A Dantzig-Wolfe Decomposition Algorithm for Economic MPC of Distributed Energy Systems

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Abstract: In economic model predictive control of distributed energy systems, the constrained optimal control problem can be expressed as a linear program with a block-angular structure. In this paper, we present an efficient Dantzig-Wolfe decomposition algorithm specifically tailored to problems of this type. Simulations show that a MATLAB implementation of the algorithm is significantly faster than several state-of-the-art linear programming solvers and that it scales in a favorable way.

Keywords: Predictive control, Decomposition methods, Renewable energy systems

1. INTRODUCTION

Due to global concerns related to environmental issues and security of supply, an increasing amount of renewable energy sources is being integrated in the power grid. Accordingly, methods for power production planning that can handle the volatile and unpredictable power generation associated with technologies such as wind, solar and hydropower are required. For this reason, energy systems management has emerged as a promising application area for economic model predictive control (MPC). In economic MPC of energy systems, the power production planning is handled in real-time by an optimization algorithm that computes an optimal production plan based on the most recent information available such as forecasts of energy prices, wind power production, and district heating consumption. Examples of economic MPC in energy systems management include cost-efficient control of refrigeration systems (Hovgaard et al., 2011), building climate control (Ma et al., 2011; Halvgaard et al., 2012a), and optimal charging strategies for batteries in electric vehicles (Halvgaard et al., 2012b).

Economic MPC is a receding horizon control strategy, and requires the solution of a linear program in every sampling instant. In energy systems management, the solution to this linear problem, known as the optimal control problem, provides a sequence of control moves that yields the most cost-efficient power generation respecting system dynamics, capacity constraints and electricity demand, with respect to a process model of the power system. To compensate for non-predictable disturbances and discrepancies between the process model and the true system only the first input in the sequence of control moves is applied to the system, and the optimization procedure is repeated using updated information at the following sampling instant.

As the control moves are computed in real-time, one of the key challenges in economic MPC is to solve the optimal control problem in an efficient and reliable way. The main contribution of this paper is an algorithm for control of distributed energy systems that satisfies these criteria. Our algorithm exploits that the units in a distributed energy system are dynamically decoupled. This gives rise to a block-angular structure in the optimal control problem that allows it to be decomposed, into a master problem and a number of subproblems, using Dantzig-Wolfe decomposition (Dantzig and Wolfe, 1960, 1961). To solve the decomposed problem efficiently, we use a column generation procedure, which is warm-started by a strategy that utilizes problem specific features.

Previously, Dantzig-Wolfe decomposition algorithms have been applied to MPC applications in Edlund et al. (2011); Cheng et al. (2008, 2007); Morosan et al. (2011). Cheng et al. (2008) uses Dantzig-Wolfe decomposition to coordinate the target calculation in set-point based MPC with ℓ_1-penalty, and similar work for ℓ_2-penalty is conducted in Cheng et al. (2007). Examples in energy systems management are provided in e.g. Edlund et al. (2011) in which a hierarchical control structure based on Dantzig-Wolfe decomposition is proposed, and in Morosan et al. (2011) that uses a Dantzig-Wolfe decomposition algorithm for building climate control.

1.1 Paper Organization

This paper is organized as follows. In Section 2, we introduce the optimal control problem solved in economic MPC, and a compact problem formulation is derived. We apply Dantzig-Wolfe decomposition to this problem in Section 3. In Section 4, we present optimality conditions for the decomposed problem, and we propose a warm-started column generation procedure for solving the problem. Performance benchmarks for the proposed algorithm based on a conceptual energy systems management case study are provided in Section 5. We give concluding remarks in Section 6.

2. PROBLEM DEFINITION

We consider an electrical grid with M distributed power generating units. The units are modelled as discrete state space systems in the form

$$x_{j,k+1} = A_j x_{j,k} + B_j u_{j,k}, \qquad j \in \mathcal{M}, \tag{1a}$$
$$y_{j,k} = C_j x_{j,k}, \qquad j \in \mathcal{M}, \tag{1b}$$

where $\mathcal{M} = \{1, 2, \dots, M\}$. The state space matrices are denoted as (A_j, B_j, C_j), the states as $x_{j,k} \in \mathbb{R}^{n_x(j)}$, the inputs as $u_{j,k} \in \mathbb{R}^{n_u(j)}$, and the outputs as $y_{j,k} \in \mathbb{R}^{n_y(j)}$.

Assuming that the power production is available as a linear combination of the outputs in (1), the total power production can be written as

$$y_{T,k} = \sum_{j\in\mathcal{M}} \Upsilon_j y_{j,k} = \sum_{j\in\mathcal{M}} \Upsilon_j C_j x_{j,k}, \tag{2}$$

in which $\Upsilon_j \in \mathbb{R}^{1\times n_y(j)}$ is a row vector such that $\Upsilon_j C_j x_{j,k}$ is the power production of unit j at time step k.

Economic MPC defines a control law for the generating units (1), that optimizes the inputs (control moves) with respect to an economic objective function, input limits, input rate limits and soft output limits. Evaluating this control law requires the solution to the minimization problem

$$\min_{u,x,y,y_T,\rho,\gamma} \quad \sum_{k\in\mathcal{N}_0} q_{k+1}^T \rho_{k+1} + \sum_{j\in\mathcal{M}} p_{j,k}^T u_{j,k} + r_{j,k+1}^T \gamma_{j,k+1}, \tag{3a}$$

subject to the constraints

$$x_{j,k+1} = A_j x_{j,k} + B_j u_{j,k}, \qquad k \in \mathcal{N}_0, \ j \in \mathcal{M}, \tag{3b}$$
$$y_{j,k} = C_j x_{j,k}, \qquad k \in \mathcal{N}_1, \ j \in \mathcal{M}, \tag{3c}$$
$$y_{T,k} = \sum_{j\in\mathcal{M}} \Upsilon_j C_j x_{j,k}, \qquad k \in \mathcal{N}_1, \tag{3d}$$
$$\underline{u}_{j,k} \le u_{j,k} \le \overline{u}_{j,k}, \qquad k \in \mathcal{N}_0, \ j \in \mathcal{M}, \tag{3e}$$
$$\Delta\underline{u}_{j,k} \le u_{j,k} - u_{j,k-1} \le \Delta\overline{u}_{j,k}, \qquad k \in \mathcal{N}_0, \ j \in \mathcal{M}, \tag{3f}$$
$$\underline{y}_{j,k} - \gamma_{j,k} \le y_{j,k} \le \overline{y}_{j,k} + \gamma_{j,k}, \qquad k \in \mathcal{N}_1, \ j \in \mathcal{M}, \tag{3g}$$
$$0 \le \gamma_{j,k} \le \overline{\gamma}_{j,k}, \qquad k \in \mathcal{N}_1, \ j \in \mathcal{M}, \tag{3h}$$
$$\underline{y}_{T,k} - \rho_k \le y_{T,k} \le \overline{y}_{T,k} + \rho_k, \qquad k \in \mathcal{N}_1, \tag{3i}$$
$$0 \le \rho_k \le \overline{\rho}, \qquad k \in \mathcal{N}_1, \tag{3j}$$

where $\mathcal{N}_i = \{0+i, 1+i, \dots, N-1+i\}$, with N being the length of the prediction horizon. The input data are the input limits, $(\underline{u}_{j,k}, \overline{u}_{j,k})$, the input rate limits, $(\Delta\underline{u}_{j,k}, \Delta\overline{u}_{j,k})$, the output limits associated with the generating units, $(\underline{y}_{j,k}, \overline{y}_{j,k})$, the output limits associated with the total power production, $(\underline{y}_{T,k}, \overline{y}_{T,k})$, the input prices, $p_{j,k}$, the price for violating the output constraints associated with the generating units, $r_{j,k}$, and the price for violating the output constraints associated with the total power production q_k. We also include upper limits on the variables $\gamma_{j,k}$ and ρ_k, as this simplifies later computations considerably.

Notice that if process noise or measurement noise is present in the model (1), an optimization problem in the form (3) can be derived using the Kalman filter under the certainty equivalence assumption.

2.1 Compact Formulation

By eliminating the states using equation (1a), we can write the output equation, (1b), as

$$y_{j,k} = C_j A_j^k x_{j,0} + \sum_{i\in\mathcal{N}_0} H_{j,k-i} u_{j,i}, \qquad j \in \mathcal{M},$$

where the impulse response coefficients are given by

$$H_{j,k} = C_j A_j^{k-1} B_j, \qquad j \in \mathcal{M}.$$

Consequently

$$y_{T,k} = \sum_{j\in\mathcal{M}} \left(\Upsilon_j C_j A_j^k x_{j,0} + \sum_{i\in\mathcal{N}_0} \Upsilon_j H_{j,k-i} u_{j,i} \right).$$

Define the vectors

$$y_j = \begin{bmatrix} y_{j,1}^T & y_{j,2}^T & \cdots & y_{j,N}^T \end{bmatrix}^T, \qquad j \in \mathcal{M}, \tag{4a}$$
$$u_j = \begin{bmatrix} u_{j,0}^T & u_{j,1}^T & \cdots & u_{j,N-1}^T \end{bmatrix}^T, \qquad j \in \mathcal{M}, \tag{4b}$$

and the matrices

$$\Gamma_j = \begin{bmatrix} H_{j,1} & 0 & \cdots & 0 \\ H_{j,2} & H_{j,1} & & \\ \vdots & \vdots & \ddots & \\ H_{j,N} & H_{j,N-1} & \cdots & H_{j,1} \end{bmatrix}, \quad \Phi_j = \begin{bmatrix} C_j A_j \\ C_j A_j^2 \\ \vdots \\ C_j A_j^{N-1} \end{bmatrix},$$

for $j \in \mathcal{M}$.

We can then write the outputs, (4a), for each of the generating units as

$$y_j = \Gamma_j u_j + \Phi_j x_{j,0}, \qquad j \in \mathcal{M}. \tag{5}$$

Moreover, by introducing $\tilde{\Gamma}_j$ and $\tilde{\Phi}_j$ accordingly, it follows that

$$y_T = \sum_{j\in\mathcal{M}} \tilde{\Gamma}_j u_j + \tilde{\Phi}_j x_{j,0}. \tag{6}$$

We simplify the notation further by introducing

$$\underline{u}_j = \begin{bmatrix} \underline{u}_{j,0} \\ \underline{u}_{j,1} \\ \vdots \\ \underline{u}_{j,N-1} \end{bmatrix}, \quad \overline{u}_j = \begin{bmatrix} \overline{u}_{j,0} \\ \overline{u}_{j,1} \\ \vdots \\ \overline{u}_{j,N-1} \end{bmatrix}, \qquad j \in \mathcal{M},$$

and similarly we define $\Delta\underline{u}_j$, $\Delta\overline{u}_j$, $\underline{y}_j$, $\overline{y}_j$, $\underline{y}_T$, $\overline{y}_T$, $\bar{\gamma}_j$, $\bar{\rho}$, ρ, q, p_j, r_j and γ_j. Using this notation, the optimal control problem, (3), can be written as

$$\min_{u,\rho,\gamma} \ q^T \rho + \sum_{j\in\mathcal{M}} p_j^T u_j + r_j^T \gamma_j, \tag{7a}$$

subject to a set of decoupled constraints

$$\underline{u}_j \le u_j \le \overline{u}_j, \qquad j \in \mathcal{M}, \tag{7b}$$
$$\Delta\underline{u}_j \le \Delta u_j \le \Delta\overline{u}_j, \qquad j \in \mathcal{M}, \tag{7c}$$
$$\underline{y}_j - \gamma_j \le \Gamma_j u_j + \Phi_j x_{j,0} \le \overline{y}_j + \gamma_j, \qquad j \in \mathcal{M}, \tag{7d}$$
$$0 \le \gamma_j \le \overline{\gamma}, \qquad j \in \mathcal{M}, \tag{7e}$$
$$0 \le \rho \le \overline{\rho}, \tag{7f}$$

and a set of linking constraints

$$\underline{y}_T - \rho \le \sum_{j\in\mathcal{M}} \tilde{\Gamma}_j u_j + \tilde{\Phi}_j x_{j,0} \le \overline{y}_T + \rho. \tag{7g}$$

In a compact form, (7) can be stated by

$$
\begin{aligned}
\min_{z} \quad & \sum_{j\in\bar{\mathcal{M}}} c_j^T z_j, && && \text{(8a)}\\
\text{s.t.} \quad & G_j z_j \geq g_j, && j\in\bar{\mathcal{M}}, && \text{(8b)}\\
& \sum_{j\in\bar{\mathcal{M}}} H_j z_j \geq h, && && \text{(8c)}
\end{aligned}
$$

where $\bar{\mathcal{M}} = 1, 2, \dots, M+1$ and

$$
c = \left[\, c_1^T \middle| \cdots \middle| c_M^T \middle| c_{M+1}^T \,\right]^T = \left[\, p_1^T \;\, r_1^T \middle| \cdots \middle| p_M^T \;\, r_M^T \middle| q^T \,\right]^T,
$$

$$
z = \left[\, z_1^T \middle| \cdots \middle| z_M^T \middle| z_{M+1}^T \,\right]^T = \left[\, u_1^T \;\, \gamma_1^T \middle| \cdots \middle| u_M^T \;\, \gamma_M^T \middle| \rho^T \,\right]^T.
$$

In (8), (8b) represents the decoupled constraints (7b)-(7f), and (8c) represents the linking constraints (7g). The data structures in (8) are defined as

$$
G_j = \begin{bmatrix} \bar{G}_j \\ -\bar{G}_j \end{bmatrix}, \quad g_j = \begin{bmatrix} \underline{g}_j \\ -\overline{g}_j \end{bmatrix}, \quad H_j = \begin{bmatrix} \bar{H}_j \\ -\bar{H}_j \end{bmatrix}, \quad h = \begin{bmatrix} \underline{h} \\ -\overline{h} \end{bmatrix},
$$

where

$$
\left[\, \bar{G}_j \middle| \underline{g}_j \middle| \overline{g}_j \,\right] = \left[\begin{array}{cc|c|c}
I & 0 & \underline{u}_j & \overline{u}_j \\
\Lambda & 0 & \Delta\underline{u}_j + I_0 u_{j,-1} & \Delta\overline{u}_j + I_0 u_{j,-1} \\
\Gamma_j & I & \underline{y}_j - \Phi_j x_{j,0} & \infty \\
\Gamma_j & -I & -\infty & \overline{y}_j - \Phi_j x_{j,0} \\
0 & I & 0 & \overline{\gamma}_j
\end{array}\right],
$$

$$
\left[\, \bar{H}_j \middle| \underline{h} \middle| \overline{h} \,\right] = \left[\begin{array}{cc|c|c}
\tilde{\Gamma}_j & 0 & \underline{y}_T - \sum_{j\in\mathcal{M}} \tilde{\Phi}_j x_{j,0} & \infty \\
\tilde{\Gamma}_j & 0 & -\infty & \overline{y}_T - \sum_{j\in\mathcal{M}} \tilde{\Phi}_j x_{j,0}
\end{array}\right],
$$

for $j \in \mathcal{M}$, with Λ and I_0 defined as

$$
\Lambda_j = \begin{bmatrix} I & & & \\ -I & I & & \\ & \ddots & \ddots & \\ & & -I & I \end{bmatrix}, \qquad I_0 = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}.
$$

In the special case $j = M+1$

$$
\left[\, \bar{G}_{M+1} \middle| \underline{g}_{M+1} \middle| \overline{g}_{M+1} \,\right] = \left[\, I | 0 | \overline{\rho} \,\right].
$$

$$
\bar{H}_{M+1} = \left[I \;\; -I \right]^T.
$$

3. DANTZIG WOLFE DECOMPOSITION

Dantzig-Wolfe decomposition (Dantzig and Wolfe, 1960, 1961; Nemhauser and Wolsey, 1988; Martin, 1999) exploits that a convex set can be characterized by its extreme points and its extreme rays. In particular, for each $j \in \bar{\mathcal{M}}$, the set of points satisfying the decoupled constraints (8b) may be written as

$$
\begin{aligned}
\mathcal{G}_j &= \{ z_j | G_j z_j \geq g_j \}, \\
&= \left\{ z_j | z_j = \sum_{i\in\mathcal{P}} \lambda_j^i z_j^i, \sum_{i\in\mathcal{P}} \lambda_j^i = 1, \lambda_j^i \geq 0 \;\; \forall i \in \mathcal{P} \right\},
\end{aligned}
$$

where z_j^i are the extreme points of $\mathcal{G}_j$, and λ_j^i are convex combination multipliers. Notice that since each of the sets, $\mathcal{G}_j$, are bounded, extreme rays are not needed to characterize these sets.

By replacing the decision variables in (8) by convex combination multipliers, we obtain the master problem formulation

	Original Problem
#constraints	$6N + N\sum_{j\in\mathcal{M}} (4n_u(j) + 6n_y(j))$
#variables	$N + N\sum_{j\in\mathcal{M}} (n_u(j) + n_y(j))$
	Master Problem
#constraints	$4N + M + 1 + \sum_{j\in\bar{\mathcal{M}}} \lvert\mathcal{P}\rvert$
#variables	$\sum_{j\in\bar{\mathcal{M}}} \lvert\mathcal{P}\rvert$

Table 1. Dimensions of the original problem, (8), and the master problem (9).

$$
\begin{aligned}
\min_{\lambda} \quad & \phi = \sum_{j\in\bar{\mathcal{M}}} \sum_{i\in\mathcal{P}} c_j^i \lambda_j^i, && && \text{(9a)}\\
\text{s.t.} \quad & \sum_{j\in\bar{\mathcal{M}}} \sum_{i\in\mathcal{P}} H_j^i \lambda_j^i \geq h, && && \text{(9b)}\\
& \sum_{i\in\mathcal{P}} \lambda_j^i = 1, && j\in\bar{\mathcal{M}}, && \text{(9c)}\\
& \lambda_j^i \geq 0, && j\in\bar{\mathcal{M}}, \;\; i \in \mathcal{P}, && \text{(9d)}
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
H_j^i &= H_j z_j^i, && j\in\bar{\mathcal{M}}, \;\; i \in \mathcal{P}, && \text{(10a)}\\
c_j^i &= c_j^T z_j^i, && j\in\bar{\mathcal{M}}, \;\; i \in \mathcal{P}. && \text{(10b)}
\end{aligned}
$$

Given a solution, λ^*, to the master problem (9), a solution to the original problem, (8) can be obtained as

$$
z_j^* = \sum_{i\in\mathcal{P}} (\lambda^*)_j^i z_j^i, \qquad j \in \bar{\mathcal{M}}.
$$

In Table 1, we have compared the dimensions of the original problem, (8), and the master problem (9). Since the number of extreme points, $|\mathcal{P}|$, can increase exponentially with the size of the original problem, solving the master problem directly is inefficient. As demonstrated in the following section, however, the problem can be solved in an attractive way using a column generation procedure that replaces $\mathcal{P}$ by a subset $\tilde{\mathcal{P}}$.

4. COLUMN GENERATION

The dual linear program of (9) can be stated as

$$
\begin{aligned}
\max_{\alpha,\beta} \quad & \alpha^T h + \sum_{j\in\bar{\mathcal{M}}} \beta_j, && && \text{(11a)}\\
\text{s.t.} \quad & (H_j^i)^T \alpha + \beta_j \leq c_j^i, && j\in\bar{\mathcal{M}}, \;\; i \in \mathcal{P}, && \text{(11b)}\\
& \alpha \geq 0, && && \text{(11c)}
\end{aligned}
$$

in which $\alpha \in \mathbb{R}^{4N}$ and $\beta \in \mathbb{R}^{M+1}$ are the Lagrange multipliers associated with the linking constraints, (9b), and the convexity constraints, (9c), respectively. The necessary and sufficient optimality conditions for (9) and (11) are

$$
\begin{aligned}
\sum_{j\in\bar{\mathcal{M}}} \sum_{i\in\mathcal{P}} H_j^i \lambda_j^i &\geq h, && && \text{(12a)}\\
\sum_{i\in\mathcal{P}} \lambda_j^i &= 1, && j\in\bar{\mathcal{M}}, && \text{(12b)}\\
\lambda_j^i &\geq 0, && j\in\bar{\mathcal{M}}, \;\; i \in \mathcal{P}, && \text{(12c)}\\
c_j^i - (H_j^i)^T \alpha - \beta_j &\geq 0, && j\in\bar{\mathcal{M}}, \;\; i \in \mathcal{P}, && \text{(12d)}\\
\alpha &\geq 0, && && \text{(12e)}\\
\lambda_j^i (c_j^i - (H_j^i)^T \alpha - \beta_j) &= 0, && j\in\bar{\mathcal{M}}, \;\; i \in \mathcal{P}, && \text{(12f)}
\end{aligned}
$$

In Proposition 1 we derive conditions for which a solution satisfying this set of optimality conditions, can be obtained by solving the master problem (9) over a subset of the original variables.

Proposition 1. Let $\tilde{\mathcal{P}} \subseteq \mathcal{P}$ for all $j \in \bar{\mathcal{M}}$, and define $(\tilde{\lambda}, \tilde{\alpha}, \tilde{\beta})$ as a primal-dual solution to (9) and (11) restricted to the subset $\tilde{\mathcal{P}}$. Then the solution

$$\begin{aligned} \alpha^* &= \alpha, & & \\ \beta_j^* &= \beta_j, & & j \in \bar{\mathcal{M}}, \\ (\lambda^*)_j^i &= \begin{cases} \tilde{\lambda}_j^i & \text{if } i \in \tilde{\mathcal{P}} \\ 0 & \text{if } i \in \mathcal{P} \setminus \tilde{\mathcal{P}} \end{cases}, & & j \in \bar{\mathcal{M}}, \;\; i \in \mathcal{P}, \end{aligned}$$

satisfies the conditions, (12), if the optimal objective value of the subproblem

$$\min_{\tilde{z}_j} \;\; \varphi_j = (c_j - H_j^T \alpha^*)^T \tilde{z}_j - \beta_j^* \tag{13a}$$

$$\text{s.t.} \;\; G_j \tilde{z}_j \geq g_j, \tag{13b}$$

is non-negative for each $j \in \bar{\mathcal{M}}$.

Proof The solution $(\lambda^*, \alpha^*, \beta^*)$ satisfies (12a) since

$$\sum_{j \in \bar{\mathcal{M}}} \sum_{i \in \mathcal{P}} H_j^i (\lambda^*)_j^i = \sum_{j \in \bar{\mathcal{M}}} \sum_{i \in \tilde{\mathcal{P}}} H_j^i \tilde{\lambda}_j^i \geq h,$$

which follows from the definition of $(\tilde{\lambda}, \tilde{\alpha}, \tilde{\beta})$. Similarly, it is easy to verify that the conditions (12c), (12b), (12e) and (12f) are fulfilled.

Provided that $(\lambda^*, \alpha^*, \beta^*)$ is optimal, (12d) yields

$$c_j^i - (H_j^i)^T \alpha^* - \beta_j^* = (c_j - H_j^T \alpha^*)^T z_j^i - \beta_j^* \geq 0, \tag{14}$$

for all $j \in \bar{\mathcal{M}}$ and $i \in \mathcal{P}$. By construction of the solution, (14) is satisfied for all $i \in \tilde{\mathcal{P}}$. To check that the condition holds for all $i \in \mathcal{P} \setminus \tilde{\mathcal{P}}$, we consider the optimization problem (13). Since this linear program minimizes the left hand side of (14) over all possible extreme points, $\tilde{z}_j$, of $\mathcal{G}_j$, the solution $(\lambda^*, \alpha^*, \beta^*)$ also satisfies the remaining optimality condition (14) if φ_j is non-negative for all $j \in \bar{\mathcal{M}}$. ■

In Algorithm 1, we have outlined a column generation procedure based on Proposition 1. The algorithm exploits that if (14) is violated, then the solution to the subproblems, (13), provides a set of extreme points that can be added to the master problem. Table 1 shows that the master problem is much smaller than the original problem when $\mathcal{P}$ is restricted to the subset $\tilde{\mathcal{P}}$. Therefore, the column generation procedure requires less memory than conventional linear programming methods. We also notice that solving the subproblems is computationally inexpensive as they do not grow with the number of units M. This step may even be performed in parallel.

4.1 Warm-Starting

Algorithm 1 requires a set of initial points $\{z_j^0\}_{j=1}^{\bar{\mathcal{M}}}$ that are feasible for both the subproblems (13) and the original problem (8). As economic MPC requires running the algorithm in a closed-loop fashion, we can generate such a set of points by exploiting the solution from a previous time step.

Given the solution to (13)

$$z_j^* = \begin{bmatrix} u_{j,0}^{*T} & \cdots & u_{j,N-1}^{*T} & \gamma_{j,1}^{*T} \cdots & \gamma_{j,N}^{*T} \end{bmatrix}^T, \quad j \in \mathcal{M},$$

$$z_{M+1}^* = \begin{bmatrix} \rho_1^{*T} & \cdots & \rho_N^{*T} \end{bmatrix}^T,$$

we build a set of initial points in the following sampling instant as

$$z_j^0 = \begin{bmatrix} u_{j,1}^{*T} & \cdots & u_{j,N-1}^{*T} & \breve{u}_j^T & \gamma_{j,2}^{*T} & \cdots & \gamma_{j,N}^{*T} & \breve{\gamma}_j^T \end{bmatrix}^T, \;\; j \in \mathcal{M}$$

$$z_{M+1}^0 = \begin{bmatrix} \rho_2^{*T} & \cdots & \rho_N^{*T} & \breve{\rho}^T \end{bmatrix}^T.$$

Hence, the original solution values are shifted forward in time, and the variables $\breve{u}_j$, $\breve{\gamma}_j$ and $\breve{\rho}$ are appended to the initial points. In our implementation, we let

$$\breve{u}_j = u_{j,N-1}^*, \qquad j \in \mathcal{M}. \tag{15}$$

which leads to an initial input sequence with constant input in the two final sampling intervals. Using the state space equations (1)-(2), we compute the outputs $\breve{y}_{j,N}$ and $\breve{y}_{T,N}$ associated with this input sequence. Based on these values we let

$$\breve{\gamma}_j = \max(\underline{y}_{j,N} - \breve{y}_{j,N}, 0) + \max(\breve{y}_{j,N} - \overline{y}_{j,N}, 0),$$

$$\breve{\rho} = \max(\underline{y}_{T,N} - \breve{y}_{T,N}, 0) + \max(\breve{y}_{T,N} - \overline{y}_{T,N}, 0).$$

where the maximum function is evaluated element-wise.

Assuming that the inputs (15) satisfy the input constraints for the updated problem data, and that the upper limits on γ_j and ρ are sufficiently large, the strategy above yields a set of feasible initial points for Algorithm 1, $\{z_j^0\}_{j=1}^{\bar{\mathcal{M}}}$, which exploits the solution obtained in the previous time step. As the solution in successive time steps are closely related in MPC applications, this approach provides a warm-start

Algorithm 1 Column generation procedure for the solution of the master problem (9).

```
Require: {z_j^0}_{j=1}^{M̄}
  i = 0, converged = 0
  while not converged do
    P̃ = {0, 1, ..., i}
    COMPUTE PROBLEM DATA
    for j ∈ M̄ do
      for i ∈ P̃ do
        H_j^i = H_j z_j^i
        c_j^i = c_j^T z_j^i
      end for
    end for
    SOLVE RESTRICTED MASTER PROBLEM
    (φ*, λ*, α*, β*) ← solve (9) with P = P̃
    SOLVE SUBPROBLEMS
    for j ∈ M̄ do
      (φ_j*, z̃_j*) ← solve (13)
    end for
    CHECK IF CONVERGED
    if φ_j ≥ 0 ∀j ∈ M̄ then
      converged = 1
    else
      UPDATE EXTREME POINTS
      for j ∈ M̄ do
        z_j^{i+1} = z̃_j*
        i = i + 1
      end for
    end if
  end while
```

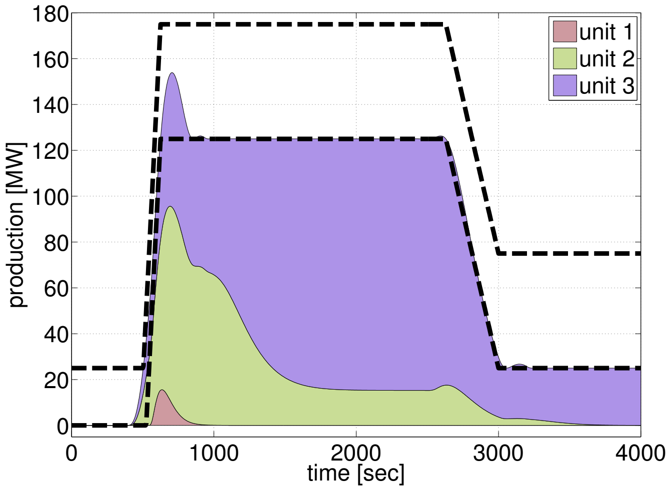

Fig. 1. Closed-loop simulation study of economic MPC. The marginal price of using the units is decreasing with the unit number.

for Algorithm 1. In case no previous solution is available, a similar strategy can be used to adjust the slack variables for an arbitrary feasible input sequence.

5. RESULTS

In this section, we compare a MATLAB implementation of Algorithm 1, denoted `DWempc`, to linear programming solvers from the following software packages: `CPLEX`, `Gurobi` and `MOSEK`. For each solver, the computation time of solving the optimal control problem (3) is measured. The algorithms are run on an Intel(R) Core(TM) i5-2520M CPU @ 2.50GHz with 4 GB RAM running a 64-bit Windows 7 Enterprise operating system. In `DWempc`, the restricted master problem and the subproblems are solved using `CPLEX`.

As a conceptual case study, we consider a collection of power generating units in the form

$$Y_j(s) = \frac{1}{(\tau_j s + 1)^3} U_j(s), \qquad j \in \mathcal{M}, \qquad (16)$$

where $U_j(s)$ is the fuel input and the $Y_j(s)$ is the power production. The third order model, (16), has been validated against actual measurement data in Edlund et al. (2010). In our study, we vary the time constant, τ_j, to represent different types of power generating units. Time constants in the range 80-120 are associated with slow units such as centralized thermal power plants, while time constants in the range 20-60 represent units with faster dynamics such as diesel generators and gas turbines.

To control the units, (16), using economic MPC we realize the system in the discrete state space form (1)-(2). In the resulting model structure, $u_{j,k} \in \mathbb{R}$ is fuel input, $y_{j,k} \in \mathbb{R}$ is the power production, and $y_{T,k} \in \mathbb{R}$ is the total power production. Thus, $\Upsilon_j = 1$, for all $j \in \mathcal{M}$. Fig. 1 demonstrates the production plan obtained using economic MPC in a case study with $M = 3$ power generating units. The graphs show the individual outputs, as well as the output limits for the total production. The case study parameters are listed in Table 2. All the parameters listed, are kept constant over the entire horizon. The values, p_j, are the prices pr. unit of fuel (e.g oil, natural gas or coal). We have defined these parameters such that the fuel price for fast units is higher than the fuel price for slow units. The price for imbalances is fixed to $q_k = 10000$. Fig. 1 shows that the cheapest generating unit accounts for the main load whereas the more expensive generating units are used only when faster dynamics are required to satisfy the constraints. This represents a common situation in the power industry, where large thermal power plants typically produce a majority of the electricity, while units with faster dynamics such as diesel generators help balancing the system when needed.

Table 2. Case study parameters.

	τ_j	p_j	$\underline{u}_j$	$\overline{u}_j$	$\Delta\underline{u}_j$	$\Delta\overline{u}_j$
Generating Unit 1	40	24	0	50	-30	30
Generating Unit 2	90	12	0	100	-20	20
Generating Unit 3	100	6	0	200	-5	5

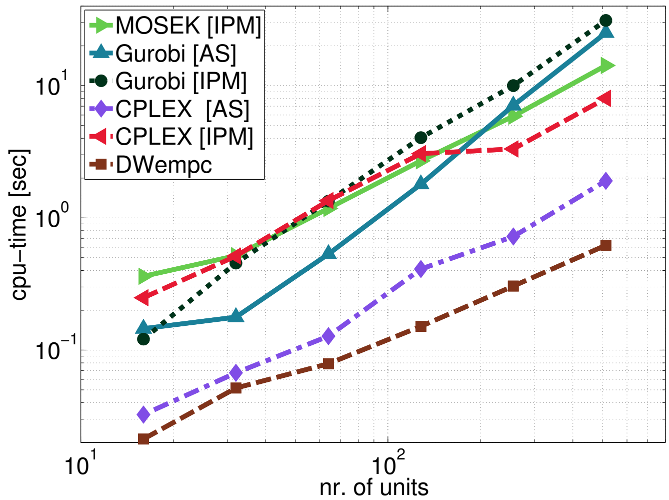

Fig. 2. CPU-time for solving (3) as a function of the number of power generating units, and fixed $N = 50$. Active-set methods are denoted by (AS) and interior-point methods are denoted by (IPM).

Fig. 2 depicts the computation time of solving the constrained optimal control problem, (3), using `DWempc`, `CPLEX`, `Gurobi` and `DWempc` for an increasing number of generating units. The problem data was generated in a similar way as in the case study above. Our results show that `DWempc` outperforms conventional linear programming solvers with a significant margin, and that the difference grows with the number of units controlled. This demonstrates that the column generation procedure outlined in Algorithm 1 is a promising method for economic MPC of distributed energy systems.

6. CONCLUSION

In this paper, we have presented a detailed description of a warm-started Dantzig-Wolfe decomposition algorithm for economic MPC of distributed energy system. Our results show that a MATLAB implementation of the algorithm is significantly faster than both active-set methods and interior-point methods provided by `MOSEK`, `CPLEX` and `Gurobi`. Moreover, `DWempc` has several desirable features such as low memory costs and parallelization capabilities, which makes it well suited for economic MPC applications with a decentralized structure such distributed energy systems.

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