ACTIVE FAULT TOLERANT CONTROL OF THE CONTINUOS-TIME SYSTEMS

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Abstract: Reformulated principle for designing the observer-based regonfiguration control in the continuos-time linear MIMO systems is treated in this paper. Based on the linear matrix inequalities the problem addressed can be indicated as the approach giving the sufficient conditions to design the stable estimator and controller with the on-line based fault estimation and accommodation support. The system model based numerical example is presented in the paper to illustrate properties of the proposed design method.

Keywords: Control reconfiguration, fault estimation, linear matrix inequality, convex optimization, state control.

1 INTRODUCTION

The complexity of control systems requires the fault tolerance schemes to provide control of the faulty system. The fault tolerant systems are that one of the more fruitful applications with potential significance for those domains in which control must proceed while the controlled system is operative and testing opportunities are limited by given operational considerations. The real problem is usually to fix the system with faults so that it can continue its mission for some time with some limitations of functionality. These large problems are known as the fault detection, identification and reconfiguration (FDIR) systems. The practical benefits of the integrated approach to FDIR seem to be considerable, especially when knowledge of the available fault isolations and the system reconfigurations is used to reduce the cost and to increase the control reliability and utility. Reconfiguration can be viewed as the task to select these elements whose reconfiguration is sufficient to do the acceptable behavior of the system. If an FDIR system is designed properly, it will be able to deal with the specified faults and maintain the system stability and acceptable level of performance in the presence of faults.

The main contribution of the paper is present the reformulated design method for the state estimator based reconfiguration control in the continuous-time linear MIMO systems. In contradiction to the adaptive systems there don't exist much structures to solve this problem [Blanke at all 2003], [Krokavec, Filasová 2008], especially using the linear matrix inequality (LMI) approach. To make formalism simpler, Lyapunov inequality is used as the design starting point to demonstrate the application suitability of the unified algebraic approach in these design tasks. Two LMIs are outlined to posse the sufficient conditions for a solution and the others LMI can be possibly introduced to adapt these for the control constrain parameters in the given estimator and controller structure. An additional control law with the fault estimation is used in this structure to compensate the fault effect. The used structure is motivated by the standard structure [Dong at all 2009], and in this presented form enables to design systems with the modified controller structure.

2 PROBLEM DESCRIPTION

Through this paper the task is concerned with the computation of the adaptive state feedback $\boldsymbol{u}(t)$, which control the faulty linear dynamic system given by the set of equations

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}_u\boldsymbol{u}(t) + \boldsymbol{B}_f\boldsymbol{f}(t) \tag{1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t) + \boldsymbol{D}_u\boldsymbol{u}(t) + \boldsymbol{D}_f\boldsymbol{f}(t) \tag{2}$$

where $\boldsymbol{q}(t) \in \mathbb{R}^n$, $\boldsymbol{u}(t) \in \mathbb{R}^r$, $\boldsymbol{y}(t) \in \mathbb{R}^m$, and $\boldsymbol{f}(t) \in \mathbb{R}^l$ are vectors of the state, input, output and fault variables, respectively, matrices $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}_u \in \mathbb{R}^{n\times r}$, $\boldsymbol{C} \in \mathbb{R}^{m\times n}$, $\boldsymbol{D}_u \in \mathbb{R}^{m\times r}$, $\boldsymbol{B}_f \in \mathbb{R}^{n\times l}$, $\boldsymbol{D}_f \in \mathbb{R}^{m\times l}$ are real matrices. Problem of the interest is to design the asymptotically stable closed-loop system with the linear memoryless state feedback controller of the form

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{q}_e(t) - \boldsymbol{L}\boldsymbol{f}_e(t) \tag{3}$$

Here $\boldsymbol{K} \in \mathbb{R}^{r\times n}$ is the nominal controller gain matrix, $\boldsymbol{L} \in \mathbb{R}^{r\times l}$ is the compensate controller gain matrix, $\boldsymbol{q}_e(t) \in \mathbb{R}^n$ is the system state estimation vector, and $\boldsymbol{f}_e(t) \in \mathbb{R}^l$ is the fault estimation vector. This method can be applied for such systems, where

$$\begin{bmatrix} \boldsymbol{B}_f \\ \boldsymbol{D}_f \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_u \\ \boldsymbol{D}_u \end{bmatrix} \boldsymbol{L} \tag{4}$$

and the additive term $\boldsymbol{B}_f\boldsymbol{f}(t)$ is compensated by the term

$$-\boldsymbol{B}_f\boldsymbol{f}_e(t) = -\boldsymbol{B}_u\boldsymbol{L}\boldsymbol{f}_e(t) \tag{5}$$

which implies (3). The estimators are then given by the set of the state equations

$$\dot{\boldsymbol{q}}_e(t) = \boldsymbol{A}\boldsymbol{q}_e(t) + \boldsymbol{B}_u\boldsymbol{u}(t) + \boldsymbol{B}_f\boldsymbol{f}_e(t) + \boldsymbol{J}(\boldsymbol{y}(t) - \boldsymbol{y}_e(t)) \tag{6}$$

$$\dot{\boldsymbol{f}}_e(t) = \boldsymbol{M}\boldsymbol{f}_e(t) + \boldsymbol{N}(\boldsymbol{y}(t) - \boldsymbol{y}_e(t)) \tag{7}$$

$$\boldsymbol{y}_e(t) = \boldsymbol{C}\boldsymbol{q}_e(t) + \boldsymbol{D}_u\boldsymbol{u}(t) + \boldsymbol{D}_f\boldsymbol{f}_e(t) \tag{8}$$

where $\boldsymbol{J} \in \mathbb{R}^{n\times m}$ is the state estimator gain matrix, and $\boldsymbol{M} \in \mathbb{R}^{l\times l}$, $\boldsymbol{N} \in \mathbb{R}^{l\times m}$ are the system and input matrices of the fault estimator, respectively.

3 BASIC PRELIMINARIES

Proposition 1. (Schur Complement) *Let* $\boldsymbol{Q} > \boldsymbol{0}$, $\boldsymbol{R} > \boldsymbol{0}$, $\boldsymbol{S}$ *are real matrices of appropriate dimensions, then the next inequalities are equivalent*

$$\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^T & -\boldsymbol{R} \end{bmatrix} < 0 \;\Leftrightarrow\; \begin{bmatrix} \boldsymbol{Q} + \boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}^T & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{R} \end{bmatrix} < 0 \;\Leftrightarrow\; \boldsymbol{Q} + \boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}^T < 0, \boldsymbol{R} > 0 \tag{9}$$

Proof. Let the linear matrix inequality takes form

$$\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^T & -\boldsymbol{R} \end{bmatrix} < 0 \tag{10}$$

then using Gauss elimination it yields

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{S}\boldsymbol{R}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^T & -\boldsymbol{R} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{R}^{-1}\boldsymbol{S}^T & \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q} + \boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}^T & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{R} \end{bmatrix}, \quad \det \begin{bmatrix} \boldsymbol{I} & \boldsymbol{S}\boldsymbol{R}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} = 1 \tag{11}$$

and it is evident that this transform doesn‘t changed negativity of (10), and so (11) implies (9). This concludes the proof.

Proposition 2. (Bounded real lemma) *For given* $\gamma \in \mathbb{R}$ *and the linear system (1), (2) with* $\boldsymbol{f}(t) = 0$ *if there exists the symmetric positive definite matrix* $\boldsymbol{P} > 0$ *such that*

$$\begin{bmatrix} \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} & \boldsymbol{P}\boldsymbol{B}_u & \boldsymbol{C}^T \\ * & -\gamma^2\boldsymbol{I}_r & \boldsymbol{D}^T \\ * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{12}$$

then the given autonomous system is asymptotically stable. In (12) $\boldsymbol{I}_r \in \mathbb{R}^{r\times r}$, $\boldsymbol{I}_m \in \mathbb{R}^{m\times m}$ *are the identity matrices, respectively,*

Hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Proof. Defining Lyapunov function candidate as follows

$$v(\boldsymbol{q}(t)) = \boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \int_0^t (\boldsymbol{y}^T(r)\boldsymbol{y}(r) - \gamma^2\boldsymbol{u}^T(r)\boldsymbol{u}(r))dr > 0 \tag{13}$$

where $\boldsymbol{P} = \boldsymbol{P}^T > 0, \boldsymbol{P} \in \mathbb{R}^{n\times n}, \gamma \in \mathbb{R}$ and evaluating derivative of $v(\boldsymbol{q}(t))$ with respect to t then it yields

$$\dot{v}(\boldsymbol{q}(t)) = \dot{\boldsymbol{q}}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}\dot{\boldsymbol{q}}(t) + \boldsymbol{y}^T(t)\boldsymbol{y}(t) - \gamma^2\boldsymbol{u}^T(t)\boldsymbol{u}(t) - (\boldsymbol{y}^T(0)\boldsymbol{y}(0) - \gamma^2\boldsymbol{u}^T(0)\boldsymbol{u}(0)) < 0 \tag{14}$$

Thus, substituting (1), (2) for $\boldsymbol{f}(t) = 0$ it can be written

$$\begin{gathered} \dot{v}(\boldsymbol{q}(t)) = (\boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}_u\boldsymbol{u}(t))^T\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}(\boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}_u\boldsymbol{u}(t)) + \\ +(\boldsymbol{C}\boldsymbol{q}(t) + \boldsymbol{D}_u\boldsymbol{u}(t))^T(\boldsymbol{C}\boldsymbol{q}(t) + \boldsymbol{D}_u\boldsymbol{u}(t)) - \gamma^2\boldsymbol{u}^T(t)\boldsymbol{u}(t) - (\boldsymbol{y}^T(0)\boldsymbol{y}(0) - \gamma^2\boldsymbol{u}^T(0)\boldsymbol{u}(0)) < 0 \end{gathered} \tag{15}$$

and with notation

$$\boldsymbol{q}_c^T(t) = \begin{bmatrix} \boldsymbol{q}^T(t) & \boldsymbol{u}^T(t) \end{bmatrix} \tag{16}$$

it is obtained

$$\dot{v}(\boldsymbol{q}(t)) = \boldsymbol{q}_c^T(t)\boldsymbol{P}_c\boldsymbol{q}_c(t) - (\boldsymbol{y}^T(0)\boldsymbol{y}(0) - \gamma^2\boldsymbol{u}^T(0)\boldsymbol{u}(0)) < 0 \tag{17}$$

where

$$\boldsymbol{P}_c = \begin{bmatrix} \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} & \boldsymbol{P}\boldsymbol{B}_u \\ * & -\gamma^2\boldsymbol{I}_r \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{C}^T\boldsymbol{D} \\ * & \boldsymbol{D}^T\boldsymbol{D} \end{bmatrix} < 0 \tag{18}$$

Since

$$\begin{bmatrix} \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{C}^T\boldsymbol{D} \\ * & \boldsymbol{D}^T\boldsymbol{D} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \boldsymbol{C}^T \\ \boldsymbol{D}^T \end{bmatrix} & \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \end{bmatrix} \geq 0 \tag{19}$$

Schur complement property implies

$$\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{C}^T \\ * & \boldsymbol{0} & \boldsymbol{D}^T \\ * & * & -\boldsymbol{I}_m \end{bmatrix} \geq 0 \tag{20}$$

Supposing $\boldsymbol{y}^T(0)\boldsymbol{y}(0) - \gamma^2\boldsymbol{u}^T(0)\boldsymbol{u}(0) \geq 0$ and using (20) the LMI condition (18) can be written compactly as (12). This concludes the proof.

4 DESIGN CONDITIONS

4.1. System compact description form

Using equality $\dot{\boldsymbol{f}}(t) = \dot{\boldsymbol{f}}(t)$ and assembling this equality with (1)-(3) and (6)-(8) gives

$$\dot{\boldsymbol{q}}_\alpha(t) = \boldsymbol{A}_\alpha \boldsymbol{q}_\alpha(t) + \boldsymbol{f}_\alpha(t) \tag{21}$$

$$\boldsymbol{y} = \boldsymbol{C}_\alpha \boldsymbol{q}_\alpha(t) \tag{22}$$

where

$$\boldsymbol{q}_\alpha^T(t) = \begin{bmatrix} \boldsymbol{q}^T(t) & \boldsymbol{q}_e^T(t) & \boldsymbol{f}^T(t) & \boldsymbol{f}_e^T(t) \end{bmatrix}, \quad \boldsymbol{f}_\alpha^T(t) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \dot{\boldsymbol{f}}^T(t) & \boldsymbol{0} \end{bmatrix} \tag{23}$$

$$\boldsymbol{A}_\alpha = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{\text{-}B_u K} & \boldsymbol{B}_f & \boldsymbol{\text{-}B_u L} \\ \boldsymbol{JC} & \boldsymbol{A - B_u K - JC} & \boldsymbol{JD}_f & \boldsymbol{B}_f \boldsymbol{- JD}_f \boldsymbol{- B_u L} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{NC} & \boldsymbol{\text{-}NC} & \boldsymbol{ND}_f & \boldsymbol{M - ND}_f \end{bmatrix}, \quad \boldsymbol{C}_\alpha = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{\text{-}D_u K} & \boldsymbol{D}_f & \boldsymbol{\text{-}D_u L} \end{bmatrix} \tag{24}$$

Since (4) implies

$$\boldsymbol{B}_f \boldsymbol{- B_u L = 0}, \qquad \boldsymbol{D}_f \boldsymbol{- D_u L = 0} \tag{25}$$

it is possible to verify using the state transform

$$\boldsymbol{q}_\beta(t) = \boldsymbol{T}\boldsymbol{q}_\alpha(t) = \begin{bmatrix} \boldsymbol{q}(t) \\ \boldsymbol{e}_q(t) \\ \boldsymbol{f}(t) \\ \boldsymbol{e}_f(t) \end{bmatrix}, \qquad \boldsymbol{T} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{I} & \boldsymbol{\text{-}I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{\text{-}I} \end{bmatrix} \tag{26}$$

where $\boldsymbol{I} \in \mathbb{R}^{n \times n}$ is the identity matrix, that

$$\boldsymbol{e}_q(t) = \boldsymbol{q}(t) - \boldsymbol{q}_e(t), \quad \boldsymbol{e}_f(t) = \boldsymbol{f}(t) - \boldsymbol{f}_e(t) \tag{27}$$

$$\boldsymbol{f}_\beta^T(t) = (\boldsymbol{T}\boldsymbol{f}_\alpha(t))^T = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \dot{\boldsymbol{f}}^T(t) & \dot{\boldsymbol{f}}^T(t) \end{bmatrix} \tag{28}$$

$$\boldsymbol{A}_\beta = \boldsymbol{T}\boldsymbol{A}_\alpha \boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{A - B_u K} & \boldsymbol{B_u K} & \boldsymbol{0} & \boldsymbol{B_u L} \\ \boldsymbol{0} & \boldsymbol{A - JC} & \boldsymbol{0} & \boldsymbol{B}_f \boldsymbol{- JD}_f \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\text{-}NC} & \boldsymbol{\text{-}M} & \boldsymbol{M - ND}_f \end{bmatrix} \tag{29}$$

$$\boldsymbol{C}_\beta = \boldsymbol{C}_\alpha \boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{C - D_u K} & \boldsymbol{D_u K} & \boldsymbol{0} & \boldsymbol{D_u L} \end{bmatrix} \tag{30}$$

and it obvious that it can be written

$$\dot{\boldsymbol{q}}_\beta(t) = \boldsymbol{A}_\beta \boldsymbol{q}_\beta(t) + \boldsymbol{f}_\beta(t) \tag{31}$$

$$\boldsymbol{y} = \boldsymbol{C}_\beta \boldsymbol{q}_\beta(t) \tag{32}$$

Eliminating out $\dot{\boldsymbol{f}}(t) = \dot{\boldsymbol{f}}(t)$ it can be written

$$\dot{\boldsymbol{q}}_\delta(t) = \boldsymbol{A}_\delta \boldsymbol{q}_\delta(t) + \boldsymbol{B}_\delta \boldsymbol{w}_\delta(t) \tag{33}$$

$$\boldsymbol{y} = \boldsymbol{C}_\delta \boldsymbol{q}_\delta(t) + \boldsymbol{D}_\delta \boldsymbol{w}_\delta(t) \tag{34}$$

where

$$\boldsymbol{q}_\delta^T(t) = \begin{bmatrix} \boldsymbol{q}^T(t) & \boldsymbol{e}_q^T(t) & \boldsymbol{e}_f^T(t) \end{bmatrix}, \quad \boldsymbol{w}_\delta^T(t) = \begin{bmatrix} \boldsymbol{f}^T(t) & \dot{\boldsymbol{f}}^T(t) \end{bmatrix} \tag{35}$$

$$\boldsymbol{A}_\delta = \begin{bmatrix} \boldsymbol{A} - \boldsymbol{B}_u\boldsymbol{K} & \boldsymbol{B}_u\boldsymbol{K} & \boldsymbol{B}_u\boldsymbol{L} \\ \boldsymbol{0} & \boldsymbol{A} - \boldsymbol{JC} & \boldsymbol{B}_f - \boldsymbol{JD}_f \\ \boldsymbol{0} & -\boldsymbol{NC} & \boldsymbol{M} - \boldsymbol{ND}_f \end{bmatrix}, \quad \boldsymbol{B}_\delta = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \\ -\boldsymbol{M} & \boldsymbol{I} \end{bmatrix} \tag{36}$$

$$\boldsymbol{C}_\delta = \begin{bmatrix} \boldsymbol{C} - \boldsymbol{D}_u\boldsymbol{K} & \boldsymbol{D}_u\boldsymbol{K} & \boldsymbol{D}_u\boldsymbol{L} \end{bmatrix}, \quad \boldsymbol{D}_\delta = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \tag{37}$$

To apply the separability principle a block diagonal symmetric matrix $\boldsymbol{P}_\delta > 0$ is chosen, i.e.

$$\boldsymbol{P}_\delta = \operatorname{diag}\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{R} & \boldsymbol{S} \end{bmatrix} \tag{38}$$

where $\boldsymbol{Q}, \boldsymbol{R} \in \mathbb{R}^{n \times n}$, $\boldsymbol{S} \in \mathbb{R}^{l \times l}$. Thus

$$\boldsymbol{P}_\delta\boldsymbol{A}_\delta + \boldsymbol{A}_\delta^T\boldsymbol{P}_\delta = \begin{bmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{QB}_u\boldsymbol{K} & \boldsymbol{QB}_u\boldsymbol{L} \\ * & \boldsymbol{\Phi}_{22} & \boldsymbol{\Phi}_{23} \\ * & * & \boldsymbol{\Phi}_{33} \end{bmatrix}, \quad \boldsymbol{P}_\delta\boldsymbol{B}_\delta = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \\ -\boldsymbol{SM} & \boldsymbol{S} \end{bmatrix} \tag{39}$$

where

$$\boldsymbol{\Phi}_{11} = \boldsymbol{Q}(\boldsymbol{A} - \boldsymbol{B}_u\boldsymbol{K}) + (\boldsymbol{A} - \boldsymbol{B}_u\boldsymbol{K})^T\boldsymbol{Q}, \quad \boldsymbol{\Phi}_{22} = \boldsymbol{R}(\boldsymbol{A} - \boldsymbol{JC}) + (\boldsymbol{A} - \boldsymbol{JC})^T\boldsymbol{R} \tag{40}$$

$$\boldsymbol{\Phi}_{33} = \boldsymbol{S}(\boldsymbol{M} - \boldsymbol{ND}_f) + (\boldsymbol{M} - \boldsymbol{ND}_f)^T\boldsymbol{S}, \quad \boldsymbol{\Phi}_{23} = \boldsymbol{R}(\boldsymbol{B}_f - \boldsymbol{JD}_f) - \boldsymbol{C}^T\boldsymbol{N}^T\boldsymbol{S} \tag{41}$$

Inserting (36), (37), and (39) into (12) gives

$$\begin{bmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{QB}_u\boldsymbol{K} & \boldsymbol{QB}_u\boldsymbol{L} & \boldsymbol{0} & \boldsymbol{0} & (\boldsymbol{C} - \boldsymbol{D}_u\boldsymbol{K})^T \\ * & \boldsymbol{\Phi}_{22} & \boldsymbol{\Phi}_{23} & \boldsymbol{0} & \boldsymbol{0} & (\boldsymbol{D}_u\boldsymbol{K})^T \\ * & * & \boldsymbol{\Phi}_{33} & -\boldsymbol{SM} & \boldsymbol{S} & (\boldsymbol{D}_u\boldsymbol{L})^T \\ * & * & * & -\gamma^2\boldsymbol{I}_l & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & * & -\gamma^2\boldsymbol{I}_l & \boldsymbol{0} \\ * & * & * & * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{42}$$

It is evident that there are the cross parameter interactions in the structure of (42). To apply the separability principle (this determines the estimator structure - the estimation error vector is independent on the state as well as on the input variables) it is possible at the first step to compute only the controller feedback gain matrix $\boldsymbol{K}$**,** and at the next step to design the estimator gain matrices $\boldsymbol{J}$, $\boldsymbol{M}$, $\boldsymbol{N}$ if the obtained $\boldsymbol{K}$ is included.

4.2. Controller Feedback Gain Matrix Design

Theorem 1. *For the fault-free system (1), (2) the sufficient condition for the stable nominal control (3) is that there exist for given γ a positive definite symmetric matrix $\boldsymbol{X} > 0$, $\boldsymbol{X} \in \mathbb{R}^{n \times n}$ and a matrix $\boldsymbol{Y} \in \mathbb{R}^{r \times n}$ such that the following LMIs are satisfied*

$$\boldsymbol{X} = \boldsymbol{X}^T > 0 \tag{43}$$

$$\begin{bmatrix} \boldsymbol{AX} + \boldsymbol{XA}^T - \boldsymbol{Y}^T\boldsymbol{B}_u^T - \boldsymbol{B}_u\boldsymbol{Y} & \boldsymbol{B}_u\boldsymbol{L} & \boldsymbol{XC}^T - \boldsymbol{Y}^T\boldsymbol{D}_u^T \\ * & -\gamma^2\boldsymbol{I}_l & \boldsymbol{L}^T\boldsymbol{D}_u^T \\ * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{44}$$

The control law gain matrix is then given as

$$\boldsymbol{K} = \boldsymbol{YX}^{-1} \tag{45}$$

Proof. To apply separability principle $\boldsymbol{e}_q(t) = \boldsymbol{0}$ is considering. Separating $\boldsymbol{q}(t)$ from (37) gives

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}^{\circ}\boldsymbol{q}(t) + \boldsymbol{B}^{\circ}\boldsymbol{w}^{\circ}(t) \tag{46}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}^{\circ}\boldsymbol{q}(t) + \boldsymbol{D}^{\circ}\boldsymbol{w}^{\circ}(t) \tag{47}$$

where

$$\boldsymbol{w}^{\circ}(t) = \boldsymbol{e}_f(t) \tag{48}$$

$$\boldsymbol{A}^{\circ} = \boldsymbol{A} - \boldsymbol{B}_u\boldsymbol{K}, \quad \boldsymbol{B}^{\circ} = \boldsymbol{B}_u\boldsymbol{L}, \quad \boldsymbol{C}^{\circ} = \boldsymbol{C} - \boldsymbol{D}_u\boldsymbol{K}, \quad \boldsymbol{D}^{\circ} = \boldsymbol{D}_u\boldsymbol{L} \tag{49}$$

and with (49) inequality (12) can be written as

$$\begin{bmatrix} \boldsymbol{QA} + \boldsymbol{A}^T\boldsymbol{Q} - \boldsymbol{QB}_u\boldsymbol{K} - \boldsymbol{K}^T\boldsymbol{B}_u^T\boldsymbol{Q} & \boldsymbol{QB}_u\boldsymbol{L} & \boldsymbol{C}^T - \boldsymbol{K}^T\boldsymbol{D}_u^T \\ * & -\gamma^2\boldsymbol{I}_l & \boldsymbol{L}^T\boldsymbol{D}_u^T \\ * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{50}$$

Introducing the congruence transform matrix

$$\boldsymbol{H} = \mathrm{diag}\left[\boldsymbol{Q}^{-1} \;\; \boldsymbol{I}_l \;\; \boldsymbol{I}_m\right] \tag{51}$$

then multiplying (50) from the left and right side by (51) gives

$$\begin{bmatrix} \boldsymbol{AQ}^{-1} + \boldsymbol{Q}^{-1}\boldsymbol{A}^T - \boldsymbol{B}_u\boldsymbol{KQ}^{-1} - \boldsymbol{Q}^{-1}\boldsymbol{K}^T\boldsymbol{B}_u^T & \boldsymbol{B}_u\boldsymbol{L} & \boldsymbol{Q}^{-1}\boldsymbol{C}^T - \boldsymbol{Q}^{-1}\boldsymbol{K}^T\boldsymbol{D}_u^T \\ * & -\gamma^2\boldsymbol{I}_l & \boldsymbol{L}^T\boldsymbol{D}_u^T \\ * & * & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{52}$$

With notation

$$\boldsymbol{Q}^{-1} = \boldsymbol{X} > 0, \quad \boldsymbol{KQ}^{-1} = \boldsymbol{Y} \tag{53}$$

(52) implies (44). This concludes the proof.

4.3. Estimator System Matrix Design

Theorem 2. *For given γ, **K**, and **L**, the observers associated with the system (1), (2) exist if there exist symmetric positive definite matrices **R** > 0, **S** > 0, a negative definite matrix **V** < 0,* $\boldsymbol{V} \in \mathbb{R}^{l \times l}$ *, and matrices* $\boldsymbol{W} \in \mathbb{R}^{n \times l}$ *,* $\boldsymbol{Z} \in \mathbb{R}^{n \times m}$ *such that the following LMIs are satisfied*

$$\boldsymbol{R} = \boldsymbol{R}^T > 0, \quad \boldsymbol{S} = \boldsymbol{S}^T > 0, \quad \boldsymbol{V} < 0 \tag{54}$$

$$\begin{bmatrix} \boldsymbol{\Phi}_{22} & \boldsymbol{\Phi}_{23} & \boldsymbol{0} & \boldsymbol{0} & (\boldsymbol{D}_u\boldsymbol{K})^T \\ * & \boldsymbol{\Phi}_{33} & -\boldsymbol{V} & \boldsymbol{S} & (\boldsymbol{D}_u\boldsymbol{L})^T \\ * & * & -\gamma^2\boldsymbol{I}_l & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & -\gamma^2\boldsymbol{I}_l & \boldsymbol{0} \\ * & * & * & * & -\boldsymbol{I}_m \end{bmatrix} \tag{55}$$

where

$$\boldsymbol{\Phi}_{22} = \boldsymbol{RA} + \boldsymbol{A}^T\boldsymbol{R} - \boldsymbol{ZC} - \boldsymbol{Z}^T\boldsymbol{R}, \quad \boldsymbol{\Phi}_{23} = \boldsymbol{RB}_f - \boldsymbol{ZD}_f - \boldsymbol{C}^T\boldsymbol{W}^T \tag{56}$$

$$\boldsymbol{\Phi}_{33} = \boldsymbol{V} + \boldsymbol{V}^T - \boldsymbol{WD}_f - \boldsymbol{D}_f^T\boldsymbol{W}^T \tag{57}$$

The estimator matrix parameters are then given as

$$\boldsymbol{M} = \boldsymbol{S}^{-1}\boldsymbol{V}, \quad \boldsymbol{N} = \boldsymbol{S}^{-1}\boldsymbol{W}, \quad \boldsymbol{J} = \boldsymbol{R}^{-1}\boldsymbol{Z} \tag{58}$$

Proof. Supposing that $\boldsymbol{q}(t) = \boldsymbol{0}$ then (42) is reduced as follows

$$\begin{bmatrix} \boldsymbol{\Phi}_{22} & \boldsymbol{\Phi}_{23} & \boldsymbol{0} & \boldsymbol{0} & (\boldsymbol{D}_u\boldsymbol{K})^T \\ * & \boldsymbol{\Phi}_{33} & \boldsymbol{-SM} & \boldsymbol{S} & (\boldsymbol{D}_u\boldsymbol{L})^T \\ * & * & -\gamma^2\boldsymbol{I}_l & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & -\gamma^2\boldsymbol{I}_l & \boldsymbol{0} \\ * & * & * & * & -\boldsymbol{I}_m \end{bmatrix} \tag{59}$$

It is obvious that $\boldsymbol{M}$ has to be a stable matrix, i.e. $\boldsymbol{M} < 0$. Thus, with notation

$$\boldsymbol{SM} = \boldsymbol{V}, \quad \boldsymbol{SN} = \boldsymbol{W}, \quad \boldsymbol{Z} = \boldsymbol{RJ} \tag{60}$$

(40), (41) can be rewritten as (56), (57) and (59) implies (55). This concludes the proof.

5 ILLUSTRATIVE EXAMPLE

To demonstrate algorithm properties it was assumed that the system is given by (1), (2) where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -9 & -5 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 5 \end{bmatrix}, \quad \boldsymbol{B}_f = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{L} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{D}_u = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{D}_f = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving (43), (44) for the LMI matrix variables γ, $\boldsymbol{X}$, Y using Self-Dual-Minimization (SeDuMi) package for Matlab [Peaucelle at all. 2002], the feedback gain matrix design problem was solved as feasible with the result

$$\boldsymbol{X} = \begin{bmatrix} 1.7454 & -0.8739 & 0.0393 \\ -0.8739 & 1.3075 & -0.5109 \\ 0.0393 & -0.5109 & 2.0436 \end{bmatrix}, \quad \boldsymbol{Y} = \begin{bmatrix} 0.9591 & 1.2907 & -0.1049 \\ -0.1950 & -0.5166 & -0.4480 \end{bmatrix}, \quad \gamma = 1.8509$$

$$\boldsymbol{K} = \begin{bmatrix} 1.2524 & 1.7652 & 0.0436 \\ -0.0488 & -0.2624 & -0.3428 \end{bmatrix}$$

In the next step the solution to (54), (55) using design parameters $\gamma = 1.8509$ was also feasible with the LMI variables

$$\boldsymbol{V} = -1.3690, \quad \boldsymbol{S} = 1.1307, \quad \boldsymbol{W} = \begin{bmatrix} 0.9831 & 0.7989 \end{bmatrix}$$

$$\boldsymbol{R} = \begin{bmatrix} 1.7475 & 0.0013 & 0.0128 \\ 0.0013 & 1.4330 & 0.0709 \\ 0.0128 & 0.0709 & 0.6918 \end{bmatrix}, \quad \boldsymbol{Z} = \begin{bmatrix} -0.0320 & 1.0384 \\ 0.1972 & 0.1420 \\ -2.0509 & -1.1577 \end{bmatrix}$$

which gives

$$\boldsymbol{J} = \begin{bmatrix} 0.0035 & 0.6066 \\ 0.2857 & 0.1828 \\ -2.9938 & -1.7033 \end{bmatrix}, \quad \boldsymbol{N} = \begin{bmatrix} 0.8694 & 0.7066 \end{bmatrix}, \quad \boldsymbol{M} = -1.2108$$

Verifying the obtain results the system matrices were constructed as

$$\boldsymbol{A}_c = \boldsymbol{A} - \boldsymbol{BK} = \begin{bmatrix} -1.1062 & 0.0282 & 0.9847 \\ -2.4561 & -3.2659 & 1.2555 \\ -6.0087 & -9.4430 & -3.3297 \end{bmatrix}, \quad \mathrm{eig}(\boldsymbol{A}_c) = \{-0.7110 \; -3.4954 \pm \mathrm{i}\,4.3387\}$$

$$\boldsymbol{A}_e = \boldsymbol{A} - \boldsymbol{JC} = \begin{bmatrix} -0.6101 & 0.3864 & -0.0035 \\ -0.4684 & -0.7541 & 0.7143 \\ -0.3029 & -1.3092 & -2.0062 \end{bmatrix}, \quad \mathrm{eig}(\boldsymbol{A}_e) = \{-1.0000 \ \ -1.1852 \pm \mathrm{i}\, 0.7328\}$$

and it is evident that the designed observer-based control structure results the stable system.

6 CONCLUSION

An active fault tolerant control is proposed in the modified structure. Sufient conditions on the existence of such an FDIR system and a solution to both controller and fault estimator matrix parameters are derived in term of LMIs. Finally, a numerical example is given to show the effectiveness of the method.

ACKNOWLEDGMENT

The work presented in this paper was supported by VEGA, Grant Agency of Ministry of Education and Academy of Science of Slovak Republic, under Grant No. 1/0328/08. This support is very gratefully acknowledged.

REFERENCES

BLANKE, M., KINNAERT, M., LUNZE, J., STAROSWIECKI, M. (2003): *Diagnosis and Fault-tolerant Control.* Springer, Berlin.

DING, S.X. (2008): *Model-based Fault Diagnosis Techniques: Design Schemes, Alghorithms, and Tools.* Springer, Berlin.

DONG, Q., ZHONG, M., DING, S.X. (2009): On active fault tolerant control for a class of time-delay systems. In Preprints of the 7th IFAC Symposium on Fault Detection, Supervision and Safety of Technical Processes, Barcelona, Spain, pp. 882-886.

KOZÁKOVÁ, A., VESELÝ, V. (2009): Design of robust decentralized controllers using the M-Δ structure robust stability conditions. *International Journal of Systems Science*, **40**, (No. 5), pp. 497-505.

KROKAVEC, D., FILASOVÁ, A. (2007): *Dynamic Systems Diagnosis*. Elfa, Košice (in slovak).

KROKAVEC, D., FILASOVÁ, A. (2008): Diagnostics and reconfiguration of control systems, *Advances in Electrical and Electronic Engineering,* **7**, (No. 1-2), pp. 15-20.

PEAUCELLE, D., HENRION, D., LABIT, Y., TAITZ, K. (2002): *User's Guide for Sedumi Interface 1.04*. LAAS-CNRS, Toulouse.

SKELTON, R.E., IVASAKI, T., GRIGORIADIS, K. (1998): *A Unified Algebraic Approach to Linear Control Design*. Taylor & Francis, London.

VESELÝ, V., ROSINOVÁ, D. (2008): Output feedback controller design. Non-iterative Approach. *Journal of Electrical Engineering*. **59**, (No. 6), pp. 317-321.

ZHOU, K.M., REN, Z. (2001): A new controller architecture for high performance, robust and fault tolerant control. *IEEE Transactions on Automatic Control*, **40**, (No. 10), pp. 1613-1618.