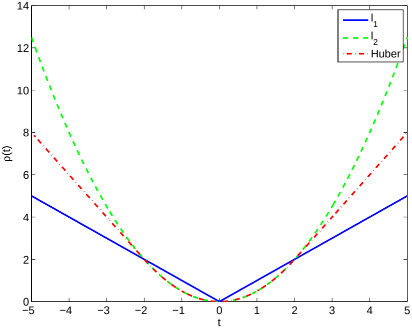
$$s.t. \quad e_k = A_k x - b_k \qquad k = 1, 2, \ldots, N$$

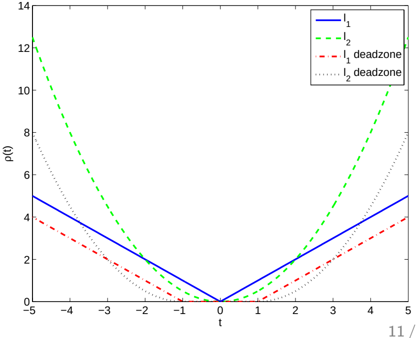
l_1 $\quad \rho(t) = \|t\|_1 \quad \rho(t) = \begin{cases} 0 & |t| \leq \gamma \\ |t| - \gamma & |t| > \gamma \end{cases}$

l_2 $\quad \rho(t) = \frac{1}{2}\|t\|_2^2 \quad \rho(t) = \begin{cases} 0 & |t| \leq \gamma \\ \frac{1}{2}(|t| - \gamma)^2 & |t| > \gamma \end{cases}$

Huber $\quad \rho(t) = \begin{cases} \frac{1}{2}t^2 & |t| \leq \gamma \\ \gamma|t| - \frac{1}{2}\gamma^2 & |t| > \gamma \end{cases}$

SOCP, SDP





Data Fitting and Approximation

$$\begin{aligned} \min_{x} \quad & \phi = \sum_{i} \rho(e_i) + \frac{1}{p} \|Lx\|_p^p \qquad p \in \{1, 2\} \\ s.t. \quad & e = Ax - b \\ & Cx \geq d \end{aligned}$$

1. Inversion Problem
2. Approximation Problem (Regression Problem)
3. Estimation Problem - parameter estimation & state estimation
4. Control Problem (Design Problem)
5. Geometric Problem

Convex Quadratic Program

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \frac{1}{2} x' H x + g' x \\ s.t. \quad & Ax \geq b \end{aligned}$$

Theorem (KKT conditions)

$$\begin{aligned} r_L &= Hx + g - A'\lambda = 0 \\ r_s &= s - Ax + b \qquad = 0 \\ r_{s\lambda} &= S\Lambda e \qquad\qquad = 0 \\ s &\geq 0 \\ \lambda &\geq 0 \end{aligned}$$

$$s = Ax - b \geq 0$$

$$S = \begin{bmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

while $Not\_Converged$ **do**

Compute: $\bar{H} = H + A'(S^{-1}\Lambda)A$

Cholesky factorization: $\bar{H} = \bar{L}\bar{L}'$

Affine Predictor Step:

Compute: $\bar{r} = A'(S^{-1}(r_{s\lambda} - \Lambda r_s))$, $-\bar{g} = -(r_L + \bar{r})$

Solve: $\bar{L}\bar{L}'\Delta x = -\bar{g}$

Compute: $\Delta s = -r_s + A\Delta x$

Compute: $\Delta\lambda = -S^{-1}(r_{s\lambda} + \Lambda\Delta s)$

Determine the maximum affine step length: $\lambda + \alpha_{\text{max}}\Delta\lambda \geq 0 \quad s + \alpha_{\text{max}}\Delta s \geq 0$

Select affine step length: $\alpha \in (0, \alpha_{\text{max}}]$

Compute affine duality gap: $\mu_a = \frac{(\lambda+\alpha\Delta\lambda)'(s+\alpha\Delta s)}{m}$

Centering parameter: $\sigma = \left(\frac{\mu_a}{\mu}\right)^3$

Center Corrector Step:

Modified complementarity: $r_{s\lambda} \leftarrow r_{s\lambda} + \Delta S\Delta\Lambda e - \sigma\mu e$

Compute $\bar{r} = A'(S^{-1}(r_{s\lambda} - \Lambda r_s))$, $-\bar{g} = -(r_L + \bar{r})$

Solve: $\bar{L}\bar{L}'\Delta x = -\bar{g}$

Compute: $\Delta s = -r_s + A\Delta x$

Compute: $\Delta\lambda = -S^{-1}(r_{s\lambda} + \Lambda\Delta s)$

Determine the maximum affine step length: $\lambda + \alpha_{\text{max}}\Delta\lambda \geq 0 \quad s + \alpha_{\text{max}}\Delta s \geq 0$

Select affine step length: $\alpha \in (0, \alpha_{\text{max}}]$

Step: $x \leftarrow x + \alpha\Delta x$, $\lambda \leftarrow \lambda + \alpha\Delta\lambda$, $s \leftarrow s + \alpha\Delta s$

Residuals and Duality Gap:

$r_L = Hx + g - A'\lambda$, $r_s = s - Ax + b$, $r_{s\lambda} = S\Lambda e$

Duality gap: $\mu = \frac{s'\lambda}{m}$

end while

Major Computations in PD Interior Point Algorithm

Standard QP

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \frac{1}{2} x' H x + g' x \\ s.t. \quad & Ax \geq b \end{aligned}$$

1. Compute modified Hessian

$$\bar{H} = H + A' D A, \quad D = S^{-1} \Lambda$$

2. Compute Cholesky factorization

$$\bar{H} = \bar{L} \bar{L}'$$

l_2-regression with a deadzone

$$\min_{x,y} \quad \phi = \frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}' \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} A & I \\ -A & I \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \geq \begin{bmatrix} b - \gamma \mathbf{1} \\ -b - \gamma \mathbf{1} \\ d \end{bmatrix}$$

1. Compute modified Hessian
2. Cholesky factorization

l_2-regression with a deadzone

$$\min_{x,y} \quad \phi = \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix}' \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} A & I \\ -A & I \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \geq \begin{bmatrix} b - \gamma \mathbf{1} \\ -b - \gamma \mathbf{1} \\ d \end{bmatrix}$$

1. Compute modified Hessian

$$\bar{H} = \begin{bmatrix} A'(D_1 + D_2)A + C'D_3C & A'(D_1 - D_2) \\ (D_1 - D_2)A & D_1 + D_2 \end{bmatrix} = \begin{bmatrix} \bar{H}_{11} & \bar{H}_{12} \\ \bar{H}_{21} & D \end{bmatrix}$$

2. Cholesky factorization - by use of Schur complement

$$\hat{H}_{xx} = \bar{H}_{11} - \bar{H}_{12}D^{-1}\bar{H}_{21} = A'\hat{D}A + C'D_3C = \hat{L}\hat{L}'$$

$$\hat{L}\hat{L}'x = \bar{b}_1 - \bar{H}_{12}D^{-1}b_2$$

$$y = D^{-1}(\bar{b}_2 - \bar{H}_{21}x)$$

Finite Impulse Response Model

$$z_k = b_k + \sum_{i=1}^{n} H_i u_{k-i}$$

The state space model

$$x_{k+1} = Ax_k + Bu_k$$
$$z_k = Cx_k$$

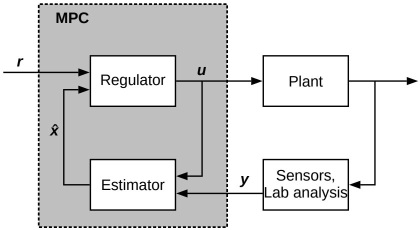
can be expressed as the impulse response coefficient model

$$z_k = CA^k x_0 + \sum_{i=1}^{k} CA^{i-1} B u_{k-i}$$
$$= b_k + \sum_{i=1}^{k} H_i u_{k-i} \approx b_k + \sum_{i=1}^{n} H_i u_{k-i}$$

with

$$H_i = CA^{i-1}B$$

FIR based MPC for Linear Systems



Plant:

$$
\begin{aligned}
\boldsymbol{x}_{k+1} &= A\boldsymbol{x}_k + Bu_k + B_d d_k + G\boldsymbol{w}_k \\
\boldsymbol{z}_k &= C\boldsymbol{x}_k \\
\boldsymbol{x}_0 &\sim N(\bar{x}_0, P_0) \quad \boldsymbol{w}_k \sim N_{iid}(0, Q)
\end{aligned}
$$

Sensors:

$$
\boldsymbol{y}_k = \boldsymbol{z}_k + \boldsymbol{v}_k \qquad \boldsymbol{v}_k \sim N_{iid}(0, R)
$$

Model:

$$
z_k = b_k + \sum_{i=1}^{n} H_i u_{k-i}
$$

Estimator:

$$
\begin{aligned}
e_k &= \Delta y_k - \sum_{i=1}^{n} H_i \Delta u_{k-i} \\
\hat{b}_k &= \hat{b}_{k-1} + e_k
\end{aligned}
$$

Regulator:

$$
\min_{\{z,u\}} \quad \phi = \frac{1}{2} \sum_{k=0}^{N-1} \|z_{k+1} - r_{k+1}\|_{Q_z}^2 + \|\Delta u_k\|_S^2
$$

$$
\begin{aligned}
s.t. \quad & z_k = b_k + \sum_{i=1}^{n} H_i u_{k-i} & k = 1, \ldots, N \\
& u_{\min} \leq u_k \leq u_{\max} & k = 0, \ldots, N-1 \\
& \Delta u_{\min} \leq \Delta u_k \leq \Delta u_{\max} & k = 0, \ldots, N-1
\end{aligned}
$$

Regulator = Convex QP

$$\min_{\{z,u\}} \quad \phi = \frac{1}{2}\sum_{k=0}^{N-1} \|z_{k+1} - r_{k+1}\|_{Q_z}^2 + \|\Delta u_k\|_S^2$$

$$\begin{aligned} s.t. \quad & z_k = b_k + \sum_{i=1}^{n} H_i u_{k-i} && k = 1, \dots, N \\ & u_{\text{min}} \le u_k \le u_{\text{max}} && k = 0, \dots, N-1 \\ & \Delta u_{\text{min}} \le \Delta u_k \le \Delta u_{\text{max}} && k = 0, \dots, N-1 \end{aligned}$$

- Finite Horizon MPC
- Gives desired performance when large disturbances renders the setpoint infeasible.
- Requires large horizon, N, and therefore efficient computational procedures

Rawlings, Bonnè, Jørgensen, Venkat, Jørgensen:
Unreachable setpoints in model predictive control

IEEE Transactions on Automatic Control, **53**, 2008: 2209-2215

FIR Model as Linear Predictor

FIR model

$$z_k = b_k + \sum_{i=1}^{n} H_i u_{k-i} \qquad k = 1, 2, \ldots, N$$

Define the vectors Z, R, and U as

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix} \quad R = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix} \quad U = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

FIR predictions as linear (affine) relation (N=6, n=3)

$$Z = c + \Gamma U$$

$$c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} = \begin{bmatrix} b_1 + (H_2 u_{-1} + H_3 u_{-2}) \\ b_2 + (H_3 u_{-1}) \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} \qquad \Gamma = \begin{bmatrix} H_1 & 0 & 0 & 0 & 0 & 0 \\ H_2 & H_1 & 0 & 0 & 0 & 0 \\ H_3 & H_2 & H_1 & 0 & 0 & 0 \\ 0 & H_3 & H_2 & H_1 & 0 & 0 \\ 0 & 0 & H_3 & H_2 & H_1 & 0 \\ 0 & 0 & 0 & H_3 & H_2 & H_1 \end{bmatrix}$$

For the case $N = 6$, define the matrices Λ and I_0 by

$$\Lambda = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ -I & I & 0 & 0 & 0 & 0 \\ 0 & -I & I & 0 & 0 & 0 \\ 0 & 0 & -I & I & 0 & 0 \\ 0 & 0 & 0 & -I & I & 0 \\ 0 & 0 & 0 & 0 & -I & I \end{bmatrix} \qquad I_0 = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

then

$$\begin{bmatrix} \Delta u_0 \\ \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \\ \Delta u_4 \\ \Delta u_5 \end{bmatrix} = \begin{bmatrix} u_0 - u_{-1} \\ u_1 - u_0 \\ u_2 - u_1 \\ u_3 - u_2 \\ u_4 - u_3 \\ u_5 - u_4 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ -I & I & 0 & 0 & 0 & 0 \\ 0 & -I & I & 0 & 0 & 0 \\ 0 & 0 & -I & I & 0 & 0 \\ 0 & 0 & 0 & -I & I & 0 \\ 0 & 0 & 0 & 0 & -I & I \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} - \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_{-1}$$

or

$$\Delta U = \Lambda U - I_0 u_{-1}$$

Define

$$\mathcal{Q}_z = \begin{bmatrix} Q_z & & & & & \\ & Q_z & & & & \\ & & Q_z & & & \\ & & & Q_z & & \\ & & & & Q_z & \\ & & & & & Q_z \end{bmatrix} \qquad \mathcal{S} = \begin{bmatrix} S & & & & & \\ & S & & & & \\ & & S & & & \\ & & & S & & \\ & & & & S & \\ & & & & & S \end{bmatrix}$$

and remember

$$Z = c + \Gamma U$$
$$\Delta U = \Lambda U - I_0 u_{-1}$$

Then the objective function may be expressed as

$$\begin{aligned} \phi &= \frac{1}{2} \sum_{k=0}^{N-1} \|z_{k+1} - r_{k+1}\|_{Q_z}^2 + \|\Delta u_k\|_S^2 \\ &= \frac{1}{2} \|Z - R\|_{\mathcal{Q}_z}^2 + \frac{1}{2} \|\Lambda U - I_0 u_{-1}\|_{\mathcal{S}}^2 \\ &= \frac{1}{2} \|c + \Gamma U - R\|_{\mathcal{Q}_z}^2 + \frac{1}{2} \|\Lambda U - I_0 u_{-1}\|_{\mathcal{S}}^2 \end{aligned}$$

Objective Function in Regulator

$$
\begin{aligned}
\phi &= \frac{1}{2}\sum_{k=0}^{N-1} \|z_{k+1} - r_{k+1}\|_{Q_z}^2 + \|\Delta u_k\|_S^2 \\
&= \frac{1}{2}\|Z - R\|_{\mathcal{Q}_z}^2 + \frac{1}{2}\|\Lambda U - I_0 u_{-1}\|_{\mathcal{S}}^2 \\
&= \frac{1}{2}\|c + \Gamma U - R\|_{\mathcal{Q}_z}^2 + \frac{1}{2}\|\Lambda U - I_0 u_{-1}\|_{\mathcal{S}}^2 \\
&= \frac{1}{2}U'\left(\Gamma'\mathcal{Q}_z\Gamma + \Lambda'\mathcal{S}\Lambda\right)U \\
&\quad + \left(\Gamma'\mathcal{Q}_z(c - R) - \Lambda'\mathcal{S}I_0 u_{-1}\right)' U \\
&\quad + \left(\frac{1}{2}\|c - R\|_{\mathcal{Q}_z}^2 + \frac{1}{2}\|I_0 u_{-1}\|_{\mathcal{S}}^2\right) \\
&= \frac{1}{2}U'HU + g'U + \rho
\end{aligned}
$$

$$
\begin{aligned}
H &= \Gamma'\mathcal{Q}_z\Gamma + \Lambda'\mathcal{S}\Lambda \\
g &= \Gamma'\mathcal{Q}_z(c - R) - \Lambda'\mathcal{S}I_0 u_{-1}
\end{aligned}
$$

Regulator = Convex QP

The regulator optimization problem

$$\min_{\{z,u\}} \quad \phi = \frac{1}{2}\sum_{k=0}^{N-1} \|z_{k+1} - r_{k+1}\|_{Q_z}^2 + \|\Delta u_k\|_S^2$$

$$\begin{aligned} s.t. \quad & z_k = b_k + \sum_{i=1}^{n} H_i u_{k-i} && k = 1, \dots, N \\ & u_{\min} \leq u_k \leq u_{\max} && k = 0, \dots, N-1 \\ & \Delta u_{\min} \leq \Delta u_k \leq \Delta u_{\max} && k = 0, \dots, N-1 \end{aligned}$$

can be expressed as a convex QP

$$\begin{aligned} \min_{U} \quad & \psi = \frac{1}{2}U'HU + g'U \\ s.t. \quad & U_{\min} \leq U \leq U_{\max} \\ & b_l \leq \Lambda U \leq b_u \end{aligned}$$

$$\begin{aligned} H &= \Gamma' \mathcal{Q}_z \Gamma + \Lambda' \mathcal{S} \Lambda \\ g &= \Gamma' \mathcal{Q}_z (c - R) - \Lambda' \mathcal{S} I_0 u_{-1} \\ b_l &= \Delta U_{\min} + I_0 u_{-1} \\ b_u &= \Delta U_{\max} + I_0 u_{-1} \end{aligned}$$

l_2 MPC quadratic program

$$\begin{aligned} \min_{U} \quad & \psi = \frac{1}{2}U'HU + g'U \\ s.t. \quad & U_{\text{min}} \leq U \leq U_{\text{max}} \\ & b_l \leq \Lambda U \leq b_u \end{aligned}$$

Structured matrix multiplication

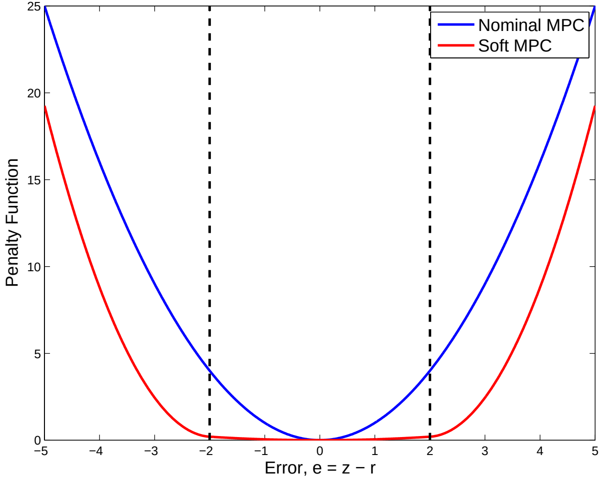
$$\Lambda = \begin{bmatrix} I & 0 & 0 & 0 \\ -I & I & 0 & 0 \\ 0 & -I & I & 0 \\ 0 & 0 & -I & I \end{bmatrix} \qquad D = \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & D_3 & \\ & & & D_4 \end{bmatrix}$$

$$\Lambda' D \Lambda = \begin{bmatrix} D_1 + D_2 & -D_2 & 0 & 0 \\ -D_2 & D_2 + D_3 & -D_3 & 0 \\ 0 & -D_3 & D_3 + D_4 & -D_4 \\ 0 & 0 & -D_4 & D_4 \end{bmatrix}$$

Major computation operations in the interior-point algorithm: $C'DC$

$$\bar{H} = H + D_{U_{\text{min}}} + D_{U_{\text{max}}} + \Lambda'(D_{\Delta U_{\text{min}}} + D_{\Delta U_{\text{max}}})\Lambda$$
$$\bar{H} = \bar{L}\bar{L}'$$

Penalty Function for Soft MPC



$$\rho(z,r) = \|z-r\|_{2,Q}^2 + \|\min(z-z_{\min},0)\|_{2,S_e}^2 + \|\max(z-z_{\max},0)\|_{2,S_e}^2$$

Soft MPC

Mathematical program

$$\begin{aligned} \min \quad & \phi = \frac{1}{2} \sum_{k=0}^{N-1} \rho(z_{k+1}, r_{k+1}) + \|\Delta u_k\|_{2,S}^2 & \\ s.t. \quad & z_k = b_k + \sum_{i=1}^{n} H_i u_{k-i} & k = 1, 2, \ldots, N \\ & u_{\text{min}} \leq u_k \leq u_{\text{max}} & k = 0, 1, \ldots, N-1 \\ & \Delta u_{\text{min}} \leq \Delta u_k \leq \Delta u_{\text{max}} & k = 1, 2, \ldots, N-1 \end{aligned}$$

with the penalty function

$$\rho(z, r) = \|z - r\|_{2,Q}^2 + \|\min(z - z_{\text{min}}, 0)\|_{2,S_e}^2 + \|\max(z - z_{\text{max}}, 0)\|_{2,S_e}^2$$

Prasath, Jørgensen:
Soft Constraints for Robust MPC of Uncertain Systems

submitted to ADCHEM 2009

Soft MPC

$$\min_{\{u_k, w_{k+1}\}_{k=0}^{N-1}} \quad \phi = \frac{1}{2} \sum_{k=0}^{N-1} \|z_{k+1} - r_{k+1}\|_{2,Q}^2 + \|w_{k+1}\|_{2,S_e}^2 + \|\Delta u_k\|_{2,S}^2$$

$$\begin{aligned}
s.t. \quad & z_k = b_k + \sum_{i=1}^{n} H_i u_{k-i} && k = 1, 2, \ldots, N \\
& u_{\text{min}} \leq u_k \leq u_{\text{max}} && k = 0, 1, \ldots, N-1 \\
& \Delta u_{\text{min}} \leq \Delta u_k \leq \Delta u_{\text{max}} && k = 0, 1, \ldots, N-1 \\
& z_k + w_k \geq z_{\text{min}} && k = 1, 2, \ldots, N \\
& z_k - w_k \geq z_{\text{max}} && k = 1, 2, \ldots, N \\
& w_k \geq 0 && k = 1, 2, \ldots, N
\end{aligned}$$

$$\begin{aligned}
\min_{U,W} \quad & \phi = \frac{1}{2}\begin{bmatrix} U \\ W \end{bmatrix}' \begin{bmatrix} H_{uu} & 0 \\ 0 & H_{ww} \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix} + \begin{bmatrix} g_u \\ g_w \end{bmatrix}' \begin{bmatrix} U \\ W \end{bmatrix} \\
s.t. \quad & U_{\mathsf{min}} \leq U \leq U_{\mathsf{max}} \\
& 0 \leq W \\
& \Delta U_{\mathsf{min}} + I_0 u_{-1} \leq \Lambda U \leq \Delta U_{\mathsf{max}} + I_0 u_{-1} \\
& \Gamma U + W \geq Z_{\mathsf{min}} - c \\
& \Gamma U - W \leq Z_{\mathsf{max}} - c
\end{aligned}$$

$$H_{uu} = \Gamma' \mathcal{Q} \Gamma + \Lambda' \mathcal{S} \Lambda \qquad H_{ww} = \mathcal{S}_e = I_N \otimes S_e$$

$$g_u = \Gamma' \mathcal{Q}_z (c - R) - \Lambda' \mathcal{S} I_0 u_{-1} \qquad g_w = 0$$

$$\Gamma = \begin{bmatrix} H_1 & 0 & 0 & 0 & 0 & 0 \\ H_2 & H_1 & 0 & 0 & 0 & 0 \\ H_3 & H_2 & H_1 & 0 & 0 & 0 \\ 0 & H_3 & H_2 & H_1 & 0 & 0 \\ 0 & 0 & H_3 & H_2 & H_1 & 0 \\ 0 & 0 & 0 & H_3 & H_2 & H_1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ -I & I & 0 & 0 & 0 & 0 \\ 0 & -I & I & 0 & 0 & 0 \\ 0 & 0 & -I & I & 0 & 0 \\ 0 & 0 & 0 & -I & I & 0 \\ 0 & 0 & 0 & 0 & -I & I \end{bmatrix} \quad I_0 = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\min_{U,W} \quad \phi = \frac{1}{2} \begin{bmatrix} U \\ W \end{bmatrix}' \begin{bmatrix} H_{uu} & 0 \\ 0 & H_{ww} \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix} + \begin{bmatrix} g_u \\ g_w \end{bmatrix}' \begin{bmatrix} U \\ W \end{bmatrix}$$

$$\begin{aligned} s.t. \quad & U_{\min} \leq U \leq U_{\max} \\ & 0 \leq W \\ & \Delta U_{\min} + I_0 u_{-1} \leq \Lambda U \leq \Delta U_{\max} + I_0 u_{-1} \\ & \Gamma U + W \geq Z_{\min} - c \\ & \Gamma U - W \leq Z_{\max} - c \end{aligned}$$

$$\Gamma = \begin{bmatrix} H_1 & 0 & 0 & 0 & 0 & 0 \\ H_2 & H_1 & 0 & 0 & 0 & 0 \\ H_3 & H_2 & H_1 & 0 & 0 & 0 \\ 0 & H_3 & H_2 & H_1 & 0 & 0 \\ 0 & 0 & H_3 & H_2 & H_1 & 0 \\ 0 & 0 & 0 & H_3 & H_2 & H_1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ -I & I & 0 & 0 & 0 & 0 \\ 0 & -I & I & 0 & 0 & 0 \\ 0 & 0 & -I & I & 0 & 0 \\ 0 & 0 & 0 & -I & I & 0 \\ 0 & 0 & 0 & 0 & -I & I \end{bmatrix} \quad I_0 = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Modified Hessian matrix in interior point algorithm

$$\hat{H}_{uu} = H_{uu} + \Gamma' \hat{D} \Gamma + D_u + \Lambda' D_{\Delta u} \Lambda$$

Example

Consider plants of the form

$$\mathbf{Z}(s) = G(s)U(s) + G_d(s)(D(s) + \mathbf{W}(s))$$
$$\mathbf{y}(t_k) = \mathbf{z}(t_k) + \mathbf{v}(t_k)$$

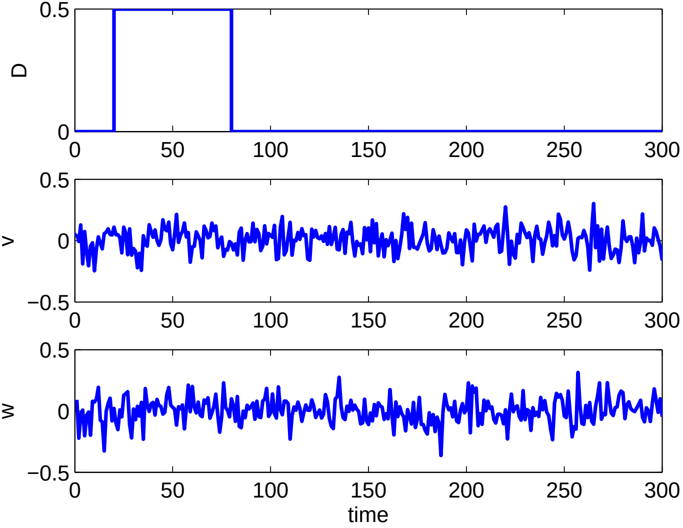
with the transfer functions

$$G(s) = \frac{K(\beta s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)} e^{-\tau s}$$
$$G_d(s) = \frac{K_d(\beta_d s + 1)}{(\tau_{d1} s + 1)(\tau_{d2} s + 1)} e^{-\tau_d s}$$

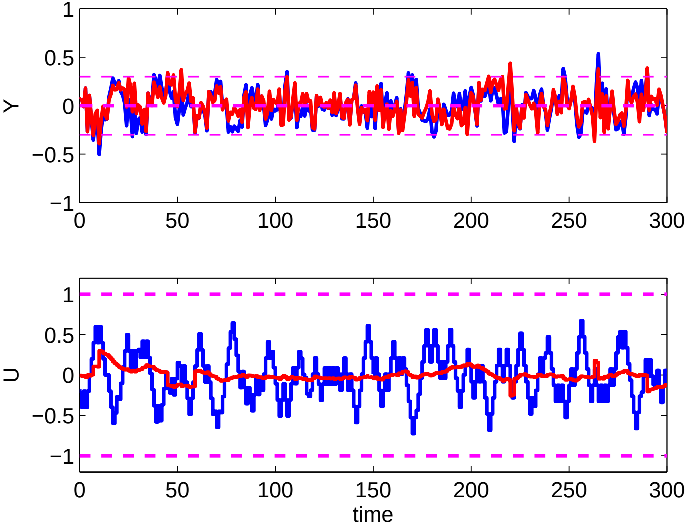
Nominal system

$$K = K_d = 1$$
$$\tau_1 = \tau_2 = \tau_{d1} = \tau_{d2} = 5$$
$$\beta = \beta_d = 2$$
$$\tau = \tau_d = 5$$

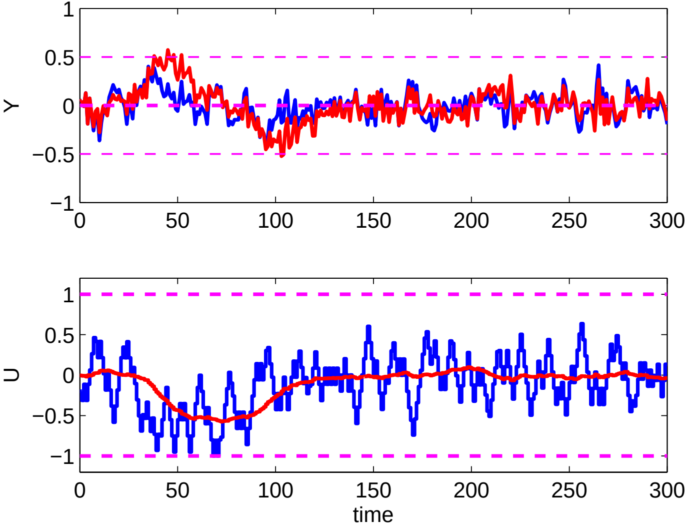
Disturbance & Process and Measurement Noise

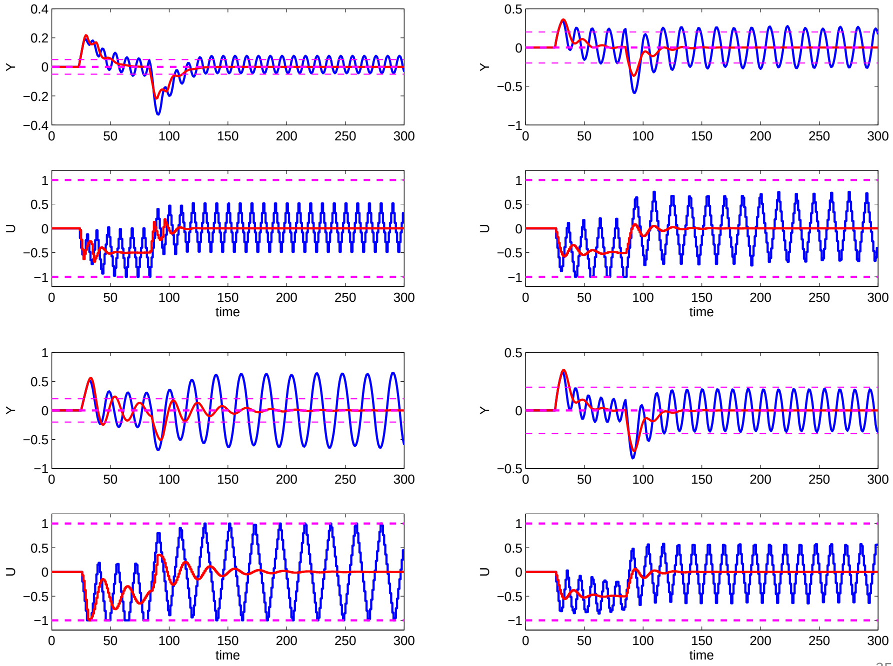


Closed Loop Performance for Stochastic Noise



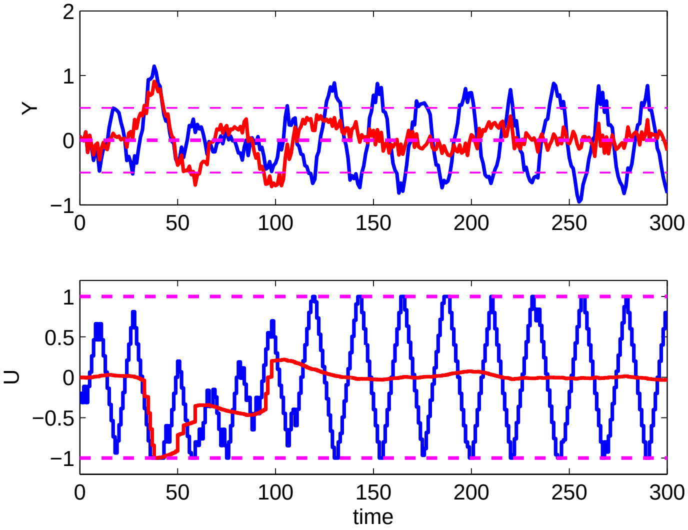
Closed Loop Performance for Nominal System





Y
U
time

Soft MPC for Stochastic System with Uncertain Gain



Conclusions

- Norms other than the 2-norm may be better suited for MPC of uncertain systems. The performance improvement can be significant.
- Contributes to longer life time / easier maintenance of MPC systems
- Primal-dual interior point algorithms tailored for MPC at least one magnitude faster than the off the shelf algorithms

Other issues not discussed in this talk

- Riccati based solvers utilizing the stair-case structure
- Issues related to nonlinear MPC
- Implementation on modern HPC architecture (e.g. GPU)

Model Predictive Control course in June at DTU

Questions and Comments

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