A new approach to explicit MPC using self-optimizing control

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Abstract

In our previous work on "self-optimizing control" we look for simple control policies to implement optimal operation. In particular, we have looked at "what should we control" (choice of controlled variables (CV's)). For quadratic problems with linear constraints, optimal linear variable combinations $c = Hy$ may be obtained. In this work, we apply these results to model predictive control and rederive the results by Bemporad et al. [2002] on explicit MPC. More importantly, we derive some new results and insights. One is that tracking the value of c (deviation from optimal feedback law) for all regions, may be used to identify changes between constraint regions. We also have new ideas on output feedback and including measurement noise.

1 Executive summary

1.1 Our starting point

In our previous work on "self-optimizing control" we look for simple control policies to implement optimal operation. In particular, we have looked at "what should we control" (choice of controlled variables (CV's)). Using off-line optimization we determine regions where different sets of active constraints are active, and implementation of optimal operation is then in each region to:

1. Control the active constraints.
2. For the remaining unconstrained degrees of freedom: Control "self-optimizing" variables c which have the property that keeping them constant ($c = c_s$) indirectly achieves close-to optimal operation (with a small loss), in spite of disturbances d. We here consider linear measurement combinations, $c = Hy$. There are here two factors that should be considered:
   (a) Disturbances d. Ideally, we want the optimal value of c (c_{opt}) to be independent of d.
   (b) Measurements errors n^y. The loss should be insensitive to these.

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1.2 Relationship to explicit MPC

Consider a simple static optimization problem $\min_u J(u,d)$, where u are the unconstrained degrees of freedom. Consider the quadratic case

$$J(u,d) = [u \quad d]^T S [u \quad d]$$
$$\text{where } S = \begin{bmatrix} J_{uu} & J_{ud} \\ J_{ud}{}^{\mathrm{T}} & J_{dd} \end{bmatrix}. \tag{1}$$

A key result, which is the basis for this report, is (see PROOF below):

For a quadratic optimization problem there exists (infinitely many) linear measurement combinations $c = Hy$ that are optimally invariant to disturbances d.

A key issue to identify a good set y that leads to a simple implementation.

One sees immediately that there may be some link to explicit MPC, because the discrete form MPC problem can be written as a static quadratic problem. The link is: If we let y contain the inputs u and the states x, then the variables $c = Hy$ give the optimal feedback law, $c = u - Kx$.

Based on this, we provide in this report some *new* ideas on explicit MPC:

1. We propose that tracking the variables c (deviation from optimal feedback law) for all regions, may be used as a local method to detect when to switch between regions.
2. We may use our results to include measurement error in y (e.g. in x and u) when deriving the optimal explicit MPC.
3. We may extend the results to output feedback ($c = u - Ky$) by including in y present and past outputs (and not present states x).
4. We can also extend the results to the case where only a subset of the states are measured (but in this case there will be a loss, which we can quantify). This may be of interest even in the unconstrained LQ case.

1.3 Some more details

First, a PROOF of the statement *For a quadratic optimization problem there exists (infinitely many) linear measurement combinations c=Hy that are optimally invariant to disturbances d*

When there is a disturbance, the optimal system state will generally depend on disturbances, and we may write $y_{\text{opt}} = Fd$, where F is the optimal sensitivity matrix. For example, it is easy to show that the optimal input is linear in d: $u_{\text{opt}} = -J_{uu}^{-1} J_{ud} d$. Then if we assume y depends linearly on u and d, $y = G^y u + G^y_d d$, we get $F = G^y J_{uu}^{-} 1 J_{du} - G^y_d$, and we see that F is a constant matrix.

1.3.1 Nullspace method

We look for linear combinations, $c = Hy$, that are optimally invariant to d, that is, we want $c_{\text{opt}} = Hy_{\text{opt}} = HFd = 0$. This should be satisfied for any d, so the requirement becomes $HF = 0$, that is, H should be in the left nullspace of F. It is always possible to find such an optimal H (in the left nullspace of F), provided the matrix $\widetilde{G}^y = [G^y \quad G^y_d]$ from $[u \quad d]$ to y has full row rank, which in particular requires $n_y \geq n_u + n_d$.

This provides a proof of the statement and also gives the "nullspace method" for finding H. Next, consider a more general method that also includes measurement noise.

1.3.2 Exact local method

With measurement error, there are no optimal invariants, so let us instead look for variable combinations $c = Hy$ which have the property that when added as extra constraints to the original optimization problem, they give a minimal extra loss, $L = z^{\mathrm{T}}z$. The "loss" variables z may be written $z = M_d d' + M_{n^y} n^{y'}$, where d' and $n^{y'}$ are the normalized disturbances and measurements errors. Minimizing the (extra) loss is then equivalent to minimizing the norm of the matrix $M = [M_d \quad M_{n^y}]$, that is we want to solve the problem $\min_H \; \|M(H)\|\|$. Some algebra gives that this may be reformulated as

$$\min_H \|H\widetilde{F}\| \quad \text{subject to } HG^y = J_{uu}^{1/2}, \tag{2}$$

where $\widetilde{F} = [FW_d \quad W_{n^y}]$.

Comments:

1. If we use the Frobenius norm, then this is a QP problem subject to linear constraint, and an analytic solution is available.
2. For the special case with no measurement error ($W_{n^y} = 0$), the (extra) loss is zero, provided we have sufficient number of measurements such that we can get $HF = 0$.
3. This general formulation applies also when we do not have sufficient number of measurements, and we can find the corresponding loss and optimal H.

This ends the "executive summary". More details are found in the rest of the report. We are grateful for any comments you may have!

2 Introduction

Consider the general static optimization problem:

$$\begin{aligned} \min_{u_0,x} \quad & J_0(x, u_0, d) \\ \text{s.t.} \quad & f_i(x, u_0, d) = 0, \quad i \in \mathcal{E} \\ & h_i(x, u_0, d) \geq 0, \quad i \in \mathcal{I}, \end{aligned} \tag{P1}$$

where $x \in \mathbb{R}^{n_x}$ are the states, $u_0 \in \mathbb{R}^{n_{u_0}}$ are the inputs, and $d \in \mathcal{D} \subset \mathbb{R}^{n_d}$ are disturbances. Usually f is a model of the physical system, whilst h is a set of inequality constraints that limits the operation (e.g., physical limits on temperature measurements or flow constraints) Alstad and Skogestad [2007a].

In addition to (P1) we have measurements on the form

$$y_0 = f^y(x, u_0, d). \tag{3}$$

Remark 2.1 (Note on notation)**.** *The "original" degrees of freedom are denoted u_0, and in regions where some constraints are active the remaining subset of unconstrained degrees of freedom are denoted u. The "original" measurements from the underlying process (which may include measured states and measured disturbances) are denoted y_0, but we often refer to an extended candidate set y of "measurements" that are used when selecting the controlled variables, $c = Hy$. The candidate "measurements" c may include, in addition to y_0, also the (original) degrees of freedom u_0.*

In this work the emphasis is on *implementation of the solution to* (P1). This means that problem (P1) is solved off-line to generate a "control policy" that is suitable for on-line implementation, especially with emphasis on remaining close to optimal solution when there are unknown disturbances. That is, we search for "control policies" such that we remain optimal or close to optimal when disturbances occur without the need to reoptimize. In terms the optimization problem (P1), the "control policy" may be viewed as an additional set of constraints that we impose, and the objective is that these extra constraints should be (1) suitable for easy implementation and (2) such that the loss in terms of the cost J_0 is acceptable. For a quadratic problem with linear constraints we will prove that the loss imposed by introducing a linear set of constraints ($c = Hy =$ constant) may be zero if H is selected optimally.

Let us now return to the optimization problem itself (and not the implementation of the optimal solution). At the solution of (P1) some of the inequalities may be active and we define the active set:

Definition 2.1 (Active set Nocedal and Wright [1999])**.** *The active set $\mathcal{A}(x, u_0, d)$ at any feasible (x, u_0, d) is the union of the set $\mathcal{E}$ with the indices of the active inequality constraints; that is,*

$$\mathcal{A}(x, u_0, d) = \mathcal{E} \cup \{i \in \mathcal{I} | h_i(x, u_0, d) = 0\} \tag{4}$$

Notice that the disturbances d may be considered to be parameters in (P1). Assume d can vary freely in the disturbance space $\mathcal{D}$. Inspired by Bemporad et al. [2002] we define critical regions as:

Definition 2.2 (Critical region)**.** *A critical region CR_i is the set of all vectors $d \in \mathcal{D}_i \subseteq D$ such that the optimal active set A_i remains unchanged.*

An important property of the solution of (P1) is the optimal active set $\mathcal{A}$. Assume that this is known *a priori* to solving (P1). Then solving (P1) is the same as solving the reduced problem in definition 2.3.

Definition 2.3 (Reduced problem). *Given* (P1), *a reduced problem is when the set of equations corresponding to the (assumed known) active set* $\mathcal{A}$ *is substituted into the original objective function* J_0. *This yields an unconstrained optimization problem*

$$\min_u J(u, d), \tag{P2}$$

where we notice that the states x *have been removed from the problem formulation. The degrees of freedom for optimisation are the dimension of* u, *which is*

$$\dim(u) = \dim(u_0) - N_A, \tag{5}$$

where N_A *is the number of active inequality constraints.*

Example 2.1 (Forming the reduced problem). *Consider the optimization problem:*

$$\begin{aligned} \min_{x,u_0} J_0 &= x_1^2 + u_1^2 + u_2^2 + u_2 d \\ s.t\ x_1 &= u_1 \\ x_1 &\geq 5 \end{aligned}$$

If d *is such that the constraint* $x_1 \geq 5$ *is active, the reduced problem will be*

$$\min_{u=u_2} J = 2 \cdot 5^2 + u_2^2 + u_2 d.$$

Observe that $\dim(u) = \dim(u_0) - N_A = 2 - 1 = 1 = \dim(u_2)$.

Locally, around the optimum, the optimization problem in definition 2.3 may be approximated by a quadratic problem (by using the first-order optimality condition $J_u = 0$ and assuming $J_d = 0$ since we cannot in any case correct for a first-order change in d on the cost):

Definition 2.4 (Reduced quadratic problem). *For the quadratic optimization problem with linear constraints,*

$$\begin{aligned} \min_{u_0,x} J(u_0, x, d) &= \begin{bmatrix} x & u_0 & d \end{bmatrix} S \begin{bmatrix} x \\ u_0 \\ d \end{bmatrix} \\ \text{subject to } & Ax + Bu + Cd = 0 \end{aligned} \tag{6}$$

we can form the following unconstrained quadratic optimization problem:

$$\min_u J(u, d) = \begin{bmatrix} u & d \end{bmatrix} \begin{bmatrix} J_{uu} & J_{ud} \\ J_{ud}^T & J_{dd} \end{bmatrix} \begin{bmatrix} u \\ d \end{bmatrix}, \tag{7}$$

when A *is invertible.*

J_{uu}, J_{ud} may be calculated from the matrices S, A, B, C. This is demonstrated in the following example.

Example 2.2. *Consider a static quadratic optimization problem in non-reduced form,*

$$\begin{aligned} \min_{u,x} \quad & (x^T Q x + u^T P u) \\ s.t.\ \ & Ax + Bu + Cd = 0. \end{aligned} \tag{8}$$

It can be written in this form (7), *with* $J_{uu} = P + B^T A^{-T} Q A^{-1} B$ *and* $J_{ud} = B^T A^{-T} Q A^{-1}$.

For a quadratic programming problem the parameter space can be partitioned into sets of critical regions using parametric programming Bemporad et al. [2002]. In each critical region the active set will then be known and the reduced problem (P2) can be formed.

Note that for general optimization problems (P2) may not be a simple function of u, d Alstad and Skogestad [2007a].

From an implementation point-of-view knowledge of the active inequalities are important because it is optimal to implement these as controlled variables Skogestad [2004].

The rest of this paper is organized as follows: In section 3 we will review earlier results from *self-optimizing control* and apply these results to quadratic programming problems, leading to new insights. In section 4 the results from section 3 will be applied to the explicit MPC problem, where we propose a method for tracking the current region. To the authors knowledge this method is new.

3 Results from self-optimizing control

The goal of self-optimizing control is to find a set of variables which, when kept at constant setpoints, indirectly lead to near-optimal operation with acceptable loss Skogestad [2000]. In this section we will present results from previous work on self-optimizing control and relate them to quadratic optimization problems. We follow the notation from Halvorsen et al. [2003].

3.1 Steady state conditions

Assume we have partitioned the disturbance space into critical regions by solving a parametric programming problem. For the i'th region we have N_A^i active constraints. The linearized steady-state input-output relationship is given by:

$$y = G^y u + G_d^y d, \tag{9}$$

where G^y and G_d^y are formed on the basis of the reduced problem (P2) (See figure 1). The unconstrained measurements y contains information about the present state and disturbances (y may include u_0 and d, but not the active constraints.) The (measured) value of y_m available for implementation is

$$y_m = y + n^y, \tag{10}$$

where n^y represents uncertainty in the measurement of y including uncertainty of implementation in u.

Theorem 3.1 (Nullspace method Alstad and Skogestad [2007a], Alstad and Skogestad [2007b]). *Assume that we have n_u independent unconstrained free variables u, n_d independent disturbances d, n_y independent measurements y, and we want to obtain $n_c = n_u$ independent controlled variables c that are linear combinations of the measurements*

$$c = Hy \tag{11}$$

If $n_y \geq n_u + n_d$ and there is no implementation error or measurement uncertainty, ($n^c = 0, n^y = 0$), there exists an optimal combination matrix H such that when controlling c and constant setpoint c_s, optimal operation is achieved locally. More specifically, by "locally" is meant in the region close to the optimum where the quadratic optimization problem (2.4) and the linear relationship (9) applies.

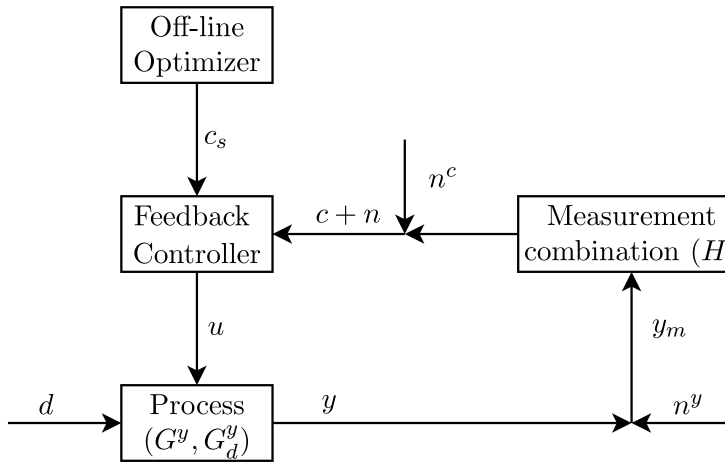

Figure 1: Block diagram of a feedback control structure including an optimization layer Alstad and Skogestad [2007b].

The optimal measurement combination matrix H is found by either:

1. *Let*

$$F = \frac{\partial y^{opt}}{\partial d^T} \tag{12}$$

be the optimal sensitivity matrix evaluated with constant active constraints. Under the assumptions stated above possible to select the matrix H in the left nullspace of F, $H \in \mathcal{N}(F^T)$, such that we get

$$HF = 0 \tag{13}$$

2. *Choose*

$$H = M_n^{-1} \widetilde{J} (\widetilde{G}^y)^{-1}, \tag{14}$$

where $\widetilde{J} = \begin{bmatrix} J_{uu}^{1/2} & J_{uu}^{-1/2} J_{ud} \end{bmatrix}$ and $\widetilde{G}^y = \begin{bmatrix} G^y & G^y_d \end{bmatrix}$ is the augmented plant. M_n^{-1} may be seen as a free parameter. (Note that $M_n = J_{cc}$ is the Hessian of the cost with respect to the c-variables; in most cases we select $M_n = I$ for convenience.)

Proof. See Alstad and Skogestad [2007a] and Alstad and Skogestad [2007b] and also the executive summary for a simple derivation of (13). □

Remark 3.1. *With this choice for H, fixing c (at its nominal optimal value) is first-order optimal for disturbances d; that is, the loss is zero as long as the sensitivity matrix F does not change.*

Remark 3.2. *The optimal F matrix is given by*

$$F = -\left(G^y J_{uu}^{-1} J_{ud} - G^y_d \right) \tag{15}$$

where J is the reduced space objective function. See the executive summary for a simple derivation of this result.

The above statements are close to the ones Alstad and Skogestad [2007a], and they are now stated in a different form more suitable for the present work. Specifically, we state two results where Theorem 3.1 is applied to first an unconstrained quadratic optimization problem (theorem 3.2), and then to a constrained optimization problem where in general the disturbance space can be partitioned into a set of critical regions (theorem 3.3).

Theorem 3.2 (Nullspace method for unconstrained quadratic optimization methods Alstad and Skogestad [2007a])**.** *Consider an unconstrained quadratic optimization problem on the form*

$$\min_u J(u,d) = \begin{bmatrix} u & d \end{bmatrix} \begin{bmatrix} J_{uu} & J_{ud} \\ J_{ud}^T & J_{dd} \end{bmatrix} \begin{bmatrix} u \\ d \end{bmatrix}, \tag{16}$$

If there exists $n_y \geq n_{u_0} + n_d$ independent measurements (where "independent" means that the matrix $\begin{bmatrix} G^y & G_d^y \end{bmatrix}$, for the effect of $\begin{bmatrix} u & d \end{bmatrix}^T$ on y, has full rank), then the optimal solution to (16) *has the property that there exists variable combinations $c = Hy$ that are invariant to the disturbances d.*

H may be obtained from the nullspace method using (13) *or* (14)*.*

Remark 3.3. *An equivalent formulation is: Assume that there exists a set of independent measurements y and that the (operational) constraint $c \triangleq Hy = c_s$ (where c_s is a constant) is added to the problem. Then there exists an H that does not change the solution to* (16)*. In terms of operation, this means that zero loss (optimal operation) is obtained by controlling $n_c = n_{u_0}$ variables $c = Hy$ with a constant set-point policy $c = c_s$, where H is selected according to theorem 3.2.*

For the case of more than one region, see theorem 3.3, this means that zero loss (optimal operation) is achieved by in each region controlling n_u^i variables $c^i = H^i y^i$ with a constant set-point policy $c = c_s$, where for each region H^i can be found using theorem 3.2.

Remark 3.4. *Note that the measurement set y is free to be chosen by the engineer.*

Theorem 3.3 (Nullspace method for constrained quadratic optimization methods)**.** *Consider an optimization problem on the form*

$$\begin{aligned} \min_{u_0,x} J_0 = \begin{bmatrix} x & u_0 & d \end{bmatrix} S \begin{bmatrix} x \\ u_0 \\ d \end{bmatrix} \\ s.t.\ f(x,u_0,d) = 0 \\ h(x,u_0,d) \leq 0 \end{aligned} \tag{17}$$

where f, h are both linear maps, $f : \mathbb{R}^{n_x+n_{u0}+n_d} \to \mathbb{R}^{n_x}$, and $h : \mathbb{R}^{n_x+n_{u_0}+n_d} \to \mathbb{R}^{n_m}$, where n_m is the number of inequality constraints. Further, let the matrices $\frac{\partial f}{\partial x^T}$ and $\frac{\partial h}{\partial x^T}$ have full rank.

Assume the disturbance space has been partitioned into n_a critical regions. In each region there is $n_u^i = n_{u_0} - n_A^i \geq 0$ unconstrained degrees of freedom, where $n_A^i \leq n_m$ is the number of optimally active constraints in region i.

If there exists a set of independent unconstrained measurements y^i in each region i, such that $n_{y^i} \geq n_{u^i} + n_d$, the optimal solution to (17) *has the property that there exists variable combinations $c^i = H^i y^i$ that for critical region i are invariant to the disturbances d. The corresponding optimal H^i may be obtained from theorem 3.2.*

Within each region, optimality requires that $c^i - c_s^i = 0$ (where c_s^i is a constant). From continuity of the solution we have that c^i is continous across the boundary of region i. This implies that the elements in the variable vector $c^i - c_s^i$ will change sign or remain zero when crossing into or from a neighboring region.

Proof. We write the linear map f as

$$Ax + Bu_0 + Cd = 0, \tag{18}$$

where $A = \frac{\partial f}{\partial x^{\mathrm{T}}}$ is invertible. Then,

$$x = -A^{-1}\left(Bu_0 + Cd\right), \tag{19}$$

and the problem can be written on the form

$$\begin{aligned} \min_{u_0} & \begin{bmatrix} u_0 & d \end{bmatrix} \begin{bmatrix} J_{u_0u_0} & J_{u_0d} \\ J_{u_0d}{}^{\mathrm{T}} & J_{dd} \end{bmatrix} \begin{bmatrix} u_0 \\ d \end{bmatrix} \\ \text{s.t } & h(x, u_0, d) \leq 0 \end{aligned} \tag{20}$$

In critical region i we can write the set of active constraints as:

$$G_i u_0 = W_i + E_i d. \tag{21}$$

Here we have substituted (19) into h, so x does no longer appear in the equations.

Assume that $G_i \in \mathbb{R}^{N_A \times (n_{u_0})}$ has rank of N_A, where $N_A = N_A^i$ is the number of active constraints in region i. (This is true if LICQ is fulfilled.) This implies that there are N_A independent columns in G_i. Without loss of generality we assume that these are the N_A last columns in G_i:

$$\begin{bmatrix} G_i^1 & G_i^2 \end{bmatrix} \begin{bmatrix} (u_0)_1 \\ (u_0)_2 \end{bmatrix} = W_i + E_i d \tag{22}$$

$$\Downarrow$$

$$u_0 = \underbrace{\begin{bmatrix} I_{(n_{u_0}-N_A)\times(n_{u_0}-N_A)} \\ (G_i^2)^{-1} G_i^1 \end{bmatrix}}_{Z} (u_0)_1 + \underbrace{\begin{bmatrix} 0_{(n_{u_0}-N_A)\times 1} \\ (G_i^2)^{-1}(W_i + E_i d) \end{bmatrix}}_{g} \tag{23}$$

$$u_0 = Z^i (u_0)_1 + g^i, \tag{24}$$

where G_i^2 consists of the last N_A columns of G_i.

We observe that for each region theorem 3.2 can be applied, as the equations for each region will be on the same form as (16) subsitute u_0 by $u = (u_0)_1$ in the objective function for each region i. □

3.2 Including measurement noise

We will now state a theorem from Halvorsen et al. [2003], which gives a method for minimizing the loss when measurement uncertainty n^y is included. The derivations in this subsection is based on Halvorsen et al. [2003] unless otherwise noticed.

The non-negative loss function is defined as

$$L(u, d) = J(u, d) - J(u_{\mathrm{opt}}(d), d). \tag{25}$$

Halvorsen et al. [2003] shows that the loss can be written as

$$L = \frac{1}{2}\|z\|_2^2 \tag{26}$$

where

$$z = J_{uu}^{1/2} \left[(J_{uu}^{-1} J_{ud} - G^{-1} G_d)(d - d^*) + G^{-1} n \right]. \tag{27}$$

Here $n = n^c + Hn^y$ is the implementation error, and $d - d^*$ is the derivation of the disturbance from the nominal operation point. n^y is the measurement/implementation error associated with y. In the following we assume zero control error, $n_c = 0$, but it is easy to extend the method to nonzero n^c.

Now, let the elements in the positive diagonal matrix W_d represent the expected magnitudes of the individual disturbances. Next, let the elements in the positive diagonal matrix W_n^y represent the magnitude of the implementation error associated with each of the candidate measurements y. Recall that we seek a measurement combination such that $c = Hy$. The expected magnitudes of the disturbances and the errors are then

$$d - d^* = W_d d' \tag{28}$$

$$n = HW_n^y n^{y'} = W_n n^{y'} \tag{29}$$

where d' and $n^{y'}$ are normalized to have norm of less than 1,

$$\| \begin{bmatrix} d' \\ n^{y'} \end{bmatrix} \| \leq 1. \tag{30}$$

(Or for the average case they have been scaled such that $\mathrm{var}([d' \quad y']^{\mathrm{T}}) = \gamma I$.) Then the loss z can be written as:

$$z = M_d d' + M_n n' = \underbrace{\begin{bmatrix} M_d & M_n \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} d' \\ n^{y'} \end{bmatrix}}_{f'} \tag{31}$$

where

$$M_d = J_{uu}^{1/2} \left(J_{uu}^{-1} J_{ud} - (HG^y)^{-1} (HG_d^y) \right) W_d \tag{32}$$

$$M_n = J_{uu}^{1/2} (HG^y)^{-1} HW_n \tag{33}$$

One observes that the matrix M may be understood as a gain-matrix from the normalized uncertain variables f' to the loss variable z.

Now consider the optimization problem of minimizing the loss $\|z\|_2^2$ subject to *the choice* (constraint) that we want to control a combination of measurements $c = Hy$. The loss by introducing this constraint for a specific choice of H is given in theorem 3.4. The optimal H that minimizes the loss is given in theorem 3.5 and 3.6. Note that this will not in general give zero loss from optimality, as there is implementation error present.

Theorem 3.4 (Loss by introducing constraint $c = Hy$ for quadratic optimization problem Halvorsen et al. [2003]). *Consider the unconstrained quadratic optimization problem stated in theorem 3.2,* (16)*:*

$$\min_u J(u, d) = \begin{bmatrix} u & d \end{bmatrix} \begin{bmatrix} J_{uu} & J_{ud} \\ J_{ud}^T & J_{dd} \end{bmatrix} \begin{bmatrix} u \\ d \end{bmatrix},$$

Consider a set of measurements y that are affected by noise, $y_m = y + n^y$. Assume that the operational constraint $c = Hy = c_s$ is added to the problem, where H is a non-trivial matrix. Consider disturbance and noise with magnitudes as given in (28)*,* (29) *and* (30)*.*

Then for a given H, the following results hold:

1. *The worst case loss is Halvorsen et al. [2003]:*

$$L_{worst\ case} = \bar{\sigma}(M)^2/2, \tag{34}$$

2. *The average loss is Kariwala et al. [2007]:*

$$L_{average} = \alpha \| \begin{bmatrix} M \end{bmatrix} \|_F^2, \tag{35}$$

where $M^T = \begin{bmatrix} M_d & M_n \end{bmatrix}$ *is given in* (32) *and* (33) *and* α *is a scaling factor depending on the probability distribution assumptions on* f*.*

Proof. The equality in (34) follows from the identity $z = Mf'$ and the fact that the induced (worst case) 2-norm of a matrix is equal to its maximum singular value.

The average loss can be written as:

$$\begin{aligned} L_{\text{average}} &= \frac{1}{2}\int (z^{\mathrm{T}}z)p(z)dz && (36)\\ &= \int (z_1^2 + z_2^2 + \cdots + z_n^2)p(z)dz && (37)\\ &= \mathrm{tr}(\mathrm{var}(z)), && (38) \end{aligned}$$

where the last equality follows from the definition of variance:

$$\mathrm{var}(z) = \int (zz^{\mathrm{T}})p(z)dz. \tag{39}$$

Further,

$$\mathrm{var}(z) = M\mathrm{var}(f)M^{\mathrm{T}} = \gamma MM^{\mathrm{T}} \tag{40}$$

where we have assumed that the variance of f is γI. One sees now that the average loss can be written as:

$$L_{\text{average}} = \alpha \mathrm{tr}(MM^{\mathrm{T}}) = \alpha \|M\|_F, \tag{41}$$

where $\alpha = \gamma/2$ is a scaling factor depending on the distribution assumptions for f.

If a uniform distribution on f is assumed, it can be shown that $\alpha = 1/(6n_u)$, where n_u is the number of unconstrained degrees of freedom (Based on Kariwala et al. [2007] and discussions with the author). □

The objective is nor to find the optimal H that minimizes the norm of M ($\bar{\sigma}(M)$ or $\|M\|_F$). This is sometimes referred to as "exact local method". However, from equations (32), (33), one sees that M depend on H in a non-linear manner. Fortunately, as shown in the following theorem, the problem can be reformulated to a constrained optimization where H enters linearly.

Theorem 3.5 (Optimal H by exact local method Alstad and Skogestad [2007b]). *The problem*

$$\min_H \|M\|, \tag{42}$$

where $M = \begin{bmatrix} M_d & M_n \end{bmatrix}$ *is given in* (32) *and* (33) $\|\cdot\|$ *is any matrix norm, can be formulated as*

$$\begin{aligned} &\min_H \|H\widetilde{F}\| \\ &s.t. \;\; (G^y)^{\,T} H^T = J_{uu}^{1/2}, \end{aligned} \tag{43}$$

where

$$\widetilde{F} \triangleq \begin{bmatrix} FW_d & W_{n^y} \end{bmatrix} \tag{44}$$

Proof. See Alstad and Skogestad [2007b] □

Theorem 3.6 (Analytical solution for the case $\|\cdot\|_F$ Alstad and Skogestad [2007b])**.** *It can be shown that when the Frobenius norm is used, and appropriate rank conditions are met, an analytical solution to the problem stated in theorem 3.5 is:*

$$H^T = (\widetilde{F}\widetilde{F}^T)^{-1} G^y \left(G^{y\,T} \left(\widetilde{F}\widetilde{F}^T \right)^{-1} G^y \right)^{-1} J_{uu}^{1/2} \tag{45}$$

Proof. See Alstad and Skogestad [2007b] □

Remark 3.5. *For the case of no implementation error,* $W_{n^y} = 0$*, we have:*

$$\|H\widetilde{F}\| = \|H \begin{bmatrix} FW_d & W_{n^y} \end{bmatrix} \|\big|_{W_{n^y}=0} = \|HFW_d\| \leq \|HF\|\|W_d\|, \tag{46}$$

and we observe that selecting H *such that* $HF = 0$ *yields zero loss (optimal operation). This is expected from theorem 3.1.*

Remark 3.6. *The author of Kariwala et al. [2007] shows that the matrix* H *minimizing the average loss* $L_{average}$ *also minimizes the worst case loss* $L_{worst\ case}$*. (The converse is not true, a matrix* $\hat{H}$ *minimizing the worst-case loss does not necessarily minimize the average loss.) This means that* (45) *is also optimal in terms of* $\bar{\sigma}(M)$*.*

3.3 Application to implementation

For the case of no measurement error, $n^y = 0$, theorems 3.2 and 3.3 show that for the solution to quadratic optimization problems variable combinations $c = Hy$ that are invariant to the disturbances can be found. In section 4 this insight will be used as a new approach to the explicit MPC problem.

4 Application to explicit MPC

We will now look at the model predictive control problem (MPC) with constraints on inputs and outputs. For a discussion on MPC in a unified theoretical framework see Muske and Rawlings [1993].

The following discrete MPC formulation is based on Pistikopoulos et al. [2002]. Consider the state-space representation of a given process model:

$$x(t+1) = Ax(t) + Bu(t) \tag{47}$$

$$y_0(t) = Cx(t), \tag{48}$$

subject to the following constraints:

$$y_{\min} \leq y_0(t) \leq y_{\max} \tag{49}$$

$$u_{\min} \leq u(t) \leq u_{\max}, \tag{50}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$ are the state, input and output vectors, respectively, subscripts min and max denote the lower and upper bounds, respectively, and (A, B) is stabilizable. MPC problems for regulating to the origin can then be posed as the following optimization problem:

$$\begin{aligned} \min_U J(U, x(t)) = x_{t+N_y|y}{}^{\mathrm{T}} P x_{t+N_y|t} + \sum_{k=0}^{N_y-1} \left[x_{t+k|t}{}^{\mathrm{T}} Q x_{t+k|t} + u_{t+k}{}^{\mathrm{T}} R u_{t+k} \right] \\ \text{s.t. } y_{\min} \leq y_{t+k|t} \leq y_{\max}, \quad k = 1, \ldots, N_c \\ u_{\min} \leq u_{t+k} \leq u_{\max}, \quad k = 0, 1, \ldots, N_c \\ x_{t|t} = x(t) \\ x_{t+k+1|t} = A x_{t+k|t} + B u_{t+k}, \quad k \geq 0 \\ y_{t+k|t} = C x_{t+k|t}, \quad k \geq 0 \\ u_{t+k} = K x_{t+k|t}, \quad N_u \leq k \leq N_y \end{aligned} \tag{51}$$

where $U \triangleq \{u_t, \ldots, u_{t+N_u-1}\}$, $Q = Q^{\mathrm{T}} \geq 0$, $R = R^{\mathrm{T}} > 0$, $P \geq 0$, $N_y \geq N_u$, and K is some feedback gain. The authors of Pistikopoulos et al. [2002] show that by substitution of the model equations, the problem can be rewritten to the form

$$\begin{aligned} \min_U \frac{1}{2} U^{\mathrm{T}} H U + x(t)^{\mathrm{T}} F U + \frac{1}{2} x(t)^{\mathrm{T}} Y x(t) \\ \text{s.t. } GU \leq W + E x(t) \end{aligned} \tag{52}$$

The MPC control law is based on the following idea: At time t, compute the optimal solution $U^*(t) = \{u_t^*, \ldots, u_{t+N_u-1}^*\}$ and apply $u(t) = u_t^*$ Bemporad et al. [2002].

If we let the initial state $x(t)$ be treated as a disturbance, (52) can be written as:

$$\begin{aligned} \min_U \frac{1}{2} \begin{bmatrix} U^{\mathrm{T}} & d^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} H & F \\ F & Y \end{bmatrix} \begin{bmatrix} U \\ d \end{bmatrix} \\ \text{s.t. } GU \leq W + Ed, \end{aligned} \tag{53}$$

and we observe that (53) is on the same form as (17), where the model equations $f(x, u_0, d) = 0$ have already been substituted into the objective function.

A property of the solution to (53) is that the disturbance space (initial state space) will be divided into critical regions. In the i'th critical region there will be $n_u^i = n_U - n_A^i$ unconstrained degrees of freedom, where n_A^i is the number of active constraints in region i. From theorem 3.3 we know that if there exists a set of independent measurements y^i (for region i), such that $n_{y^i} \geq n_u^i + n_d$, the optimal solution to (53) has the property that there exists variable combinations $c^i = H^i y^i$, where H^i is a non-trivial combination matrix, that are invariant to the parameters d $(= x(t))$.

As we will show in section 4.1, a possible set of measurements y is the current state and the inputs, $y^{\mathrm{T}} = \begin{bmatrix} x^{\mathrm{T}} & u^{\mathrm{T}} \end{bmatrix}$. We further note that causality is not an issue here, as we have the information at the current time. If however not all states are measured we can still find variable combinations that are invariant to the initial states, but these will be functions of future measurements.

4.1 Exact measurements of all states (state feedback)

The following theorem is well known, but we will there prove the theorem using the nullspace method. The reason for why we can do this is that the MPC problem at the present time $t = k$ can be seen as a static quadratic optimization problem with linear constraints.

Theorem 4.1 (Optimal state feedback Bemporad et al. [2002]). *The control law $u(t) = f(x(t))$, $f : \mathbb{R}^n \mapsto \mathbb{R}^m$, defined by the MPC problem, is continuous and piecewise affine*

$$f(x) = K^i x + g^i \quad \textit{if } H^i x \leq k^i, \quad i = 1, \ldots, N_{mpc} \tag{54}$$

where the polyhedral sets $\left\{H^i x \leq k^i\right\}$, $\quad i = 1, \ldots, N_{mpc} \leq N_r$ are a partition of the given set of states X.

Proof. We will use theorem 3.3 to prove that the optimal control is on the form $u = Kx + g$. We consider the MPC problem written on the from

$$\begin{aligned} &\min_U \frac{1}{2} \begin{bmatrix} U^{\mathrm{T}} & d^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \hat{H} & \hat{F} \\ \hat{F} & Y \end{bmatrix} \begin{bmatrix} U \\ d \end{bmatrix} \\ &\text{s.t. } GU \leq W + Ed, \end{aligned} \tag{55}$$

which is on the form of (17) (see theorem 3.3), but the states have here already been substituted into the objective function.

First we assume that no constraints are active, i.e. there are no equations that need to be substituted back. From theorem 3.2 we known that to achieve zero loss from optimality we need as many measurements as there are disturbances and inputs. We now the inputs in the "measurements" y. We assume that the disturbances enter *additively* on the states, which themselves are also measured. This means the vector of measurements y is

$$y^{\mathrm{T}} = \begin{bmatrix} x^{\mathrm{T}} & U^{\mathrm{T}} \end{bmatrix}$$

With this choice of measurements and disturbances on the present state, we form the process model:

$$\Delta y = G^y \Delta U + G^y_d \Delta d \tag{56}$$

$$G^y = \begin{bmatrix} 0_{n_x \times (n_u N_u)} \\ I_{(n_u N_u) \times (n_u N_u)} \end{bmatrix} \in \mathbb{R}^{(n_x + n_u N_u) \times (n_u N_u)} \tag{57}$$

$$G^y_d = \begin{bmatrix} I_{n_x \times n_x} \\ 0_{(n_u N_u) \times n_x} \end{bmatrix} \in \mathbb{R}^{(n_x + n_u N_u) \times n_x} \tag{58}$$

Here $I_{n \times n}$ is a square identity matrix and $0_{n \times m}$ is a matrix of zeros. n_x is the number of states, n_u is the number of inputs and N_u is the input horizon. From the objective function and the assumption that no constraints are active, we have that $J_{uu} = \hat{H}$ and $J_{ud} = \hat{F}$. Hence the optimal sensitivity matrix is

$$F = \frac{\partial y^{\mathrm{opt}}}{\partial d^{\mathrm{T}}} = -\left(G^y J_{uu}^{-1} J_{ud} - G^y_d\right) = -\left(\begin{bmatrix} 0_{n_x \times (n_u N_u)} \\ (J_{uu}^{-1} J_{ud})_{(n_u N_u) \times n_x} \end{bmatrix} - \begin{bmatrix} I_{n_x \times n_x} \\ 0_{(n_u N_u) \times n_x} \end{bmatrix} \right) \tag{59}$$

$$= \begin{bmatrix} I_{n_x \times n_x} \\ -J_{uu}^{-1} J_{ud} \end{bmatrix} \tag{60}$$

We now search for a matrix H that gives a non-trivial solution to $HF = 0$:

$$\begin{bmatrix}(H_1)_{(n_u N_u)\times n_x} & (H_2)_{(n_u N_u)\times(n_u N_u)}\end{bmatrix}\begin{bmatrix}I_{n_x\times n_x}\\ J_{uu}^{-1}J_{ud}\end{bmatrix} = \tag{61}$$

$$= H_1 - H_2\left(J_{uu}^{-1}J_{ud}\right) = 0 \tag{62}$$

To ensure a non-trivial solution we can for example choose $H_2 = I_{(n_u N_u)\times n_u N_u}$. Then we must have $H_1 = J_{uu}^{-1}J_{ud}$, and hence the *optimal combination c of x and U* becomes

$$c = Hy = J_{uu}^{-1}J_{ud}x + U = 0 \in \mathbb{R}^{(n_u N_u)} \tag{63}$$

This implies that the input at present and future times can be written on the form:

$$(u_k = K^k x_k), (u_{k+1} = K^{k+1}x_k), \ldots, (u_{k+N_u-1} = K^{k+N_u-1}x_k)$$

We will now cover the case when some constraints are optimally active. Assume that these can be expressed as

$$G_1 U = W_1 + E_1 x(t), \tag{64}$$

where $G_1 \in \mathbb{R}^{N_A\times(n_u N_u)}$ has rank of N_A, where N_A is the number of active constraints. This implies that there are N_A independent columns in G_1. Without loss of generality we assume that these are the N_A last columns in G_1:

$$\begin{bmatrix}G_1^1 & G_1^2\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} = W_1 + E_1 x(t) \tag{65}$$

$$\Downarrow$$

$$u = \begin{bmatrix}I_{(n_u N_u - N_A)\times(n_u N_u - N_A)}\\ (G_1^2)^{-1}G_1^1\end{bmatrix}u_1 + \begin{bmatrix}0_{(n_u N_u - N_A)\times 1}\\ (G_1^2)^{-1}\left(W_1 + E_1 x(t)\right)\end{bmatrix} \tag{66}$$

$$u = Zu_1 + g, \tag{67}$$

where G_1^2 consits of the last N_A columns of G_1. After removing constant terms the reduced space objective function can now be written as:

$$J = \frac{1}{2}{u_1}^{\mathrm{T}}Z^{\mathrm{T}}HZu_1 - g^{\mathrm{T}}HZu_1 + x(t)^{\mathrm{T}}FZu_1 \tag{68}$$

and we observe that $J_{uu} = Z^{\mathrm{T}}HZ$, and $J_{ud} = HZ$. The measurement model is now:

$$\Delta y = G^y Z\Delta u_1 + G_d^y \Delta d, \tag{69}$$

but due to the active constraints there is N_A dependent measurements. We can therefore use the reduced measurement model

$$\Delta\hat{y} = \hat{G}^y u_1 + \hat{G}_d^y \Delta d, \tag{70}$$

with

$$\hat{G}^y = \begin{bmatrix}0_{n_x\times(n_u N_u - N_a)}\\ I_{(n_u N_u - N_a)\times(n_u N_u - N_a)}\end{bmatrix} \in \mathbb{R}^{(n_x + n_u N_u - N_a)\times(n_u N_u - N_a)} \tag{71}$$

$$\hat{G}_d^y = \begin{bmatrix}I_{n_x\times n_x}\\ 0_{(n_u N_u - N_a)\times n_x}\end{bmatrix} \in \mathbb{R}^{(n_x + n_u N_u - N_a)\times n_x} \tag{72}$$

We observe that the reduced problem is on the same form as the full problem and hence we will get feedback laws on the same form as before in the unconstrained degrees of freedom u. We also observe that in u_0 we get affine terms, see equation (67). □

Remark 4.1 (Comparison with previous results on unconstrained MPC)**.** *In (63) the state feedback gain matrix is given as* $J_{uu}^{-1} J_{ud}$*. In the appendix we prove that this is gives the same result as conventional MPC, see equation (3) in Rawlings and Muske [1993].*

Remark 4.2. *As set up above, we consider a disturbance to the present (initial) state and find the optimal present and future inputs (within the defined input horizon) using the derived state feedback law. However, we only implement the present input, and then obtain a new measurement (state) and find the next optimal input using the new state (but with the same state feedback law). This implies that optimal state feedback solution will be optimal to any change (disturbances) on the states, and not only on the initial state.*

Remark 4.3. *These are not new results but the alternative proof leads to some new insights. The most important is probably that the "self-optimizing" variables* $c^i = u - (K^i x + g^i)$ *which are optimally zero in region* i*, may be used for identifying when to switch between regions (theorem 4.2) rather than using a "centralized" approach, for example based on a state tree structure search. This seems to be new. Another insight is to understand why a simple feedback solution must exist in the first place. A third is to allow for new extensions.*

4.2 Region detection using feedback law

Theorem 4.2 (Optimal region for explicit MPC detection using feedback law)**.** *The variables* $c = u_k - (Kx_k + g)$ *can be used to identify region changes.*

Proof. Assume there is a partition of the state space consisting of regions CR_l with different optimal feedback controllers $u_k = K^l x_k + g^l$, i.e. we have merged regions where the first optimal input is the same. From theorem 4.1, the feedback laws are continuous and affine. Let c_I be the set of feedback laws for the neighboring regions to the current region. For $i \in I$, $c_i = u_k - (K^i x_k + g^i)$. Assume further that in the current region the optimal controller is already implemented. Due to the continuity of c_i in region i, and since region i is a neighboring region to the current region, we know that if optimal control is implemented c_i is zero at the boundary between the current region and region i.

From this we realise that is it optimal to switch controller when one of the elements in the neighboring controlled variables vector c_I changes sign. □

Remark 4.4. *We have in the above derivation assumed that the disturbances enter such that the process can only move to the current region to a neighboring region between to sample times, i.e. it is assumed impossible to "jump across" regions.*

Remark 4.5. *Neighboring regions with the same feedback law (including regions where the feedback law is to keep the input saturated) can be merged (without needing to worry about the convexity properties of the regions). This may greatly reduce the number of regions compared to presently used enumeration schemes. Note that the number of c-variables that need to be tracked to detect region changes is only equal to the number of inputs* n_{u_0} *times the number of distinct merged regions. Because of the merging of regions, this may be a small number even with a large input or control horizon and with output (state) constraints.*

Algorithm 1 shows how the current region CR_k and the input u_k can be calculated by tracking the feedback laws of the neighbors to the current region. The parameter α_i takes the value of -1 or 1, in order to normalize the identification scheme. This comes from the fact that for each critical region we need to gather the following information:

Algorithm 1 Detect current region and calculate u_k

Require: CR_{k-1}, i.e. the region of the last sample time, and x_k

1: $u_k = K(CR_{k-1}) + g(CR_{k-1})$
2: $[\text{Regions}, \alpha] = \text{Neighbors}(CR_{k-1})$
3: **for** $i = 1$ to length(Regions) **do**
4: $\quad c_k(i) = \alpha_i \left(u_k - (K(\text{Regions}(i)) + g(\text{Regions}(i)))\right)$
5: **end for**
6: **if** $\text{sign}(c_k(i) \neq -1\ )$ **then**
7: $\quad CR_k = \text{Regions}(i)$
8: **else**
9: $\quad CR_k = CR_{k-1}$
10: **end if**
11: **return** $u_k = K(CR_k)x_k + g(CR_k)$, CR_k

- The neighboring regions.
- The sign of the gradient of the feedback laws of the neighboring regions in the current region.

The call $[\text{Regions}, \alpha] = \text{Neighbors}(CR_{k-1})$ calls the lookup table "Neighbors", which is an off-line generated table giving, for each region i, its neighboring regions $i_1, \ldots, i_l$ and the sign of the gradient of the neighboring regions feedback laws into the current region $(\alpha_{i_1}, \ldots, \alpha_{i_l})$, where l is the number of neighboring regions i.

In line 1 we calculate a trial input u_k. Then calculate the value of the c_k's using this input. If it was found that this input (not implemented yet) changed the current region, the current region is updated. Finally, the actual input is calculated, based on the possibly updated current region CR_k.

We next present a simple example from Bemporad et al. [2002] that confirms that our switching policy based on tracking the sign of the c-variables works in practice. Additional examples are presented in Section 4.7

Example 4.1 (Optimal switching)**.** *This example is taken from Bemporad et al. [2002] (with correction), and is included here to demonstrate optimal switching using $c = u - Kx$ as criterion. The system is:*

$$y(t) = \frac{2}{s^2 + 3s + 2} u(t).$$

With a sampling time $T = 0.1$ seconds the following state-space representation is obtained:

$$\begin{aligned} x(t+1) &= \begin{bmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{bmatrix} x(t) + \begin{bmatrix} 0.0609 \\ 0.0064 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1.4142 \end{bmatrix} x(t) \end{aligned}$$

One observes that only the last state is measured, but it will be assumed that both states are known (measured) in the remainder of this example.

The task is to regulate the system to the origin while fulfilling the input constraint

$$-2 \leq u(t) \leq 2. \tag{73}$$

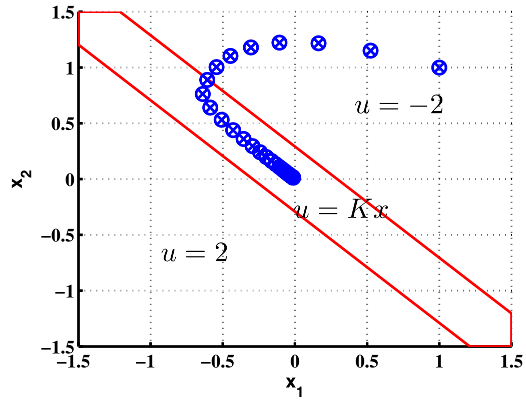

Figure 2: Partition of state space for first input. (Example 4.1.)

The objective function to be minimized is

$$\min x_{t+2|t}{}^T P x_{t+2|t} + \sum_{k=0}^{1} \left[x_{t+k|t}{}^T x_{t+k|t} + 0.01 u_{t+k}^2 \right] \tag{74}$$

subject to the constraints and $x_{t|t} = x(t)$.

P solves the Lyapunov equation $P = A^T PA + Q$, where $Q = I$ in this case. The optimal control problem can be solved for example using the MPT toolbox Kvasnica et al. [2004]. The P-matrix is numerically:

$$P = \begin{bmatrix} 5.5461 & 4.9873 \\ 4.9873 & 10.4940 \end{bmatrix}$$

To illustrate ideas a simulation from $x_0 = (1, 1)$ was done. State space trajectories and inputs are shown in figures 2 and 3. As long as the state is in the input-constrained region where $u^{opt} = -2$, the linear combination $c = u_k - Kx_k$ remains positive. One chooses to leave the input-constrained region when c becomes zero. The state trajectory is the same as in Bemporad et al. [2002].

The reason for why c never becomes negative is because both states are assumed measured at the present time and hence optimal switching is achieved. This can be understood from the algorithm 1, where we show how the current critical region (CR_k) is tracked and how the current input u_k is calculated.

So far we have not made any new contributions to explicit MPC except for proposing a new method for detecting region changes.

We now want to consider four new extensions to explicit MPC:

1. Include measurement noise (which includes implementation error for the input u).
2. Output feedback (with and without measurement noise).
3. Output feedback with fewer or additional measurements.
4. Other assumptions for the disturbance (rather than changes in the initial state).

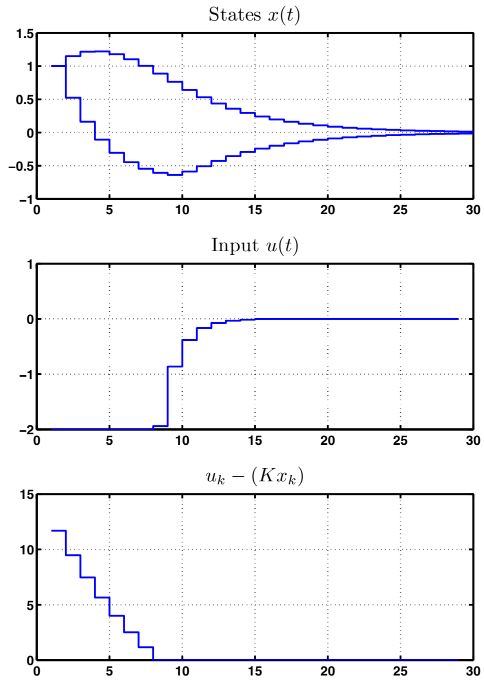

Figure 3: Closed loop MPC with region detection using $u_k - (Kx_k)$. (Example 4.1.)

4.3 State feedback with input noise

For the case with unconstrained optimal control (quadratic objective with a linear dynamic model), the separation principle applies, and with measurement noise we may for the use a Kalman filter to optimally reconstruct the states x, and otherwise use the same feedback law $u = -Kx$ as in the noise-free case. However, the separation principle does not apply when we add constraints to the inputs and outputs, at least as far as we know, and more importantly the estimation problem is not solved for this case.

In any case, we here propose an alternative approach, which avoids the need to explicitly reconstruct the states. We use the diagonal entries in the matrix W_n^y to quantify the magnitude of the noise in measurements of the states and the implementation error for the inputs u. Zero loss (in terms of obtaining the truly optimal u) is then not possible, but by imposing the constraint $c = Hy$ (which can be realized as a feedback law $u = -Kx$ by implementing only the first input) we get from theorem 3.6 (Optimal H for exact local method) the H that provides an optimal trade-off in terms of putting less emphasis on noisy measurements.

4.4 Output feedback with no noise

Consider now the case where all the states x are not measured. The objective is to find linear combinations $c = Hy$ that are optimally constant in each optimal region. From the nullspace method (Theorem 3.1) this requires that we have as many independent measurements y as there are inputs and disturbances.

We choose to do as above, and include the inputs U in y. This also ensures that we can easily implement the resulting $c = Hy$ (by setting $c = c_s$ and solving for U). With U included, we need as many additional measurements as we have disturbances.

If we assume that the disturbances are changes on the initial states (as we did above), then we need n_x additional measurements. For the case when all states are measured, these additional measurements are simply the present value of the states. However, here we consider the case when all states are not measured. Assume that we only have one measurement y, then to find an optimal linear combination (which is optimally invariant to d) we need to use (at least) n_x measurements, for example, the present and (at least) the previous $(n_x - 1)$ back in time, $[y_{k-n_x+1}, ..., y_k]$.

With no measurement error, the optimal combination $c = Hy$ can be obtained from the nullspace method using (14). This requires that $\widetilde{G}_y$ has full rank, which again implies that all d's can be observed from the outputs y. Because of causality $\widetilde{G}_y$ will not be full rank initially (just after the disturbance occurs), but the rank condition will be satisfied if we consider a disturbance entering sufficiently long ($n_x - 1$ steps) back in time. From this time and on the solution is the same as the state feedback solution. This argument may be used as an alternative approach (to the nullspace method) of generating the optimal output feedback solution. This is illustrated in the following example.

Also note that because we only implement the present input u, the solution will remain optimal with respect to new disturbances on the states, of course, with the restriction that it will not be optimal just after the change because of the causality issue mentioned above (see also remark 4.2). However, given that we only have output (and not state) measurements, the solution is presumably close to optimal.

In terms detecting region changes, we suggested for the state feedback case to use the deviation c from the optimal feedback laws as tracking variables. This simple strategy may not work as well with output feedback, partly because output feedback is not truly optimal, and partly because the outputs do not contain accurate information about the present state.

Example 4.2 (Output feedback). *Consider the discrete time state space model (see also example 4.1):*

$$\begin{aligned} x(t+1) &= \begin{bmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{bmatrix} x(t) + \begin{bmatrix} 0.0609 \\ 0.0064 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1.4142 \end{bmatrix} x(t), \end{aligned}$$

with associated constrained optimal control problem shown in equations (73) *and* (74)*:*

$$\begin{aligned} \min_{u,x} \quad & x_{t+2|t}{}^T P x_{t+2|t} + \sum_{k=0}^{1} \left[x_{t+k|t}{}^T x_{t+k|t} + 0.01 u_{t+k}^2 \right] \\ s.t. \quad & -2 \le u(t) \le 2 \text{ and proces model} \end{aligned}$$

Even though the states are not measured, it is known that the optimal solution is on the form $u_k = K^i x_k + g^i$. *Figure 2 shows how the state space is, by solving a parametric program, partitioned into 3 regions with 3 different state feedback laws. As before, let* $d = x_k$. *Since the optimal solution is known, the sensitivity matrix* F *for the measurements* $y = (u_k, y_k, y_{k+1})$ *can be established:*

$$\left(\frac{\partial \begin{bmatrix} u_k & y_k & y_{k+1} \end{bmatrix}^T}{\partial d^T} \right)^{opt} = \begin{bmatrix} K \\ C \\ C(A+BK) \end{bmatrix}, \tag{75}$$

where A, B, C *correspond to the discrete process model. An invariant combination of* (u_k, y_k, y_{k+1}) *can now be found by solving the equation* $H'F = 0$ *for* H', *yielding* $h'_1 u_k + h'_2 y_k + h'_3 y_{k+1} = c^1 = c_s^1 = 0$.

This can not be implemented due to causality. Since there are an infinite number of possible invariants to the disturbances for the solution of quadratic optimization problems (see theorem 3.2), a combination of (y_k, y_{k+1}, y_{k+2}) *can also be found, for which the optimal sensitivity matrix is:*

$$\left(\frac{\partial \begin{bmatrix} y_k & y_{k+1} & y_{k+2} \end{bmatrix}^T}{\partial d^T}\right)^{opt} = \begin{bmatrix} C(A+BK)^1 \\ C(A+BK)^2 \\ C(A+BK)^3 \end{bmatrix}. \tag{76}$$

Also for this variable combination a matrix H *can be found such that* $HF = 0$*, i.e.* $h_1 y_k + h_2 y_{k+1} + h_3 y_{k+2} = c^2 = c_s^2 = 0$*. This combination can be shifted one time step back by multiplying by* z^{-1}*, giving* $h_1 y_{k-1} + h_2 y_k + h_3 y_{k+1} = c_s^2 = 0$*. Now, by eliminating* y_{k+1}*, we get*

$$u_k = \underbrace{\frac{1}{h_1'}\left(h_2 \frac{h_3'}{h_3} - h_2'\right)}_{k_1} y_k + \underbrace{\frac{h_1}{h_1'}}_{k_2} y_{k-1}, \tag{77}$$

which gives the input-output relationship. (Both set-points were zero because the process is normalised to $x = (0, 0)$*.)*

Another way to get the same controller gain is to assume that the disturbance enters at $t = k - 1$ *rather than at* $t = k$*. Again, assuming that the state feedback* $u = Kx$ *is optional, the sensitivity matrix is:*

$$\begin{bmatrix} u_k \\ y_k \\ y_{k-1} \end{bmatrix} = \underbrace{\begin{bmatrix} K(A+BK) \\ C(A+BK) \\ C \end{bmatrix}}_{F} x_{k-1}. \tag{78}$$

After finding the invariant variable combination, it is confirmed numerically that this approach yields the same controller gains as (77)*. The controller gains for the central region are* $(k_1, k_2) = (-16.7, 13.7)$*.*

Figure 4 shows the result of a simulation of the output feedback control from $x_0 = (1, 1)$*. Note that we use the output feedback control law for the unconstrained region to decide when to leave the constrained region. The optimal control with both states assumed measured is shown in the dotted line. One observers that the optimal control scheme leaves the constrained region one time instant before the output feedback scheme. This is expected as we need to wait another time instant to "estimate" the states in the new scheme. The "discontinuity" in the* $(u_k - (k_1 y_k + k_2 y_{k-1}))$ *curve is due to initialisation issues.*

4.5 Output feedback with noise

We have so far assumed that we have as many measurements as there are disturbances, which is the basis for the original nullspace method. However, more generally, we may easily extend the method to the case with fewer measurements (to get a lower-order controller) or with extra measurements (no provide additional noise filtering). This may involve using the pseudo inverse in the optimal H in Theorem 3.6 (exact local method), or if some rank condition fails, solving the problem in Theorem 3.5 numerically.

4.6 Other disturbance models

We have so far only considered disturbances on the (initial) states. This may be extended to include any realistic disturbance by including a disturbance model and assuming some states have no disturbances (or more generally making use of the disturbance weight W_d).

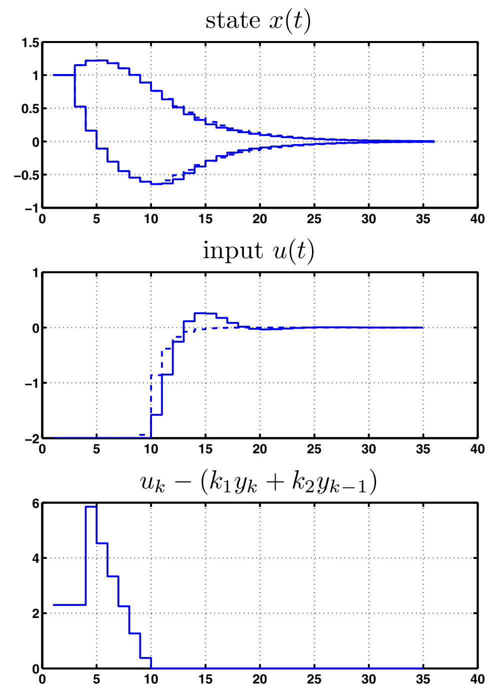

Figure 4: Simulation of output feedback configuration, where the output feedback law is used both for switching and control. Dotted line is optimal switching and control when both states assumed measured.

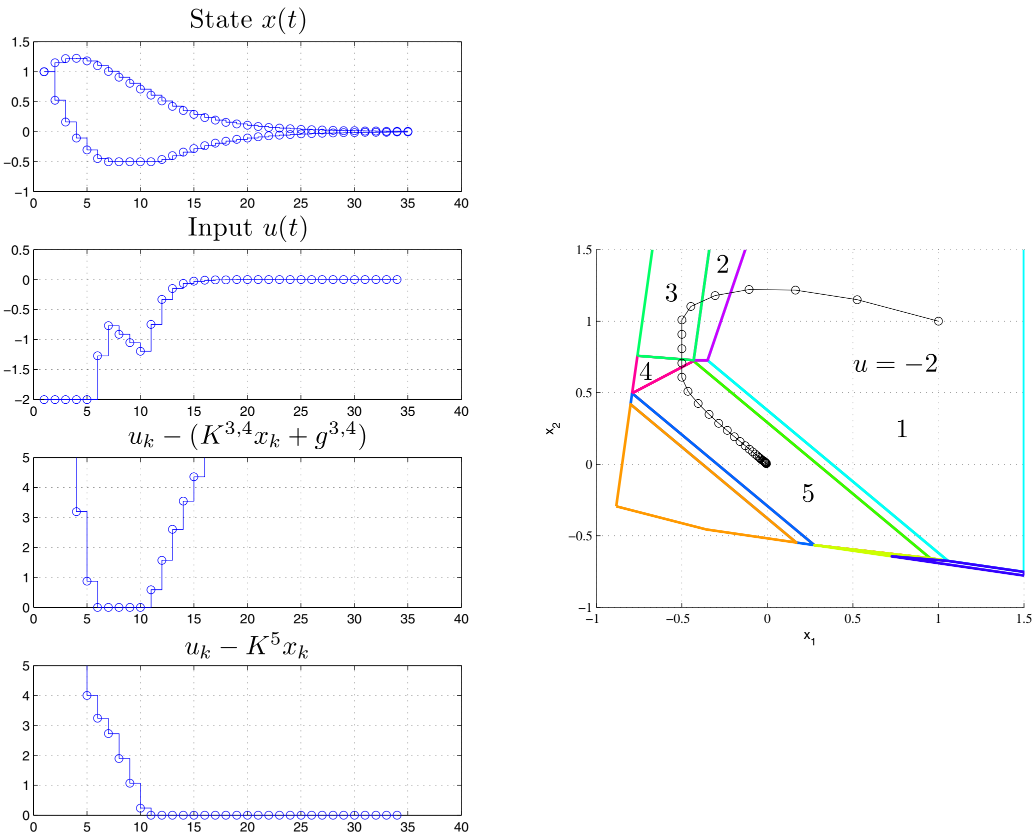

Figure 5: Simulation of state feedback, and illustration of how feedback laws can be used to identify optimal switching. Right figure shows the state trajectory in the state space. Simulation stated from $x = (1, 1)$. (Example 4.3.)

4.7 Additional examples

Example 4.3 (SISO system with state constraint)**.** *Consider the same process as in example 4.1, but with the additional state constraint* $x_{t+k|t} \geq x_{\min}$*,*

$$x_{\min} \triangleq \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}. \tag{79}$$

This problem is solved in Bemporad et al. [2002], we include it there to illustrate the switching using feedback laws and how the regions can be merged. In order to compare with the results from Bemporad et al. [2002], we start the simulation at $x = (1, 1)$ *and study how the controller brings the process back to origin.*

The right-hand-side of figure 5 shows the evaluation of the states in the state space, as well as the regions generated by solving the parametric program. The space was divided into 10 regions, and the regions that are relevant to this example have been numbered. (See corrigendum to Bemporad et al. [2002]).

Using the feedback law for optimal switching we here only need to consider 3 regions. First, in regions

1 and 2 the optimal input is $u = -2$ *and these can be merged. Further, regions 3 and 4 has the same feedback law,* $u_k = [-12.0296 \quad 1.4128]x_k + (-8.2102)$*, and can thus be merged. The last region, region 5, has the same feedback gain as the unconstrained region in example 4.1,* $K^5 = [-6.8355 \quad -6.8585]$*. The "physical" reason for why we can do this merging is that the merged regions have different future feedback laws (the difference in the optimal active set is in the form of future constraints entering or leaving). For example, in region 2,* $u_{k+1}^{opt} \neq u_k^{opt} = -2$*, but since we only consider states and input at the present time this information need not to be stored.*

For control and tracking, we see that we leave the input-constrained region when the variable $c_k^{3,4} = u_k - (K^{3,4}x_k + g^{3,4})$ *becomes zero, and we then control this variable to zero. Further, when the feedback law for the central region,* $c_k^5 = u_k - K^5 x_k$*, becomes zero we switch to this region.*

Example 4.4 (2×2 plant). *We study the* 2×2 *plant [Skogestad and Postlethwaite, 2005, p. 90]:*

$$G(s) = \frac{1}{5s+1}\begin{bmatrix} s+1 & s+4 \\ 1 & 2 \end{bmatrix}. \tag{80}$$

The plant has a RHP zero at $s = 2$*. We sample with 5/3 time-units (one third of the dominant time constant) and get the discrete-time model:*

$$x_{k+1} = 0.7165\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -0.0567 & -0.0567 \\ 0.2835 & 0.5669 \end{bmatrix} u, \tag{81}$$

$$y_k = I x_k + \begin{bmatrix} 0.2 & 0.2 \\ 0 & 0 \end{bmatrix} u. \tag{82}$$

We here assume that measurements of both states are available. Again we set up an optimal control problem on the form

$$\min_{U \triangleq u_t, \ldots, u_{t+N_u-1}} J(U, x(t)) = x_{t+N_y|t}{}^T P x_{t+N_y|t} + \sum_{k=0}^{N_y-1} [x_{t+k|t}{}^T Q x_{t+k|t} + u_{t+k}{}^T R u_{t+k}] \tag{83}$$

subject to the input constraint $-1 \leq u \leq 1$ *and the process mode. This this example we set*

$$N_c = 1, N_y = 2, N_u = 2, Q = I, R = 0.01I,$$

and $P = 2.0551I$ *is a solution of the Lyapunov equation* $P = A^T P A + Q$*.*

Figure 6 shows how the state-space is partitioned into regions when we allow the initial state to be inside $(-2, -2) \leq x_0 \leq (2, 2)$*. We observe that there is a large number of regions, even for this small example. Fortunately a lot of this regions can be merged because they have the same feedback law at the present time, see table 1. Indeed, by using the current approach the 23 original regions can be merged into 9 new ones. Note that when merging regions we don't need to worry about the convexity of the merged regions.*

Figure 7 shows the results of a simulation from $x_0 = (-1.5, 1.5)$*. In region 14 both inputs were optimally constrained at* -1*, while in reigons 7 and 2 the second input was optimally unconstrained, and finally in region 1 both inputs are unconstrained. Of the total number of 9 regions we here traversed 3. In the lower plot in figure 7 we see clearly how we change regions when one of the neighboring* c_k^i*'s changes sign.*

5 Conclusions

This paper has discussed properties of solutions to quadratic optimization problems using previous results from self-optimizing control. For the noise-free case it was shown that for quadratic optimization

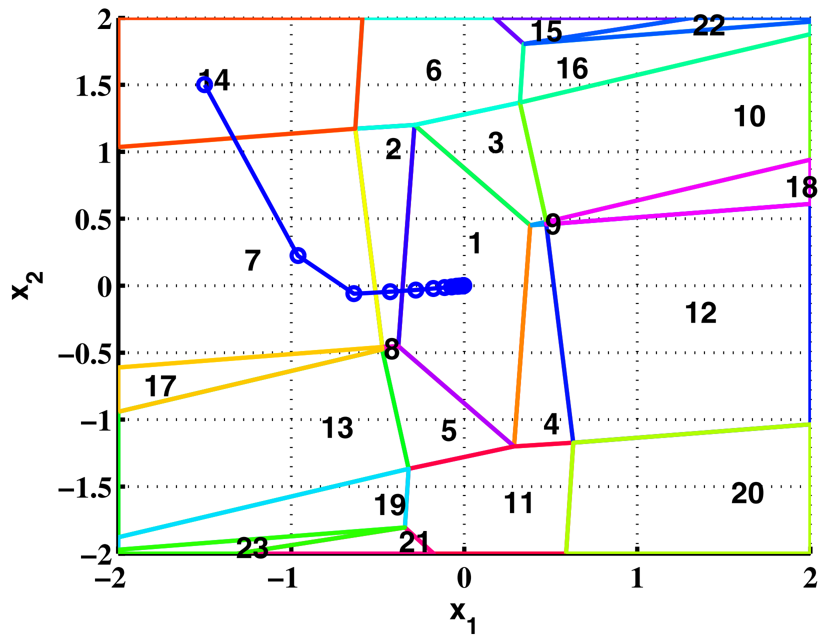

Figure 6: All regions for example 4.4, as well as state trajectory for simulation.

Region	u_1	u_2
14,6,15,22,16	-1	-1
23,19,21,11,20	1	1
17,8	-1	1
18,9	1	-1
13,5	unc.	1
10,3	unc.	-1
7,2	-1	unc.
12,4	1	unc.
1	unc.	unc.

Table 1: Regions and optimal inputs. “unc.” means that the input is not at an active constraint.

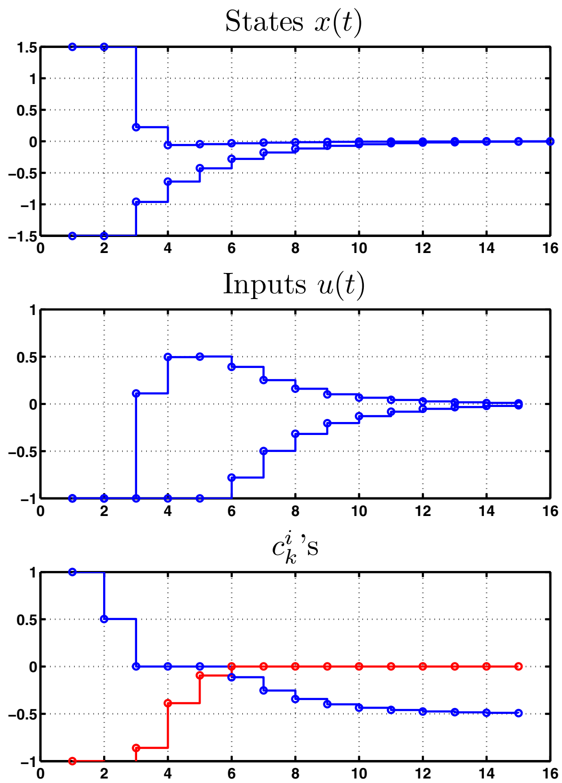

Figure 7: Simulation of MIMO system from $x_0 = (-1.5, 1.5)$ (Example 4.4).

problems there exists an infinite number of invariants to the parameters in the solution space. When the measurements are noisy (rather than perfect) methods were shown for minimizing the loss from optimality subject to a constant set-point policy of the controlled variables. These are related to the invariants for the noise-free case, but zero loss can no longer be guaranteed.

The insight of invariants were applied to the explicit MPC problem were we proposed an alternative proof of the fact that the optimal control law is on a state feedback form. The alternative proof led to a new way of calculating the feedback gain than was is used in standard solutions to optimal control problems.

By the insight of invariants a novel method for detecting when the set of active constraints shift were presented, and it was shown by an example how this can be applied to explicit MPC.

When not all states are assumed known at the present time methods were proposed for finding optimal output feedback gains. That this, given a chosen measurement combination the operational loss is minimised by choosing the correct gains. By the "exact local method" for noisy measurements (theorem 3.4), we showed that the order of the controller is actually free of choice, though optimality is not guaranteed.

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A Equivalence between $u_k = -(J_{uu}^{-1} J_{ud})x_k$ and open loop problem by Rawlings and Muske [1993]

Based on the following references: Rawlings and Muske [1993], Muske and Rawlings [1993]

Process model:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \quad k = 0, 1, 2, \dots \\ x_0 &: \text{assumed given} \\ A &: \text{assumed stable} \end{aligned} \tag{84}$$

Optimization vector is u^N, the N first input (from u_0 to u_{N-1}). Thereafter control is switched off: $u_k = 0, k \geq N$. Open-loop problem:

$$\min_{u^N} J = \sum_{k=0}^{\infty} \left({x_k}^{\mathrm{T}} Q x_k + {u_k}^{\mathrm{T}} R u_k \right) \tag{85}$$

$$= \min_{u^N} \sum_{k=0}^{N-1} \left({x_k}^{\mathrm{T}} Q x_k + {u_k}^{\mathrm{T}} R u_k \right) + \sum_{k=N}^{\infty} x_k Q x_k \tag{86}$$

$$= \min_{u^N} \sum_{k=0}^{N-1} \left({x_k}^{\mathrm{T}} Q x_k + {u_k}^{\mathrm{T}} R u_k \right) + x_N Q_N x_N \tag{87}$$

$$= \min_{u^N} {x_0}^{\mathrm{T}} Q x_0 + \sum_{k=1}^{N-1} \left({x_k}^{\mathrm{T}} Q x_k \right) + \sum_{k=0}^{N-1} \left({u_k}^{\mathrm{T}} R u_k \right) + x_N Q_N x_N \tag{88}$$

where Q_N is the solution of $Q_N = \sum_{j=N}^{\infty} \left((A^j)^{\mathrm{T}} Q A^j \right) = A^{\mathrm{T}} Q_N A + Q$.

From Bemporad et al. [2002] we have that:

$$x_k = A^k x_0 + \sum_{j=0}^{k-1} A^j B u_{k-1-j} \tag{89}$$

$$\Downarrow$$

$$x_1 = Ax_0 + A^0 B u_0 \tag{90}$$

$$x_2 = A^2 x_0 + ABu_0 + Bu_1 \tag{91}$$

$$x_3 = A^3 x_0 + A^2 B u_0 + ABu_1 + Bu_2 \tag{92}$$

$$x_{N-1} = A^{N-1} x_0 + A^{N-2} B u_0 + \cdots + Bu_{N-2} \tag{93}$$

We let $x = (x_1, x_2, \dots, x_{N-1})$. Then,

$$x = \underbrace{\begin{bmatrix} A \\ A^2 \\ A^3 \\ \vdots \\ A^{N-1} \end{bmatrix}}_{a} x_0 + \underbrace{\begin{bmatrix} B & 0 & 0 & \dots & 0 & 0 \\ AB & B & 0 & \dots & 0 & 0 \\ A^2B & AB & B & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ A^{N-2}B & A^{N-3}B & \dots & AB & B & 0 \end{bmatrix}}_{\hat{A}} u_N \tag{94}$$

Now,

$$\sum_{k=1}^{N-1} {x_k}^{\mathrm{T}} Q x_k = x^{\mathrm{T}} \hat{Q} x, \tag{95}$$

$$\hat{Q} = \mathrm{diag}(Q, Q, \dots, Q) \tag{96}$$

Further,

$$x^{\mathrm{T}} \hat{Q} x = (a x_0 + \hat{A} u_N)^{\mathrm{T}} \hat{Q} (a x_0 + \hat{A} u_N) \tag{97}$$

$$= (a x_0)^{\mathrm{T}} \hat{Q} (a x_0) + 2 {u_N}^{\mathrm{T}} \hat{A}^{\mathrm{T}} \hat{Q} a x_0 + {u_N}^{\mathrm{T}} \hat{A}^{\mathrm{T}} \hat{Q} \hat{A} u_N \tag{98}$$

We further rewrite

$$\sum_{k=0}^{N-1} {u_k}^{\mathrm{T}} R u_k = u^{\mathrm{T}} \hat{R} u, \tag{99}$$

$$\hat{R} = \mathrm{diag}(R, R, \dots, R) \tag{100}$$

The final state x_N can be written as

$$x_N = A^N x_0 + A^{N-1} B u_0 + \dots + B u_{N-1} \tag{101}$$

$$= A^N x_0 + \begin{bmatrix} A^{N-1}B & A^{N-2}B & \dots & AB & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{bmatrix} \tag{102}$$

$$= A^N x_0 + \underbrace{\begin{bmatrix} A^{N-1}B & A^{N-2}B & \dots & AB & B \end{bmatrix}}_{\Lambda} u^N, \tag{103}$$

so, the term ${x_N}^{\mathrm{T}} Q_N x_N$ can be written as

$${x_N}^{\mathrm{T}} Q x_N = (A^N x_0 + \Lambda u)^{\mathrm{T}} Q (A^N x_0 + \Lambda u) \tag{104}$$

$$= {x_0}^{\mathrm{T}} (A^N)^{\mathrm{T}} Q A^N x_0 + 2 {x_0}^{\mathrm{T}} (A^N)^{\mathrm{T}} Q_N \Lambda u + u^{\mathrm{T}} \Lambda^{\mathrm{T}} Q_N \Lambda u \tag{105}$$

We observe that

$$J_{uu} = 2 \left(\hat{A}^{\mathrm{T}} \hat{Q} \hat{A} + \Lambda^{\mathrm{T}} Q_N \Lambda + \hat{R} \right) \tag{106}$$

$$J_{ud} = 2 \left(\hat{A}^{\mathrm{T}} \hat{Q} a + (A^N)^{\mathrm{T}} Q_N \Lambda \right) \tag{107}$$

First we calculate the J_{uu} matrix. We observe that B can be "left out" for intermediate calculations, and we focus on adding $\widetilde{A}^{\mathrm{T}}\hat{Q}\widetilde{A} + \widetilde{\Lambda}^{\mathrm{T}}Q_N\widetilde{\Lambda}$, where $\hat{A} = \widetilde{A}B$ and $\Lambda = \widetilde{\Lambda}B$.

We get that

$$(\widetilde{A}^{\mathrm{T}}\hat{Q})\widetilde{A} = \begin{bmatrix} Q & A^{\mathrm{T}}Q & (A^2)^{\mathrm{T}}Q & \dots & (A^{N-2})^{\mathrm{T}}Q \\ 0 & Q & A^{\mathrm{T}}Q & \dots & \vdots \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & Q & A^{\mathrm{T}}Q \\ 0 & 0 & 0 & 0 & Q \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 & \dots & 0 & 0 \\ A & I & 0 & \dots & 0 & 0 \\ A^2 & A & I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ A^{N-2} & A^{N-3} & \dots & A & I & 0 \end{bmatrix} \tag{108}$$

and

$$\widetilde{\Lambda}^{\mathrm{T}}Q_N\widetilde{\Lambda} = \begin{bmatrix} (A^{N-1})^{\mathrm{T}}Q_N \\ (A^{N-2})^{\mathrm{T}}Q_N \\ \vdots \\ A^{\mathrm{T}}Q_N \\ Q_N \end{bmatrix} \begin{bmatrix} A^{N-1} & A^{N-2} & \dots & A & I \end{bmatrix}, \tag{109}$$

and by using the equation $Q_N = A^{\mathrm{T}}Q_NA + Q$, we see that "our" J_{uu} is equal to E_N in Rawlings and Muske [1993], that is

$$J_{uu} = 2\begin{bmatrix} B^{\mathrm{T}}Q_NB + R & B^{\mathrm{T}}A^{\mathrm{T}}KB & \cdots & B^{\mathrm{T}}(A^{N-1})^{\mathrm{T}}Q_NB \\ B^{\mathrm{T}}Q_NAB & B^{\mathrm{T}}Q_NB + R & \cdots & B^{\mathrm{T}}(A^{N-2})^{\mathrm{T}}Q_NB \\ \vdots & \vdots & \ddots & \vdots \\ B^{\mathrm{T}}Q_NA^{N-1}B & B^{\mathrm{T}}Q_NA^{N-2}B & \cdots & B^{\mathrm{T}}Q_NB + R \end{bmatrix} \tag{110}$$

$$J_ud = 2\left(\hat{A}\hat{Q}a + (A^{N\mathrm{T}}Q_N\Lambda)\right) = \left(\hat{A}\hat{Q}a + \Lambda^{\mathrm{T}}Q_NA_N\right)$$

$$\left(\hat{A}\hat{Q}\right)a = B^{\mathrm{T}}\begin{bmatrix} Q & A^{\mathrm{T}}Q & (A^2)^{\mathrm{T}}Q & \dots & (A^{N-2})^{\mathrm{T}}Q \\ 0 & Q & A^{\mathrm{T}}Q & \dots & \vdots \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & Q & A^{\mathrm{T}}Q \\ 0 & 0 & 0 & 0 & Q \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^{N-1} \end{bmatrix} \tag{111}$$

$$= \begin{bmatrix} QA + A^{\mathrm{T}}QA^2 + \dots + (A^{N-2})^{\mathrm{T}}QA^{N-1} \\ QA^2 + A^{\mathrm{T}}QA^3 + \dots + (A^{N-3})QA^{N-1} \\ \vdots \\ QA^{N-1} \\ 0 \end{bmatrix} \tag{112}$$

and

$$\Lambda^{\mathrm{T}}Q_NA_N = B^{\mathrm{T}}\begin{bmatrix} A^{N-1\mathrm{T}}Q_NA_N \\ A^{N-2\mathrm{T}}Q_NA_N \\ \vdots \\ A^{\mathrm{T}}Q_NA^N \\ Q_NA^N \end{bmatrix} \tag{113}$$

By yet again using the equation $Q_N = A^{\mathrm{T}} Q_N A + Q$, we get that $J_{ud} = 2{B_N}^{\mathrm{T}} G_N A$ in Rawlings and Muske [1993]:

$$J_{ud} = 2{B_N}^{\mathrm{T}} G_N A = 2 \begin{bmatrix} B^{\mathrm{T}} & & & \\ & B^{\mathrm{T}} & & \\ & & \ddots & \\ & & & B^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} Q_N \\ Q_N A \\ \vdots \\ Q_N A^{N-1} \end{bmatrix} A \tag{114}$$

Note that the factor 2 disappears as we form the product $J_{uu}^{-1} J_{ud}$.